ON DIVISIBILITY PROPERTIES OF SOME DIFFERENCES OF THE CENTRAL BINOMIAL COEFFICIENTS AND CATALAN NUMBERS

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Received: 5/1/12, Revised: 10/16/12, Accepted: 2/24/13, Published: 3/15/13

Abstract

We discuss divisibility properties of some differences of the central binomial coefficients and Catalan numbers. The main tool is the application of various congruences modulo high prime powers for binomial coefficients combined with some recurrences relevant to these combinatorial quantities.

1. Introduction

The differences of certain combinatorial quantities exhibit interesting divisibility properties. For example, we can consider differences of central binomial coefficients

$$c_n = \binom{2n}{n}, n \geq 0,$$

and Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}, n \geq 0.$$

We will need some basic notation. Let n and k be positive integers, p be a prime, $d_p(k)$ and $\nu_p(k)$ denote the sum of digits in the base p representation of k and the highest power of p dividing k, respectively. The latter one is often referred to as the p-adic order of k. For the rational n/k we set $\nu_p(n/k) = \nu_p(n) - \nu_p(k)$.

Our main results concern the p-adic order of the differences of central binomial coefficients $c_{ap^{n+1}+b} - c_{ap^n+b}$ (cf. Theorems 2.1 and 2.4), Catalan numbers $C_{ap^{n+1}+b} - C_{ap^n+b}$ (cf. Theorems 2.2 and 2.11, and Remark 4.1) with a prime p, $(a,p) = 1$, and $n \geq n_0$ for some integer $n_0 \geq 0$. These results are essential in obtaining the p-adic order of the differences of certain Motzkin numbers, more precisely $M_{ap^{n+1}+b} - M_{ap^n+b}$ with a prime p and different settings of a and b that are discussed in [8] and [9]. Of course, results involving the exact orders of differences

or lower bounds on them can be easily rephrased in terms of super congruences for the underlying quantities.

Section 2 collects some of the main results while Section 3 is devoted to some known results and their direct consequences regarding congruential and p-adic properties of the binomial coefficients, e.g., Corollary 3.9. Section 4 contains the proofs of the main results of Section 2 and presents some interesting congruences for central binomial coefficients (cf. Corollary 4.1 improving Corollary 3.5) and Catalan numbers (cf. Theorem 2.12). In Section 5, we derive and prove some important p-adic properties of the differences of certain harmonic numbers stated in Section 2. These properties are important in the actual use of Theorem 2.4.

2. Main Results

We state some of the main results of the paper.

Theorem 2.1. *For $p = 2$ and a odd, we have that*

$$\nu_2(c_{a2^{n+1}} - c_{a2^n}) = 3(n+1) + \nu_2\left(\binom{2a}{a}\right) - 1 = 3(n+1) + d_2(a) - 1, n \geq 1.$$

For $p = 3$ and $(a, 3) = 1$,

$$\nu_3(c_{a3^{n+1}} - c_{a3^n}) = 3(n+1) + \nu_3\left(\binom{2a}{a}\right) - 1, n \geq 0.$$

Let B_n denote the nth Bernoulli number. For any prime $p \geq 5$, $(a, p) = 1$, and $\nu_p(B_{p-3}) = 0$ or -1, we have that

$$\nu_p(c_{ap^{n+1}} - c_{ap^n}) = 3(n+1) + \nu_p\left(\binom{2a}{a}\right) + \nu_p(B_{p-3}), n \geq 0.$$

Note. The term $\nu_p(\binom{2a}{a})$ contributes to the p-adic orders above exactly if at least one of the p-ary digits of a is at least as large as $p/2$.

Remark 2.1. *It is well known that $\nu_p(B_n) \geq -1$ by the von Staudt–Clausen theorem. If the prime p divides the numerator of B_{p-3}, i.e., $\nu_p(B_{p-3}) \geq 1$, or equivalently $\binom{2p}{p} \equiv 2 \bmod p^4$, then it is sometimes called a Wolstenholme prime [2]. The only known Wolstenholme primes up to 10^9 are $p = 16843$ and 2124679. For such primes Theorem 2.1 is inconclusive and gives only the lower bound $3(n+1) + \nu_p(\binom{2a}{a}) + 1$ on the p-adic order.*

For the p-adic order of the differences of Catalan numbers we obtain the following theorem.

Theorem 2.2. *For any prime $p \geq 2$ and $(a,p) = 1$, we have*

$$\nu_p(C_{ap^{n+1}} - C_{ap^n}) = n + \nu_p\left(\binom{2a}{a}\right), n \geq 1.$$

The case with $n = 0$ is slightly different and included in

Lemma 2.3. *For $n \geq 1$ and any prime $p \geq 2$, we have $\nu_p(C_{ap^n}) = \nu_p(c_{ap^n}) = \nu_p\left(\binom{2a}{a}\right)$ and $\nu_p(C_{ap} - C_a) = \nu_p(C_a)$ if in addition $(a,p) = 1$.*

The nature of the p-adic order in Theorem 2.1 changes as we introduce an additive term b, and we get a result similar to Theorem 2.2.

Theorem 2.4. *For $p = 2$, a odd, and $n \geq n_0 = 1$ we have*

$$\nu_2(c_{a2^{n+1}+1} - c_{a2^n+1}) = n + \nu_2\left(\binom{2a}{a}\right) + 1,$$

and in general, for $b \geq 1$ and $n \geq n_0 = \lfloor \log_2 2b \rfloor$

$$\begin{aligned}\nu_2(c_{a2^{n+1}+b} - c_{a2^n+b}) &= n + \nu_2\left(\binom{2a}{a}\right) + \nu_2(f(b)) \\ &= n + d_2(a) + d_2(b) - \lfloor \log_2 b \rfloor,\end{aligned}$$

where $f(b) = 2\binom{2b}{b}(H_{2b} - H_b)$ with $H_n = \sum_{j=1}^{n} 1/j$ being the nth harmonic number.

For any prime $p \geq 3$, $(a,p) = 1$, and $b \geq 1$ we have that

$$\nu_p(c_{ap^{n+1}+b} - c_{ap^n+b}) = n + \nu_p\left(\binom{2a}{a}\right) + \nu_p(f(b)),$$

for $n \geq n_0 = \max\{\nu_p(f(b)) + 2r - \nu_p\left(\binom{2b}{b}\right) + 1, r+1\} = \max\{\nu_p(2(H_{2b} - H_b)) + 2r + 1, r+1\}$ and $r = \lfloor \log_p 2b \rfloor$.

In general, for any prime $p \geq 3$, $(a,p) = 1, b \geq 1$, and $n > \lfloor \log_p 2b \rfloor$, we have

$$\nu_p(c_{ap^{n+1}+b} - c_{ap^n+b}) \geq n + \nu_p\left(\binom{2a}{a}\right) + \nu_p\left(\binom{2b}{b}\right) - \lfloor \log_p 2b \rfloor$$

and for $p = 2$, a odd, and $n \geq \lfloor \log_2 2b \rfloor$

$$\nu_2(c_{a2^{n+1}+b} - c_{a2^n+b}) \geq n + \nu_2\left(\binom{2a}{a}\right) + \nu_2\left(\binom{2b}{b}\right) - \lfloor \log_2 b \rfloor.$$

