# REPRESENTATIONS OF SQUARES BY CERTAIN SEPTENARY QUADRATIC FORMS

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*Received: 11/14/12, Accepted: 5/1/13, Published: 6/14/13*

**Abstract**

For any positive integer $n$ and for certain fixed positive integers $a_1,\ a_2,\dots,a_7$, we study the number of solutions in integers of

$$a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2+a_6x_6^2+a_7x_7^2=n^2.$$

When $a_1=a_2=\cdots=a_7=1$, this reduces to the classical formula for the number of representations of a square as a sum of seven squares. A further eighteen analogous results will be given.

## 1. Introduction

Let $n$ be a positive integer and let its prime factorization be given by $n=\prod_p p^{\lambda_p}$. Let $r_k(n)$ denote the number of solutions in integers of $x_1^2+x_2^2+\cdots+x_k^2=n$.

Three classical results are given by Hurwitz (see [6] and [5]) and Sandham (see [9]), respectively:

$$r_3(n^2) = 6 \prod_{p \geq 3} \left( \frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-1}{p}\right) \frac{p^{\lambda_p}-1}{p-1} \right), \tag{1.1}$$

$$r_5(n^2) = 10 \left( \frac{2^{3\lambda_2+3}-1}{2^3-1} \right) \prod_{p \geq 3} \left( \frac{p^{3\lambda_p+3}-1}{p^3-1} - p\, \frac{p^{3\lambda_p}-1}{p^3-1} \right) \tag{1.2}$$

and

$$r_7(n^2) = 14 \left( \frac{5 \times 2^{5\lambda_2+3}-9}{2^5-1} \right) \prod_{p \geq 3} \left( \frac{p^{5\lambda_p+5}-1}{p^5-1} - p^2 \left(\frac{-1}{p}\right) \frac{p^{5\lambda_p}-1}{p^5-1} \right), \tag{1.3}$$

where the values of the Legendre symbol are given by

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

In recent work [3], the number of solutions in integers of

$$n^2 = x_1^2 + bx_2^2 + cx_3^2 \tag{1.4}$$

was investigated for certain values of $b$ and $c$. When $b = c = 1$, the number of solutions of (1.4) is given by (1.1). The number of solutions of (1.4) in the case $b = 1$, $c = 2$ is given by

**Theorem 1.1.** [3, Theorem 1.2] *The number of* $(x_1, x_2, x_3) \in \mathbf{Z}^3$ *such that*

$$n^2 = x_1^2 + x_2^2 + 2x_3^2$$

*is given by*

$$4\, b(\lambda_2) \prod_{p \geq 3} \left[ \frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-8}{p}\right) \frac{p^{\lambda_p}-1}{p-1} \right], \tag{1.5}$$

*where*

$$b(\lambda_2) = \begin{cases} 1 & \text{if } \lambda_2 = 0, \\ 3 & \text{if } \lambda_2 \geq 1 \end{cases}$$

*and the values of the Legendre symbol are given by*

$$\left(\frac{-8}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 3 \pmod 8, \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod 8. \end{cases}$$

Further three-variable analogues of Theorem 1.1 were given in [3] and five-variable analogues were analyzed in [4]. In this work we study seven-variable analogues of Theorem 1.1. That is, we investigate the number of solutions in integers of

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2 + a_7x_7^2 = n^2$$

for certain fixed positive integers $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$ and $a_7$. The number of solutions in the case $a_1 = a_2 = \cdots = a_7 = 1$ is given by Sandham's identity (1.3).

This work is organized as follows. In Section 2 we define some notation and list all the results by grouping them into three theorems. The results in the first theorem are different from the others and they are treated in Section 3. Proofs of results in the second theorem are given in Section 4. Finally, results in the third theorem can be deduced from the results in the second theorem. A proof of one of them is given as an illustration in Section 5.