We can make some useful statements on the magnitude of $\nu_p(f(b))$. By taking the common denominator in $H_{2b} - H_b$, we note that $\nu_p(f(b)) \geq \nu_p\left(\binom{2b}{b}\right) - \lfloor \log_p 2b \rfloor$ for $p \geq 3$ and $\nu_2(f(b)) = d_2(b) - \lfloor \log_2 b \rfloor$. This implies that $\nu_p(f(b)) \geq 0$ for $1 \leq b \leq (p-1)/2$. In this range the binomial factor of $f(b)$ can be dropped, leaving only $\nu_p(H_{2b} - H_b)$. It appears that $\nu_p(f(b)) = 0$ in many cases, however $f(2) = 7$ and $f(15) = 450351518582/2145$; thus, $\nu_p(f(2)) = 1$ and $\nu_p(f(15)) = 2$ with $p = 7$. Also note that for $p \geq 3$ we have $\nu_p(f(b)) = 0$ if $b + 1 \leq p \leq 2b$ and $\nu_p(f(p)) = -1$. In fact, a much stronger statement about $\nu_p(f(b))$ is given in

Theorem 2.5. *For $p = 2$ and $c \geq 1$, we have $\nu_2(f(c)) = d_2(c) - \lfloor \log_2 c \rfloor$. For $p \geq 3$, we have $\nu_p(f(p^k)) = -k, k \geq 1$, and in general, for $c \geq 1$ and $k \geq 0$, we have $\nu_p(f(cp^k)) = -k + \nu_p(f(c))$ provided that $\nu_p(H_{2c} - H_c) \leq 0$.*

We observe that $\nu_p(f(b))$; and therefore, the p-adic order of the difference $c_{ap^{n+1}+b} - c_{ap^n+b}$ changes only slightly for bs that are small relative to p.

Clearly, $\nu_2(f(b)) \leq 1$ and equality holds only if $b+1$ is a power of two. We observe that $\nu_3(f(b)) \leq 0$ and equality holds exactly if $b = 1$ or all ternary digits of b are at least one and $b \equiv 2 \bmod 3$, i.e., the least significant digit is two, as these facts follow from Theorem 2.8. Clearly, the study of $\nu_p(f(b))$ requires the understanding of the behavior of $\nu_p(H_{2b} - H_b)$. The next lemma and theorem are straightforward.

Lemma 2.6. *For any prime p and integer $b \geq 1$, we have $\nu_p(H_{2b} - H_b) \geq -\lfloor \log_p 2b \rfloor$.*

This follows immediately as the exponent in the largest power of p can not exceed $\lfloor \log_p 2b \rfloor$. Also note the following

Theorem 2.7. *For the positive integers $b < a$, $H_a - H_b$ is never an integer.*

We include the standard proof of this theorem which, with some tweaking, leads to the proof of the following theorem. Indeed, for the exact orders, we get

Theorem 2.8. *For $p = 2$ and 3, we have $\nu_p(H_{2b} - H_b) = -\lfloor \log_p 2b \rfloor$. For $p = 5$ we have*

$$\nu_5(H_{2b} - H_b) = -\lfloor \log_5 2b \rfloor + \chi_{\exists m: \frac{3}{2} \cdot 5^m < b < 2 \cdot 5^m} \tag{2.1}$$

with the indicator variable $\chi_{\exists m: \frac{3}{2} \cdot 5^m < b < 2 \cdot 5^m}$ which is 1 if for b there exists an m so that $\frac{3}{2} \cdot 5^m < b < 2 \cdot 5^m$ and otherwise, it is 0.

For primes larger than 5 it seems more complicated to establish the exact order of $\nu_p(H_{2b} - H_b)$. Also note that somewhat related investigations have been initiated by [4] in which the set $J(p) = \{b \mid \nu_p(H_b) \geq 1\}$ is analyzed and determined for some primes including $3, 5, 13, 17, 23$, and 67, e.g., $J(5) = \{4, 20, 24\}$, and $\{p-1, p(p-1), p^2-1\} \subseteq J(p)$ for $p > 3$. It has been conjectured that the set $J(p)$ is finite for all primes.

We also note that we could not find any b for which $\nu_p(f(b))$ exceeded two and thus suggest

Conjecture 2.9. *We have $\nu_p(f(b)) \leq 2$ for $p \geq 5$ and $b \geq 1$.*

In terms of the harmonic numbers, Conjecture 2.9 can be rephrased as

Conjecture 2.10. *We have $\nu_p(H_{2b} - H_b) \leq 2 - \nu_p(\binom{2b}{b})$ for $p \geq 5$ and $b \geq 1$.*

Finally, for the Catalan numbers we obtain

Theorem 2.11. *For $p = 2$, a odd, and $n \geq n_0 = 2$ we have*

$$\nu_2(C_{a2^{n+1}+1} - C_{a2^n+1}) = n + \nu_2\left(\binom{2a}{a}\right) - 1,$$

and in general, for $b \geq 1$ and $n \geq n_0 = \lfloor \log_2 2b \rfloor + 1$

$$\begin{aligned}\nu_2(C_{a2^{n+1}+b} - C_{a2^n+b}) &= n + \nu_2\left(\binom{2a}{a}\right) + \nu_2(g(b)) \\ &= n + d_2(a) + d_2(b) - \lceil \log_2(b+2) \rceil - \nu_2(b+1) + 1\end{aligned}$$

where $g(b) = 2\binom{2b}{b}(b+1)^{-1}(H_{2b} - H_b - 1/(2(b+1))) = 2C_b(H_{2b} - H_b - 1/(2(b+1))) = (f(b) - C_b)/(b+1)$ with $H_n = \sum_{j=1}^{n} 1/j$.

For any prime $p \geq 3$, $(a,p) = 1$, and $b \geq 1$ we have that

$$\nu_p(C_{ap^{n+1}+b} - C_{ap^n+b}) = n + \nu_p\left(\binom{2a}{a}\right) + \nu_p(g(b)),$$

with $n \geq n_0 = \max\{\nu_p(g(b)) + 2r - \nu_p(C_b) + 1, r+1\} = \max\{\nu_p(2(H_{2b} - H_b - 1/(2(b+1)))) + 2r + 1, r+1\}$ and $r = \lfloor \log_p 2b \rfloor$.

In general, for any prime $p \geq 2$, $(a,p) = 1, b \geq 1$, and $n > \lfloor \log_p 2b \rfloor$, we have

$$\nu_p(C_{ap^{n+1}+b} - C_{ap^n+b}) \geq n + \nu_p\left(\binom{2a}{a}\right) + \nu_p\left(\binom{2b}{b}\right) - \lfloor \log_p 2b \rfloor - \nu_p(b+1).$$

Note. Clearly, $\nu_p(g(b)) \geq 0$ for $1 \leq b \leq (p-1)/2$. We note that in general, for $b \geq 1$ we have $\nu_p(g(b)) \geq \nu_p(\binom{2b}{b}) - \lfloor \log_p 2b \rfloor - \nu_p(b+1)$ if $p \geq 2$ while $\nu_2(g(b)) = d_2(b) - \lceil \log_2(b+2) \rceil - \nu_2(b+1) + 1 = d_2(b+1) - \lceil \log_2(b+2) \rceil$ if $p = 2$.

Remark 2.2. *The combination of Theorems 2.2 and 2.11 proves some kind of generalization of the observation in [11] that for any $n \geq 2$ the remainders $C_{2^{n+m-1}-1} \bmod 2^n$ are equal for each $m \geq 0$. The latter fact has been proven in [10] and [13] and extended in [14] recently.*

Our result focuses on $C_{ap^m+b} \bmod p^n$, with any large enough m, for non-negative values of b only.

Theorem 2.12. *For any prime $p \geq 2$, $(a,p) = 1, b \geq 0$, we have that $C_{ap^m+b} \bmod p^n$ is constant for $m \geq n + \nu_p(b+1) + \max\{0, \lfloor \log_p 2b \rfloor\}, n \geq 1$.*

3. Preparation

We note that there are many places in the literature where relevant divisibility and congruential properties of the binomial coefficients are discussed. Excellent surveys can be found in [6] and [12].

The following three theorems comprise the most basic facts regarding divisibility and congruence properties of the binomial coefficients. We assume that $0 \le k \le n$.

Theorem 3.1 (Kummer, 1852). *The power of a prime p that divides the binomial coefficient $\binom{n}{k}$ is given by the number of carries when we add k and $n-k$ in base p.*

Theorem 3.2 (Legendre, 1830). *We have $\nu_p\big(\binom{n}{k}\big) = \frac{n-d_p(n)}{p-1} - \frac{k-d_p(k)}{p-1} - \frac{n-k-d_p(n-k)}{p-1} = \frac{d_p(k)+d_p(n-k)-d_p(n)}{p-1}$. In particular, $\nu_2\big(\binom{n}{k}\big) = d_2(k) + d_2(n-k) - d_2(n)$ represents the carry count in the addition of k and $n-k$ in base 2.*

From now on M and N will denote integers such that $0 \le M \le N$.

Theorem 3.3 (Lucas, 1877). *Let $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d$ and $M = m_0 + m_1 p + \cdots + m_d p^d$ with $0 \le n_i, m_i \le p-1$ for each i, be the base p representations of N and M, respectively. Then*

$$\binom{N}{M} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_d}{m_d} \bmod p.$$

Lucas' theorem has some remarkable extensions.

Theorem 3.4 (Anton, 1869, Stickelberger, 1890, Hensel, 1902). *Let $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d, M = m_0 + m_1 p + \cdots + m_d p^d$ and $R = N - M = r_0 + r_1 p + \cdots + r_d p^d$ with $0 \le n_i, m_i, r_i \le p-1$ for each i, be the base p representations of N, M, and $R = N - M$, respectively. Then with $q = \nu_p\big(\binom{N}{M}\big)$,*

$$(-1)^q \frac{1}{p^q}\binom{N}{M} \equiv \left(\frac{n_0!}{m_0! r_0!}\right)\left(\frac{n_1!}{m_1! r_1!}\right)\cdots\left(\frac{n_d!}{m_d! r_d!}\right) \bmod p.$$

An immediate consequence of this theorem is

Corollary 3.5. *For $(a,p) = 1, b \ge 1, n \ge 1$, with $q = \nu_p\big(\binom{2a}{a}\big)$, $t = \nu_p\big(\binom{2b}{b}\big)$, and $t \le \lfloor \log_p 2b \rfloor < n$, we have that*

$$c_{ap^n+b} = \binom{2ap^n + 2b}{ap^n + b} \equiv \binom{2a}{a}\binom{2b}{b} \bmod p^{q+t+1}.$$

This congruence also holds for $b = 0$ with $n \ge 0$.

Davis and Webb (1990) and Granville (1995) have independently generalized Lucas' theorem and its extension Theorem 3.4. Here we use the latter version whose strength is in the choice of modulus which can be arbitrarily large. For a given integer n and prime p, we define $(n!)_p = n!/(p^{\lfloor n/p \rfloor}\lfloor n/p \rfloor!)$ to be the product of positive integers not exceeding n and not divisible by p, and which is closely related to the p-adic Morita gamma function.

Theorem 3.6. (Granville, 1995 in [6]) *Let* $N = (n_d, \dots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d, M = m_0 + m_1 p + \cdots + m_d p^d$ *and* $R = N - M = r_0 + r_1 p + \cdots + r_d p^d$ *with* $0 \le n_i, m_i, r_i \le p-1$ *for each* i*, be the base* p *representations of* $N, M,$ *and* $R = N - M$*, respectively. Let* $N_j = n_j + n_{j+1} p + \cdots + n_{j+k-1} p^{k-1}$ *for each* $j \ge 0$*, i.e., the least positive residue of* $\lfloor N/p^j \rfloor \bmod p^k$ *with some integer* $k \ge 1$*; also make the corresponding definitions for* M_j *and* R_j*. Let* ε_j *be the number of carries when adding* M *and* R *on and beyond the* j*th digit. Then with* $q = \varepsilon_0 = \nu_p\big(\binom{N}{M}\big)$*,*

$$\frac{1}{p^q}\binom{N}{M} \equiv (\pm 1)^{\varepsilon_{k-1}} \left(\frac{(N_0!)_p}{(M_0!)_p (R_0!)_p}\right)\left(\frac{(N_1!)_p}{(M_1!)_p (R_1!)_p}\right) \cdots \left(\frac{(N_d!)_p}{(M_d!)_p (R_d!)_p}\right) \bmod p^k$$

where ± 1 *is* -1 *except if* $p = 2$ *and* $k \ge 3$*.*

We also need another related theorem.

Theorem 3.7. (Granville, 1995, (39) in [6]) *We have*

$$(p^{K+1}!)_p \equiv (p^K!)_p^p \bmod p^{3K+1}, \quad \textit{for} \quad p \ge 5, \textit{and}$$
$$(p^{K+1}!)_p \equiv (p^K!)_p^p \bmod p^{3K-1}, \quad \textit{for} \quad p = 2, 3 \quad \textit{except if} \quad p^K = 2.$$

The following generalization of the Jacobstahl–Kazandzidis [2] congruences will be helpful in proving Theorem 2.1 and congruence (4.5).

Theorem 3.8. (Corollary 11.6.22 [2]) *Let* M *and* N *such that* $0 \le M \le N$ *and* p *prime. We have*

$$\binom{pN}{pM} \equiv \begin{cases} \left(1 - \frac{B_{p-3}}{3} p^3 NM(N-M)\right)\binom{N}{M} & \bmod\ p^4 NM(N-M)\binom{N}{M},\ \textit{if } p \ge 5, \\ (1 + 45NM(N-M))\binom{N}{M} & \bmod\ p^4 NM(N-M)\binom{N}{M},\ \textit{if } p = 3, \\ (-1)^{M(N-M)} P(N,M)\binom{N}{M} & \bmod\ p^4 NM(N-M)\binom{N}{M},\ \textit{if } p = 2, \end{cases}$$

where $P(N,M) = 1 + 6NM(N-M) - 4NM(N-M)(N^2 - NM + M^2) + 2(NM(N-M))^2$*.*

We note that a stronger version of Corollary 3.5 also follows by Theorem 3.8.

Corollary 3.9. *For* $(a,p) = 1, b \ge 1, n \ge 1$*, with* $q = \nu_p\big(\binom{2a}{a}\big)$*,* $t = \nu_p\big(\binom{2b}{b}\big)$*, and* $\lfloor \log_p 2b \rfloor + h(p) < n$*, we have that*

$$c_{ap^n+b} = \binom{2ap^n + 2b}{ap^n + b} \equiv \binom{2a}{a}\binom{2b}{b} \bmod p^{q+t+h(p)},$$

where

$$h(p) = \begin{cases} 3 + \nu_p(B_{p-3}), & if\ p \ge 5, \\ 2, & if\ p = 3, \\ 1, & if\ p = 2. \end{cases}$$

The congruence also holds for $b = 0$ *with* $n \ge 1$*. The exponents of* p *are best possible in the sense that* $\nu_p\big(\binom{2ap^n+2b}{ap^n+b} - \binom{2a}{a}\binom{2b}{b}\big) = q + t + h(p)$*.*

4. Proofs and More Results

This section contains the proofs of the main theorems (Theorems 2.1, 2.2, 2.4, and 2.11) and present some additional results.

To shed light on the nature of the problems at hand, we note that by Legendre's theorem $\nu_2(c_k) = d_2(k)$, i.e., $\nu_2(c_{2^{n+1}}) = \nu_2(c_{2^n}) = 1$. In a similar fashion,

$$\nu_2(C_k) = d_2(k) - \nu_2(k+1) = d_2(k+1) - 1, \tag{4.1}$$

i.e., $\nu_2(C_{2^{n+1}}) = \nu_2(C_{2^n}) = 1$. It also follows that C_k is odd if and only if $k = 2^q - 1$ for some integer $q \geq 0$.