## 2. Notation and Results

Let $n$ be a positive integer and let its prime factorization be $n = 2^{\lambda_2} \prod_{p \geq 3} p^{\lambda_p}$. Let $m = \prod_{p\geq 3} p^{\lambda_p}$ so that $n = 2^{\lambda_2} m$ and define $s(n)$ and $t(n)$ by

$$s(n) = \prod_{p \geq 3} \left( \frac{p^{5\lambda_p+5} - 1}{p^5 - 1} - p^2 \left( \frac{-8}{p} \right) \frac{p^{5\lambda_p} - 1}{p^5 - 1} \right) \tag{2.1}$$

and

$$t(n) = \prod_{p \geq 3} \left( \frac{p^{5\lambda_p+5} - 1}{p^5 - 1} - p^2 \left( \frac{-1}{p} \right) \frac{p^{5\lambda_p} - 1}{p^5 - 1} \right). \tag{2.2}$$

Let $b(n)$ be defined by

$$\sum_{n=1}^{\infty} b(n) q^n = q \prod_{i=1}^{\infty} \left(1 - q^{2i}\right)^{12}. \tag{2.3}$$

Then let $h(n)$ and $k(n)$ be defined by

$$h(n) = b(m) \prod_{p \geq 3} \left( 1 - p^2 \left( \frac{-8}{p} \right) \frac{b(m/p)}{b(m)} \right) \tag{2.4}$$

and

$$k(n) = b(m) \prod_{p \geq 3} \left( 1 - p^2 \left( \frac{-1}{p} \right) \frac{b(m/p)}{b(m)} \right). \tag{2.5}$$

We note that $s(n)$, $t(n)$, $h(n)$ and $k(n)$ do not depend on $\lambda_2$, thus $s(n) = s(m)$, $t(n) = t(m)$, $h(n) = h(m)$ and $k(n) = k(m)$. The theta functions $\varphi(q)$ and $\psi(q)$ are defined for $|q| < 1$ by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{and} \quad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}$$

and for any positive integer $k$ we define $\varphi_k$ and $\psi_k$ by

$$\varphi_k = \varphi(q^k) \qquad \text{and} \qquad \psi_k = \psi(q^k).$$

In addition, for positive integers $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$ and $a_7$ and for any non-negative integer $n$ let $r_{(a_1,a_2,a_3,a_4,a_5,a_6,a_7)}(n)$ denote the number of solutions in integers of

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2 + a_7x_7^2 = n.$$

We note that Sandham's result (1.3) is equivalent to

$$r_{(1,1,1,1,1,1,1)}(n^2) = 14\left(\frac{5\times 2^{5\lambda_2+3}-9}{2^5-1}\right)t(n). \tag{2.6}$$

Analogously, we have the following results:

**Theorem 2.1.**

$$r_{(1,1,1,1,1,1,2)}(n^2) = 12\left|\frac{2^{5\lambda_2+5}-63}{2^5-1}\right|s(n). \tag{2.7}$$

$$r_{(1,1,1,1,2,2,2)}(n^2) = \begin{cases} 6s(n)+2h(n) & \textit{if } n \textit{ is odd,} \\ \\ \frac{198\times 2^{5\lambda_2}-756}{2^5-1}s(n) & \textit{if } n \textit{ is even.} \end{cases} \tag{2.8}$$

$$r_{(1,1,2,2,2,2,2)}(n^2) = \begin{cases} 3s(n)+h(n) & \textit{if } n \textit{ is odd,} \\ \\ \frac{105\times 2^{5\lambda_2}-756}{2^5-1}s(n) & \textit{if } n \textit{ is even.} \end{cases} \tag{2.9}$$

**Theorem 2.2.**

$$r_{(1,2,2,2,2,2,2)}(n^2) = 2\left|\frac{2^{5\lambda_2+5}-63}{2^5-1}\right|t(n). \tag{2.10}$$

$$r_{(1,1,1,1,1,2,2)}(n^2) = \begin{cases} 8t(n)+2k(n) & \textit{if } n \textit{ is odd,} \\ \\ \frac{250\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even.} \end{cases} \tag{2.11}$$