A natural way to start proving Theorem 2.1 is to apply a generalization of Lucas' theorem. However, it turns out that it will provide only lower bounds on $\nu_p(c_{ap^{n+1}} - c_{ap^n})$ although it will suffice to prove Theorem 2.2. For example, with $p = 2, a = 1, q = 1, K = n+1$, and $k = 3(n+1)$, and combining Theorems 3.6 and 3.7, it follows that

$$\nu_2(c_{2^{n+1}} - c_{2^n}) \geq 3(n+1), n \geq 1.$$

In fact, we set $N = 2^{n+2}$ and $M = 2^{n+1}$ and obtain that

$$\frac{1}{2}\binom{2^{n+2}}{2^{n+1}} \equiv \frac{(2^{n+2}!)_2}{(2^{n+1}!)_2^2} \prod_{j=1}^{n+1} \left(\frac{(2^{n+2-j}!)_2}{(2^{n+1-j}!)_2^2} \right) \bmod 2^{3(n+1)}$$

and

$$\frac{1}{2}\binom{2^{n+1}}{2^n} \equiv \prod_{j=1}^{n+1} \left(\frac{(2^{n+2-j}!)_2}{(2^{n+1-j}!)_2^2} \right) \bmod 2^{3(n+1)}.$$

From here on, o_i, e_i, and $a_i, i \geq 1$, denote odd, even, or arbitrary non-negative integers, respectively, whose actual values are of no significance. By Theorem 3.7, we get

$$\begin{aligned} c_{2^{n+1}} - c_{2^n} = \binom{2^{n+2}}{2^{n+1}} - \binom{2^{n+1}}{2^n} &\equiv 2\left(\frac{(2^{n+2}!)_2}{(2^{n+1}!)_2^2} - 1 \right) o_1 \equiv \\ &\equiv 2 \cdot 2^{3(n+1)-1} a_1 \bmod 2^{3(n+1)+1}. \end{aligned} \tag{4.2}$$

It turns out that a_1 is always an odd number as it can be derived by Theorem 3.8. In fact, this will be already sufficient to obtain a full

Proof of Theorem 2.1. We set $N = 2ap^n$ and $M = ap^n$ and proceed as follows. If $p \geq 5$ then we get that

$$\binom{pN}{pM} - \binom{N}{M} \equiv -\frac{B_{p-3}}{3} p^3 NM(N-M)\binom{N}{M} \bmod p^4 NM(N-M)\binom{N}{M};$$

thus, $\nu_p(c_{ap^{n+1}} - c_{ap^n}) = 3(n+1) + \nu_p(\binom{2a}{a}) + \nu_p(B_{p-3})$ by the assumption on $\nu_p(B_{p-3})$ and $\nu_p(\binom{N}{M}) = \nu_p(\binom{2a}{a})$.

If $p = 3$ then

$$\binom{pN}{pM} - \binom{N}{M} \equiv 5 \cdot 3^2 \cdot NM(N-M)\binom{N}{M} \bmod p^4 NM(N-M)\binom{N}{M};$$

thus, $\nu_3(c_{a3^{n+1}} - c_{a3^n}) = 3(n+1) - 1 + \nu_p(\binom{2a}{a})$.

If $p = 2$ and $n \geq 1$ then

$$\binom{pN}{pM} - \binom{N}{M} \equiv 2 \cdot 3 \cdot NM(N-M)\binom{N}{M} \bmod p^4 NM(N-M)\binom{N}{M};$$

thus, $\nu_2(c_{a2^{n+1}} - c_{a2^n}) = 1 + (n+1) + n + n + \nu_2(\binom{2a}{a}) = 3(n+1) - 1 + d_2(a)$. Note that here we need $n \geq 1$ otherwise the factor $(-1)^{M(N-M)}$ evaluates to -1 in Theorem 3.8. □

Proof of Theorem 2.2. For $p = 2$ we can prove the statement by Theorem 3.8 but in fact, Theorem 3.7 suffices. As in (4.2), we get

$$\begin{aligned} C_{2^{n+1}} - C_{2^n} &= \frac{c_{2^{n+1}}}{2^{n+1}+1} - \frac{c_{2^n}}{2^n+1} \equiv \left(\frac{1}{2^{n+1}+1} 2 \frac{(2^{n+2}!)_2}{(2^{n+1}!)_2^2} - \frac{1}{2^n+1} 2\right) o_1 \\ &\equiv 2\left(\frac{1 + a_2 2^{3(n+1)-1}}{2^{n+1}+1} - \frac{1}{2^n+1}\right) o_1 \equiv o_3 2^{n+1} \bmod 2^{3(n+1)}. \end{aligned}$$

If $p \geq 3$ then we cannot apply Theorem 3.7. However, by taking all factors in the numerator and denominator modulo p^{n+1}, we get that

$$\frac{((2ap^{n+1})!)_p}{((ap^{n+1})!)_p^2} \equiv 1 \bmod p^{n+1}.$$

By the repeated application of the definition of $(m!)_p$, we get that

$$\binom{2ap^n}{ap^n} = \frac{((2ap^n)!)_p}{((ap^n)!)_p^2}\binom{2ap^{n-1}}{ap^{n-1}} = \ldots = \prod_{j=1}^{n} \frac{((2ap^j)!)_p}{((ap^j)!)_p^2} \cdot \binom{2a}{a},$$

and thus,

$$\binom{2ap^n}{ap^n} = f^* p^q \tag{4.3}$$

with $q = \nu_p\big(\binom{2a}{a}\big)$ and

$$f^* = \frac{\binom{2a}{a}}{p^q} \prod_{j=1}^{n} \frac{((2ap^j)!)_p}{((ap^j)!)_p^2}.$$

(This also follows by Theorem 3.6 applied with a sufficiently large k.) We set $f' = a(1-p)f^*$ and observe that $\nu_p(f^*) = \nu_p(f') = 0$ and

$$\begin{aligned} C_{ap^{n+1}} - C_{ap^n} &= \frac{c_{ap^{n+1}}}{ap^{n+1}+1} - \frac{c_{ap^n}}{ap^n+1} \equiv p^q\left(\frac{1}{ap^{n+1}+1}\frac{((2ap^{n+1})!)_p}{((ap^{n+1})!)_p^2} - \frac{1}{ap^n+1}\right)f^* \\ &\equiv p^q\left(\frac{1+a_3p^{n+1}}{ap^{n+1}+1} - \frac{1}{ap^n+1}\right)f^* \equiv \frac{a(1-p)+a_4p}{(ap^{n+1}+1)(ap^n+1)}f^*p^{n+q} \\ &\equiv f'p^{n+q} \bmod p^{n+1+q}, \end{aligned} \tag{4.4}$$

and the proof is complete. □

Remark 4.1. *Theorems 2.1 and 2.2 can be viewed as corollaries of Theorem 3.8. In fact, by using Theorem 3.8 we can improve (4.4) if we replace the term a_3p^{n+1} by a_5p^{3n} (although a_5p^{2n} suffices) and we obtain*

$$C_{ap^{n+1}} - C_{ap^n} \equiv f'p^{n+q} \bmod p^{2n+q}. \tag{4.5}$$

Proof of Theorem 2.4. First we prove the lower bounds then we derive the exact orders by applying a recurrence. We observe for $M = N/2$ that

$$\binom{pN+2b}{pM+b} = \binom{pN}{pM}\frac{(pN+1)(pN+2)\cdots(pN+2b)}{\big((pM+1)(pM+2)\cdots(pM+b)\big)^2} \tag{4.6}$$

and

$$\binom{N+2b}{M+b} = \binom{N}{M}\frac{(N+1)(N+2)\cdots(N+2b)}{\big((M+1)(M+2)\cdots(M+b)\big)^2}. \tag{4.7}$$

We use the setting $N = 2ap^n$ and $M = ap^n$ from the proof of Theorem 2.1 and take the factors of the second factor modulo p^n on the right-hand side in the above equations (4.6) and (4.7) for $2 \le 2b < p$ (and thus, $r = \lfloor \log_p 2b \rfloor = 0$ and $t = \nu_p(\binom{2b}{b}) = 0$). By Theorem 2.1, with some f_i so that $\nu_p(f_i) \ge 0$ for each $i \ge 1$, and $q = \nu_p(\binom{N}{M}) = \nu_p(\binom{2a}{a})$, we get that

$$\begin{aligned} c_{ap^{n+1}+b} - c_{ap^n+b} &= \binom{pN+2b}{pM+b} - \binom{N+2b}{M+b} \\ &= \binom{pN}{pM}\binom{2b}{b}\left(1+f_1p^n\right) - \binom{N}{M}\binom{2b}{b}\left(1+f_2p^n\right) \\ &= \binom{N}{M}\left(1+f_3p^{3n}\right)\binom{2b}{b}\left(1+f_1p^n\right) - \binom{N}{M}\binom{2b}{b}\left(1+f_2p^n\right) \\ &\equiv 0 \bmod p^{n+q} \end{aligned}$$

which already guarantees that $\nu_p(c_{ap^{n+1}+b} - c_{ap^n+b}) \ge n + \nu_p(\binom{2a}{a})$ for $1 \le b \le (p-1)/2$.

For larger values of b, with some modifications that also allow us to handle terms of the second factor in (4.6) and (4.7) that are multiples of p, we get that

$$\begin{aligned} c_{ap^{n+1}+b} - c_{ap^n+b} &= \binom{pN+2b}{pM+b} - \binom{N+2b}{M+b} \\ &= \binom{N}{M}\left(1+f_3p^{3n}\right)\binom{2b}{b}\left(1+f_4p^{n-r}\right) \\ &- \binom{N}{M}\binom{2b}{b}\left(1+f_5p^{n-r}\right) \\ &\equiv 0 \bmod p^{n+q+t-r} \end{aligned} \tag{4.8}$$

with $r = \lfloor \log_p 2b \rfloor < n$. Indeed, if $i : 1 \le i \le 2b$, $s(i) = \nu_p(i) \le r < n$, and $t = \nu_p\left(\binom{2b}{b}\right)$ then e.g., with some f_i' and f_i" so that $\nu_p(f_i'), \nu_p(f_i") \ge 0$ for each $i \ge 1$, we obtain that

$$\begin{aligned} &\frac{(N+1)(N+2)\cdots(N+2b)}{((M+1)(M+2)\cdots(M+b))^2} = \frac{\prod_{i=1}^{2b} p^{s(i)}(Np^{-s(i)}+ip^{-s(i)})}{(\prod_{i=1}^{b} p^{s(i)}(Mp^{-s(i)}+ip^{-s(i)}))^2} = \\ &p^t\prod_{i=1}^{b}(1+f_i'p^{n-s(i)})\frac{\prod_{i=b+1}^{2b}(f_i"p^{n-s(i)}+ip^{-s(i)})}{\prod_{i=1}^{b}(f_i"p^{n-s(i)}+ip^{-s(i)})} = p^t\frac{\binom{2b}{b}}{p^t}(1+f_6p^{n-r}) \end{aligned} \tag{4.9}$$

with r being the maximum value of $s(i), 1 \le i \le 2b$.

It follows that

$$\nu_p(c_{ap^{n+1}+b} - c_{ap^n+b}) \ge n + \nu_p\left(\binom{2a}{a}\right) + \nu_p\left(\binom{2b}{b}\right) - \lfloor \log_p 2b \rfloor$$

for $b \ge 1$. Meanwhile we also proved another useful version of Corollary 3.5 which involves n in the exponent of the modulus.

Corollary 4.1. *For $(a,p) = 1, b \ge 1, n \ge 1$, with $q = \nu_p\left(\binom{2a}{a}\right)$, $t = \nu_p\left(\binom{2b}{b}\right)$, and $r = \lfloor \log_p 2b \rfloor < n$, we have that*

$$c_{ap^n+b} = \binom{2ap^n+2b}{ap^n+b} \equiv \binom{2ap^n}{ap^n}\binom{2b}{b} \bmod p^{n+q+t-r}.$$

To find the exact order $\nu_p(\Delta_b)$ of $\Delta_b = c_{ap^{n+1}+b} - c_{ap^n+b}$, another approach is suggested by the part of the statement that the p-adic order only slightly depends

on b in the given range. We set

$$\begin{aligned}\Delta_{b+1} &= c_{ap^{n+1}+b+1} - c_{ap^n+b+1} = \binom{2ap^{n+1}+2b+2}{ap^{n+1}+b+1} - \binom{2ap^n+2b+2}{ap^n+b+1} \\ &= \binom{2ap^{n+1}+2b}{ap^{n+1}+b} \frac{(2ap^{n+1}+2b+1)(2ap^{n+1}+2b+2)}{(ap^{n+1}+b+1)^2} \\ &- \binom{2ap^n+2b}{ap^n+b} \frac{(2ap^n+2b+1)(2ap^n+2b+2)}{(ap^n+b+1)^2} \\ &= 2\Delta_b \frac{2ap^{n+1}+2b+1}{ap^{n+1}+b+1} + \binom{2ap^n+2b}{ap^n+b} \frac{2a(p-1)p^n}{(ap^{n+1}+b+1)(ap^n+b+1)}\end{aligned}$$

which yields

$$\frac{\Delta_{b+1}}{2^{b+1}} = \frac{\Delta_b}{2^b} \frac{2ap^{n+1}+2b+1}{ap^{n+1}+b+1} + \binom{2ap^n+2b}{ap^n+b} \frac{ap^n(p-1)}{2^b(ap^{n+1}+b+1)(ap^n+b+1)}. \tag{4.10}$$

In order to determine $\nu_p(\Delta_b)$, we first deal with the case $1 \le b \le (p-1)/2+1$. The case with $p = 2$, a odd, and $b = 1$ immediately follows by setting b to 0 in (4.10) and using Theorem 2.1 on $\nu_2(\Delta_0)$.

For $p \ge 3$, Theorem 2.1 already yields $\nu_p(\Delta_0)$, $\nu_p(\Delta_1) = n + \nu_p\big(\binom{2a}{a}\big)$, and thus, $\nu_p(\Delta_b) \ge n + \nu_p\big(\binom{2a}{a}\big)$. Also, if we can justify that $\nu_p(\Delta_{(p-1)/2}) = n + \nu_p\big(\binom{2a}{a}\big)$ then identity (4.10) implies that $\nu_p(\Delta_{(p-1)/2+1}) = n + \nu_p\big(\binom{2a}{a}\big)$, too, since now $\nu_p(2b+1) = 1$. The latter comment will take care of the largest value of b in the given range.