**Theorem 2.3.**

$$r_{(1,1,1,2,2,2,2)}(n^2) = \begin{cases} 4t(n)+2k(n) & \textit{if } n \textit{ is odd,} \\ \\ \frac{126\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even.} \end{cases} \tag{2.12}$$

$$r_{(1,1,1,1,1,1,4)}(n^2) = \begin{cases} 6t(n) + 6k(n) & \textit{if } n \textit{ is odd}, \\ \frac{250\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.13}$$

$$r_{(1,1,1,1,1,4,4)}(n^2) = \begin{cases} 3t(n) + 7k(n) & \textit{if } n \textit{ is odd}, \\ \frac{95\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.14}$$

$$r_{(1,1,1,1,4,4,4)}(n^2) = \begin{cases} 2t(n) + 6k(n) & \textit{if } n \textit{ is odd}, \\ \frac{33\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.15}$$

$$r_{(1,1,1,4,4,4,4)}(n^2) = \begin{cases} \frac{3}{2}t(n) + \frac{9}{2}k(n) & \textit{if } n \textit{ is odd}, \\ \frac{35\times 2^{5\lambda_2-1}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.16}$$

$$r_{(1,1,4,4,4,4,4)}(n^2) = \begin{cases} t(n) + 3k(n) & \textit{if } n \textit{ is odd}, \\ \frac{35\times 2^{5\lambda_2-1}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.17}$$

$$r_{(1,4,4,4,4,4,4)}(n^2) = \begin{cases} \frac{1}{2}t(n) + \frac{3}{2}k(n) & \textit{if } n \textit{ is odd}, \\ \frac{35\times 2^{5\lambda_2-1}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.18}$$

$$r_{(1,1,1,1,2,2,4)}(n^2) = \begin{cases} 4t(n) + 4k(n) & \textit{if } n \textit{ is odd}, \\ \frac{126\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.19}$$

$$r_{(1,1,1,2,2,4,4)}(n^2) = \begin{cases} 2t(n) + 4k(n) & \textit{if } n \textit{ is odd}, \\ \frac{64\times 2^{5\lambda_2}-126}{2^5-1}t(n) & \textit{if } n \textit{ is even}. \end{cases} \tag{2.20}$$

$$r_{(1,1,2,2,4,4,4)}(n^2) = \begin{cases} t(n) + 3k(n) & \text{if } n \text{ is odd,} \\ \\ \frac{33\times 2^{5\lambda_2}-126}{2^5-1} t(n) & \text{if } n \text{ is even.} \end{cases} \tag{2.21}$$

$$r_{(1,1,2,2,2,2,4)}(n^2) = \begin{cases} 2t(n) + 2k(n) & \text{if } n \text{ is odd,} \\ \\ \frac{64\times 2^{5\lambda_2}-126}{2^5-1} t(n) & \text{if } n \text{ is even.} \end{cases} \tag{2.22}$$

$$r_{(1,2,2,2,2,4,4)}(n^2) = \begin{cases} t(n) + k(n) & \text{if } n \text{ is odd,} \\ \\ \frac{33\times 2^{5\lambda_2}-126}{2^5-1} t(n) & \text{if } n \text{ is even.} \end{cases} \tag{2.23}$$

$$r_{(1,2,2,4,4,4,4)}(n^2) = \begin{cases} \frac{1}{2}t(n) + \frac{3}{2}k(n) & \text{if } n \text{ is odd,} \\ \\ \frac{35\times 2^{5\lambda_2-1}-126}{2^5-1} t(n) & \text{if } n \text{ is even.} \end{cases} \tag{2.24}$$

All three theorems give values of $r_{(a_1,\ldots,a_7)}(n^2)$. Theorems 2.2 and 2.3 account for all instances for which $1 = a_1 \le a_2 \le \cdots \le a_7 \le 4$ with $a_1, \ldots, a_7 \in \{1, 2, 4\}$ and for which the product $a_1 a_2 \cdots a_7$ is an even power of 2. Theorem 2.1 accounts for those instances for which $1 = a_1 \le a_2 \le \cdots \le a_7 = 2$ with $a_1, \ldots, a_7 \in \{1, 2\}$ and for which the product $a_1 a_2 \cdots a_7$ is an odd power of 2. Some further comments about other values of $a_1, \ldots, a_7$ are given in the concluding remarks at the end of the paper.