By the recurrence (4.10), we obtain that

$$\begin{aligned}\frac{\Delta_{b+1}}{2^{b+1}} &= \sum_{i=0}^{b} \binom{2ap^n+2i}{ap^n+i} \frac{ap^n(p-1)}{2^i(ap^{n+1}+i+1)(ap^n+i+1)} \prod_{j=i+1}^{b} \frac{2ap^{n+1}+2j+1}{ap^{n+1}+j+1} \\ &+ \Delta_0 \prod_{j=0}^{b} \frac{2ap^{n+1}+2j+1}{ap^{n+1}+j+1}.\end{aligned} \tag{4.11}$$

We start with the first summand on the right-hand side. The first factor $\binom{2ap^n+2i}{ap^n+i}, 0 \le i \le b$, can be handled by Corollary 4.1, and it contributes a constant factor $\binom{2ap^n}{ap^n}$ with p-adic order $q = \nu_p\big(\binom{2ap^n}{ap^n}\big) = \nu_p\big(\binom{2a}{a}\big)$, and a factor $\binom{2i}{i} \bmod p^m$ with $m = n + t - r \le n$ (since $t \le r$) and any large enough n so that $\max\{0, \nu_p(f(b))\} < m \le n$. (We can choose $m = 1$ if $\nu_p(f(b)) \le 0$, which in fact means $\nu_p(f(b)) = 0$ in the range $1 \le b \le (p-1)/2$.)

The second factor contributes an extra p-adic order n. We factor out the constant factors (i.e., those that are independent of i) including $a(p-1)$ and the one with p at the exponent $n+q$. After this, we take everything modulo p^m and get

$$\sum_{i=0}^{b}\binom{2i}{i}\frac{1}{2^i(i+1)^2}\prod_{j=i+1}^{b}\frac{2j+1}{j+1}=\frac{\binom{2b+2}{b+1}}{2^{b+1}}\sum_{i=0}^{b}\frac{1}{(2i+1)(i+1)}$$
$$=\frac{\binom{2b+2}{b+1}}{2^b}(H_{2b+2}-H_{b+1}). \tag{4.12}$$

Finally, the second summand on the right-hand side of (4.11) has a p-adic order larger than the other terms by Theorem 2.1. We cross multiply with 2^{b+1} to complete the proof for $\nu_p(\Delta_{b+1})$.

Note that n can be selected so that $n \geq n_0 = \nu_p(f(b)) + 1$ to get $\nu_p(\Delta_b)$.

From now on in this section $f_k, k \geq 7$, with $\nu_p(f_k) \geq 0$, may depend on i and j. For larger values of b, i.e., if $b \geq (p-1)/2+2$, the equation (4.12) turns into

$$\sum_{i=0}^{b}\binom{2i}{i}\left(1+f_7p^{n-r}\right)\frac{1}{2^i(i+1)^2}\left(1+f_8p^{n-r}\right)\prod_{j=i+1}^{b}\frac{2j+1}{j+1}\left(1+f_9p^{n-r}\right)$$
$$=\frac{\binom{2b+2}{b+1}}{2^{b+1}}\sum_{i=0}^{b}\left(\frac{1}{(2i+1)(i+1)}\left(1+f_{10}p^{n-r}\right)\right)$$
$$=\frac{\binom{2b+2}{b+1}}{2^b}(H_{2b+2}-H_{b+1})+f_{11}p^{n-r-r'+\nu_p\left(\binom{2b+2}{b+1}\right)} \tag{4.13}$$

with $r' = \lfloor \log_p(2b+1) \rfloor$ via Corollary 4.1 and in a similar fashion to the case with smaller bs and using that $\nu_p((2i+1)(i+1)) \leq \lfloor \log_p(2b+1) \rfloor = r'$ for $2i+1 \leq 2b+1$. Therefore, $n \geq n_0 = \max\{\nu_p(f(b)) + 2r - \nu_p\left(\binom{2b}{b}\right)+1, r+1\}$ suffices in order to get $\nu_p(\Delta_b)$.

Note that in (4.13) we used the fact that we could write e.g.,

$$\frac{1}{ap^n+i+1}=\frac{1}{p^{s(i+1)}(ap^{n-s(i+1)}+(i+1)p^{-s(i+1)})}$$
$$=\frac{1}{p^{s(i+1)}}\left(\frac{1}{(i+1)p^{-s(i+1)}}+f_{12}p^{n-s(i+1)}\right)$$
$$=\frac{1}{(i+1)}\left(1+f_{13}p^{n-s(i+1)}\right)$$

which guarantees that in the range $0 \leq i \leq b$, $b \geq 1$, we get

$$\frac{1}{ap^n+i+1}=\frac{1}{(i+1)}\left(1+f_{14}p^{n-r}\right).$$

Finally, for $p = 2$ and $1 \le b < 2^n$, i.e., $n \ge \lfloor \log_2 2b \rfloor$, equations (4.11), (4.12), (4.13) and the above argument with some modifications yield that

$$\nu_2(\Delta_b) = n + \nu_2\left(\binom{2a}{a}\right) + \nu_2(f(b)) = n + d_2(a) + d_2(b) - \lfloor \log_2 b \rfloor.$$

□

We can modify the above proof and obtain the

Proof of Theorem 2.11. The proof is similar to that of Theorem 2.4. We set $N = 2ap^n$ and $M = ap^n$ again and rewrite (4.8) for $b \ge 1$,

$$\begin{aligned}
\Delta_b^* = \frac{c_{ap^{n+1}+b}}{ap^{n+1}+b+1} - \frac{c_{ap^n+b}}{ap^n+b+1} &= \frac{\binom{pN+2b}{pM+b}}{pM+b+1} - \frac{\binom{N+2b}{M+b}}{M+b+1} \\
&= \binom{N}{M}\left(1+f_3p^{3n}\right)\binom{2b}{b}\left(1+f_4p^{n-r}\right)\frac{1}{b+1}\left(1+f_{15}p^{n-s(b+1)}\right) \\
&- \binom{N}{M}\binom{2b}{b}\left(1+f_5p^{n-r}\right)\frac{1}{b+1}\left(1+f_{16}p^{n-s(b+1)}\right) \\
&\equiv 0 \bmod p^{n+q+t-r-\nu_p(b+1)}
\end{aligned}$$

which guarantees the inequality on the order $\nu_p(\Delta_b^*)$.

Now we also have

$$\begin{aligned}
\Delta_{b+1}^* = \frac{c_{ap^{n+1}+b+1}}{ap^{n+1}+b+2} - \frac{c_{ap^n+b+1}}{ap^n+b+2} &= \frac{\binom{2ap^{n+1}+2b+2}{ap^{n+1}+b+1}}{ap^{n+1}+b+2} - \frac{\binom{2ap^n+2b+2}{ap^n+b+1}}{ap^n+b+2} \\
&= \binom{2ap^{n+1}+2b}{ap^{n+1}+b}\frac{(2ap^{n+1}+2b+1)(2ap^{n+1}+2b+2)}{(ap^{n+1}+b+1)^2(ap^{n+1}+b+2)} \\
&- \binom{2ap^n+2b}{ap^n+b}\frac{(2ap^n+2b+1)(2ap^n+2b+2)}{(ap^n+b+1)^2(ap^n+b+2)} \\
&= 2\Delta_b^*\frac{2ap^{n+1}+2b+1}{ap^{n+1}+b+2} + \frac{\binom{2ap^n+2b}{ap^n+b}}{ap^n+b+1}\frac{6a(p-1)p^n}{(ap^{n+1}+b+2)(ap^n+b+2)}.
\end{aligned}$$

It follows that

$$\frac{\Delta_{b+1}^*}{2^{b+1}} = \frac{\Delta_b^*}{2^b}\frac{2ap^{n+1}+2b+1}{ap^{n+1}+b+2} + \frac{\binom{2ap^n+2b}{ap^n+b}}{ap^n+b+1}\frac{3ap^n(p-1)}{2^b(ap^{n+1}+b+2)(ap^n+b+2)} \quad (4.14)$$

which can be rewritten as

$$\frac{\Delta^*_{b+1}}{2^{b+1}} = \sum_{i=0}^{b} \frac{\binom{2ap^n+2i}{ap^n+i}}{ap^n+i+1} \frac{3ap^n(p-1)}{2^i(ap^{n+1}+i+2)(ap^n+i+2)} \prod_{j=i+1}^{b} \frac{2ap^{n+1}+2j+1}{ap^{n+1}+j+2} \tag{4.15}$$

$$+ \Delta^*_0 \prod_{j=0}^{b} \frac{2ap^{n+1}+2j+1}{ap^{n+1}+j+2}.$$

We set $q = \nu_p(\binom{2ap^k}{ap^k}) = \nu_p(\binom{2a}{a}), k \geq 0$. Since $\nu_p(\Delta^*_0) = n+q$ by (4.4), the terms in (4.15) are of similar magnitude and it requires a more careful analysis than the treatment of (4.11) did.

We start with the second summand on the right-hand side of (4.15) and obtain

$$\Delta^*_0 \equiv p^{n+q}a(1-p)f^* \bmod p^{2n+q}$$

by the congruence (4.5). Now we use Corollary 4.1 and equation (4.3).

If $p \neq 2$ then

$$\Delta^*_{b+1} = 2^{b+1}p^{n+q}a(p-1)f^*\Bigg(3\sum_{i=0}^{b}\Bigg(\frac{\binom{2i}{i}}{i+1}\frac{1}{2^i(i+2)^2}\prod_{j=i+1}^{b}\frac{2j+1}{j+2}\Big(1+f_{17}p^{n-r}\Big)\Bigg)$$
$$-\prod_{j=0}^{b}\frac{2j+1}{j+2}\Big(1+f_{18}p^{n-r}\Big)\Bigg).$$

With some simplifications we obtain

$$\Delta^*_{b+1} = p^{n+q}a(p-1)f^*\frac{\binom{2b+2}{b+1}}{b+2}\Bigg(\Big(3\sum_{i=0}^{b}\frac{1}{(i+2)(2i+1)}\Big)-1\Bigg)+f_{19}p^{2n+q-r-r'+\nu_p(C_{b+1})} \tag{4.16}$$

with $r' = \lfloor \log_p(2b+1) \rfloor$, and we are done since the product of the last two factors of the first term on the right-hand side of (4.16) simplifies to $g(b+1) = 2\binom{2b+2}{b+1}(b+2)^{-1}\big(H_{2b+2} - H_{b+1} - 1/(2(b+2))\big)$.

Similarly to the proof of Theorem 2.4, although now using the extra factor 3 in (4.16), we also get that $n \geq n_0 = \max\{\nu_p(g(b)) + 2r - \nu_p(C_b) + 1, r+1\}$ suffices in order to get $\nu_p(\Delta^*_b)$.

Note that the case with $p = 2$, a odd, and $b = 1$ immediately follows by setting b to 0 in (4.14) and $g(b+1)$ which is $g(1) = 1/2$. The 2-adic error term in (4.16) comes with the modified exponent $2n+q-2$.

The general case with $p = 2$ and $n \geq \lfloor \log_2 2b \rfloor + 1$ follows in a similar manner. □

5. Other Proofs

Now we prove some p-adic properties of the differences of the harmonic numbers $H_{2b} - H_b$ (cf. Theorems 2.5, 2.7, and 2.8) and a related quantity (cf. (5.3)). Although these properties are not essential to other parts of the paper, they are important in the actual use of Theorem 2.4.

The proof of Theorem 2.5. For $p = 2$, the definition obviously yields that $\nu_2(f(c)) = 1 + d_2(c) - \lfloor \log_2 2c \rfloor$. If $p \geq 3$ then we proceed with induction on k. Note that

$$f(dp) = 2\binom{2dp}{dp}(H_{2dp} - H_{dp}).$$

This implies that

$$f(dp) = 2\binom{2dp}{dp}\left(\frac{1}{p}(H_{2d} - H_d) + \sum_{\substack{i=dp+1 \\ (i,p)=1}}^{2dp} \frac{1}{i}\right), \tag{5.1}$$

and clearly,

$$\nu_p\left(\sum_{\substack{i=dp+1 \\ (i,p)=1}}^{2dp} \frac{1}{i}\right) \geq 0.$$

For $k = 1$, we set $d = 1$ and the statement follows. For $k \geq 2$ we have two cases.

Since $\nu_p(\binom{2dp}{dp}/\binom{2d}{d}) = 0$, by identity (5.1) we get that

$$\nu_p(f(dp)) = \nu_p\left(\binom{2dp}{dp}\right) + \nu_p\left(\frac{1}{p}(H_{2d} - H_d)\right) = \nu_p(f(d)) - 1 \tag{5.2}$$

provided that $\nu_p(H_{2d} - H_d) \leq 0$. By the inductive hypothesis, with $d = p^{k-1}$, we have that $\nu_p(f(p^{k-1})) = -(k-1), k \geq 2$, which yields $\nu_p(f(p^k)) = -k$ by recurrence (5.2). Similarly, with $d = cp^{k-1}, c \geq 1$ and $k \geq 1$, we obtain $\nu_p(f(cp^k)) = -k + \nu_p(f(c))$ for $k \geq 0$ provided that $\nu_p(H_{2c} - H_c) \leq 0$. □

Note that finer details of divisibility by p can be recovered from

$$\nu_p\left(H_{(a+1)p^n} - H_{ap^n} - \frac{1}{p}\big(H_{(a+1)p^{n-1}} - H_{ap^{n-1}}\big)\right) \geq 2n,$$

for $p \geq 5$ prime, $n \geq 1$, and $a \geq 0$, e.g., if $a = 1$, cf. [7, p245: Solution to Problem 1997, B3]. In fact, more can be said according to Theorem 5.2 as stated in Theorem 5.1.

Theorem 5.1. *For $p \geq 3$ prime, $a \geq 0$, and $n \geq 1$, we have that*

$$\nu_p\left(H_{(a+1)p^n} - H_{ap^n} - \frac{1}{p}\big(H_{(a+1)p^{n-1}} - H_{ap^{n-1}}\big)\right) = 2n + \nu_p(2a+1) - \chi_{p=3}. \tag{5.3}$$

This follows by

Theorem 5.2. (Theorem 5.2 in [1]) *Let p be an odd prime. Then there is a sequence $c_k \in \mathbb{Q}_p$ such that, for all $m \geq 1$,*

$$H_{pm} - \frac{1}{p}H_m = \sum_{k=1}^{\infty} c_k p^{2k} m^{2k},$$

where the series converges in p-adic norm. The c_k are p-adic integers unless $(p-1)|2k$ or $p|k$. In general, $\nu_p(c_k) = -1 + \nu_p(1/k)$ if $(p-1)|2k$, and $\nu_p(c_k) = \nu_p(1/k)$ otherwise.

Proof of Theorem 5.1. By setting $m = bp^{n-1}, n \geq 1$, and $b \geq 1$ in Theorem 5.2, we obtain that

$$\nu_p(H_{bp^n} - \frac{1}{p}H_{bp^{n-1}}) = 2n + 2\nu_p(b) - \chi_{p=3}$$

and (5.3), too. □

Before we turn to the proof of Theorem 2.8, as a starting point, we include

The proof of Theorem 2.7 (attributed to J. Kürschák, cf. [1]). Let $m \geq 1$ be the largest integer so that 2^m divides k for some k with $b+1 \leq k \leq a$. We prove that there is a unique integer $k, b+1 \leq k \leq a$, for which $\nu_2(k) = m$ and thus, $\nu_2(H_a - H_b) = -m < 0$ already guaranteeing the statement. Assume to the contrary that there are two such integers: $2^m o_1 < 2^m o_2$ with odd integers o_1 and o_2. Clearly, $2^m(o_1+1)$ falls between $b+1$ and a and it has 2-adic order at least $m+1$ which contradicts the definition of m. □

The proof of Theorem 2.8. The case of $p = 2$ is included in Theorem 2.5. If $p = 3$ then we take the highest power of three not exceeding $2b$, i.e., let $3^m < 2b$. We prove that there is a unique term in the sum $\sum_{k=b+1}^{2b} \frac{1}{k}$ with k of 3-adic order $m = \lfloor \log_3 2b \rfloor$. In fact, for this term we have either $b+1 \leq 3^m < 2b \leq 2 \cdot 3^m$ or $3^m < b+1 \leq 2 \cdot 3^m \leq 2b < 3^{m+1}$.

If $p = 5$ then we take the highest power of five not exceeding $2b$, i.e., let $5^m < 2b$. We set $m = \lfloor \log_5 2b \rfloor$. We have the following cases:

$$\begin{aligned}
&(1) && b+1 \leq 5^m < 2b < 2 \cdot 5^m,\\
&(2) && 5^m < b+1 \leq 2 \cdot 5^m \leq 2b < 3 \cdot 5^m,\\
&(3) && 5^m < b+1 \leq 2 \cdot 5^m < 3 \cdot 5^m < 2b < 4 \cdot 5^m,\\
&(4) && 2 \cdot 5^m < b+1 \leq 3 \cdot 5^m < 4 \cdot 5^m \leq 2b < 5^{m+1}.
\end{aligned}$$

Clearly, in Cases (1), (2), and (4), we have $\nu_5(H_{2b} - H_b) = -\lfloor \log_5 2b \rfloor$ and thus, (2.1) holds.

In case (3) we have $m \geq 1$ and we face the problem that $\frac{1}{2\cdot 5^m} + \frac{1}{3\cdot 5^m} = \frac{1}{6\cdot 5^{m-1}}$ and other terms might also have the 5-adic order $-(m-1)$. Note, however, that this is exactly the case when there exists an m so that $\frac{3}{2}\cdot 5^m < b < 2\cdot 5^m$. We can finish the proof by proceeding with induction on m. The cases with $m = 1$ and 2 are easily tested.

We assume that $m \geq 3$ and write

$$H_{2b} - H_b = \sum_{k=b+1}^{2b} \frac{1}{k} = \sum_{\substack{k=b+1 \\ (k,5)=1}}^{2b} \frac{1}{k} + \frac{1}{5}\sum_{k=b'}^{b"} \frac{1}{k} \tag{5.4}$$

with $b' = \lceil (b+1)/5 \rceil$ and $b" = \lfloor 2b/5 \rfloor$. Clearly, $\lceil \frac{3}{2}\cdot 5^m \rceil \equiv 3 \bmod 5$ with $m \geq 1$. We have two cases.

`Case 1.` If $b \equiv 0, 1$ or $2 \bmod 5$ then $b" = 2b' - 2$ and $\frac{3}{2}\cdot 5^{m-1} < b' - 1 < 2\cdot 5^{m-1}$. Therefore,

$$\begin{aligned}\nu_5\left(\frac{1}{5}\sum_{k=b'}^{b"} \frac{1}{k}\right) &= \nu_5\left(\frac{1}{5}\sum_{k=(b'-1)+1}^{2(b'-1)} \frac{1}{k}\right) = -1 - \lfloor \log_5 2(b'-1) \rfloor + 1 \\ &= -1 - (m-1) + 1 = -m + 1 = -\lfloor \log_5 2b \rfloor + 1\end{aligned}$$

by the induction hypothesis. Therefore, (5.4) guarantees the statement.

`Case 2.` If $b \equiv 3$ or $4 \bmod 5$ then we have two subcases.

`Case 2.1.` This case deals with b values within a distance of 2 of the boundaries in (3). If $b = \lceil \frac{3}{2}\cdot 5^m \rceil$ or $\lceil \frac{3}{2}\cdot 5^m \rceil + 1$ then $b' = \lceil \frac{3}{2}\cdot 5^{m-1} \rceil$, $b" = 2b' - 1$, and $\frac{3}{2}\cdot 5^{m-1} < b' < 2\cdot 5^{m-1}$ as well as $2b' \equiv 1 \bmod 5$ and hence, $\nu_5(2b') = 0$. Thus, we have

$$\begin{aligned}\nu_5\left(\frac{1}{5}\sum_{k=b'}^{b"} \frac{1}{k}\right) &= \nu_5\left(\frac{1}{5}\sum_{k=b'+1}^{2b'} \frac{1}{k} + \frac{1}{5}\frac{1}{b'} - \frac{1}{5}\frac{1}{2b'}\right) = \nu_5\left(\frac{1}{5}\sum_{k=b'+1}^{2b'} \frac{1}{k} + \frac{1}{5}\frac{1}{2b'}\right) \\ &= -1 - \lfloor \log_5 2b' \rfloor + 1 = -1 - (m-1) + 1 = -m + 1 = -\lfloor \log_5 2b \rfloor + 1\end{aligned}$$

by the induction hypothesis. In a similar fashion, if $b = 2\cdot 5^m - 1$ or $2\cdot 5^m - 2$ then $b' = 2\cdot 5^{m-1}$, $b" = 2b' - 1$, and $\frac{3}{2}\cdot 5^{m-1} < b' - 1 < 2\cdot 5^{m-1}$ as well as $2b' - 1 \equiv 4 \bmod 5$ and thus, $\nu_5(2b' - 1) = 0$. We have

$$\begin{aligned}\nu_5\left(\frac{1}{5}\sum_{k=b'}^{b"} \frac{1}{k}\right) &= \nu_5\left(\frac{1}{5}\sum_{k=(b'-1)+1}^{2(b'-1)} \frac{1}{k} + \frac{1}{5}\frac{1}{2b'-1}\right) = -1 - \lfloor \log_5 2(b'-1) \rfloor + 1 \\ &= -1 - (m-1) + 1 = -m + 1 = -\lfloor \log_5 2b \rfloor + 1\end{aligned}$$

by the induction hypothesis.

Case 2.2. If $\lceil \frac{3}{2} \cdot 5^m \rceil + 1 < b < 2 \cdot 5^m - 2$ then $b" = 2b' - 1$ and $\frac{3}{2} \cdot 5^{m-1} + 1 < b' < 2 \cdot 5^{m-1}$, so both $b' - 1$ and b' will be in the right range to apply the induction hypothesis. We either have $\nu_5(2b' - 1) = 0$ or $\nu_5(2b') = 0$. We proceed as in Case 2.1. In the former case, we get that

$$\nu_5\left(\frac{1}{5}\sum_{k=b'}^{b"}\frac{1}{k}\right) = \nu_5\left(\frac{1}{5}\sum_{k=(b'-1)+1}^{2(b'-1)}\frac{1}{k} + \frac{1}{5}\frac{1}{2b'-1}\right) = -1 - \lfloor \log_5 2(b'-1) \rfloor + 1$$
$$= -1 - (m-1) + 1 = -m + 1 = -\lfloor \log_5 2b \rfloor + 1$$

while in the latter case,

$$\nu_5\left(\frac{1}{5}\sum_{k=b'}^{b"}\frac{1}{k}\right) = \nu_5\left(\frac{1}{5}\sum_{k=b'+1}^{2b'}\frac{1}{k} + \frac{1}{5}\frac{1}{b'} - \frac{1}{5}\frac{1}{2b'}\right) = \nu_5\left(\frac{1}{5}\sum_{k=b'+1}^{2b'}\frac{1}{k} + \frac{1}{5}\frac{1}{2b'}\right)$$
$$= -1 - \lfloor \log_5 2b' \rfloor + 1 = -1 - (m-1) + 1 = -m + 1 = -\lfloor \log_5 2b \rfloor + 1$$

by the induction hypothesis.

The proof is complete by (5.4). □

Acknowledgements The author wishes to thank the referee and Gregory P. Tollisen for helpful comments and suggestions that improved the organization of the paper.

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