## 3. Proof of Theorem 2.1

In this section we will outline the proof of Theorem 2.1.

**Lemma 3.1.** *Fix an odd integer $m$. For any nonnegative integer $k$, let*

$$u_1(k) = r_{(1,1,1,1,1,1,2)}(2^{2k}m^2),$$
$$u_2(k) = r_{(1,1,1,1,2,2,2)}(2^{2k}m^2)$$

*and*

$$u_3(k) = r_{(1,1,2,2,2,2,2)}(2^{2k}m^2).$$

*Then*

$$u_i(k+3) = 33u_i(k+2) - 32u_i(k+1) \quad \text{for } i{=}1, 2 \text{ or } 3.$$

*Moreover*

$$u_1(1) = 31u_1(0), \qquad u_1(2) = 1055u_1(0)$$
$$u_2(1) = 15u_1(0), \qquad u_2(2) = 543u_1(0)$$

*and*

$$u_3(1) = 7u_1(0), \qquad u_3(2) = 287u_1(0).$$

*Hence, on solving the recurrence relation, we have: for* $k \geq 1$

$$r_{(1,1,1,1,1,1,2)}(2^{2k}m^2) = \left|\frac{2^{5k+5}-63}{2^5-1}\right| r_{(1,1,1,1,1,1,2)}(m^2),$$
$$r_{(1,1,1,1,2,2,2)}(2^{2k}m^2) = \frac{33 \times 2^{5k} - 126}{(2^5-1) \times 2} r_{(1,1,1,1,1,1,2)}(m^2)$$

*and*

$$r_{(1,1,2,2,2,2,2)}(2^{2k}m^2) = \frac{35 \times 2^{5k} - 252}{(2^5-1) \times 4} r_{(1,1,1,1,1,1,2)}(m^2).$$

*Proof.* These can all be deduced by the methods in [3, Section 4]. □

It remains to determine the values of $r_{(1,1,1,1,1,1,2)}(m^2)$, $r_{(1,1,1,1,2,2,2)}(m^2)$ and $r_{(1,1,2,2,2,2,2)}(m^2)$ in the case that $m$ is odd.

**Proposition 3.2.** *Let* $m$ *be a positive odd number and let its prime factorization be given by*

$$m = \prod_{p \geq 3} p^{\lambda_p}.$$

*Let* $s(m)$ *and* $h(m)$ *be defined by* (2.1) *and* (2.4)*. Then*

$$r_{(1,1,1,1,1,1,2)}(m^2) = 12s(m), \tag{3.1}$$
$$r_{(1,1,1,1,2,2,2)}(m^2) = 6s(m) + 2h(m) \tag{3.2}$$

*and*

$$r_{(1,1,2,2,2,2,2)}(m^2) = 3s(m) + h(m). \tag{3.3}$$

We may note that Lemma 3.1 and Proposition 3.2 immediately imply (2.7)–(2.9) in Theorem 2.1.

To prove Proposition 3.2, we will need:

**Lemma 3.3.** *Let $f_1$, $f_2$ and $f_3$ be defined by*

$$f_1(q) = \frac{1}{8}\varphi_1^5\varphi_2 - \frac{1}{8}\varphi_1\varphi_2^5,$$
$$f_2(q) = \frac{1}{2}\varphi_1^5\varphi_2 - 2\varphi_1\varphi_2^5$$

*and*

$$f_3(q) = -\frac{1}{4}\varphi_1^5\varphi_2 + \frac{3}{4}\varphi_1^3\varphi_2^3 - \frac{1}{2}\varphi_1\varphi_2^5.$$

*And let their series expansions be given by*

$$f_1(q) = \sum_{n=0}^{\infty} a_1(n)q^n, \quad f_2(q) = \sum_{n=0}^{\infty} a_2(n)q^n \tag{3.4}$$

*and*

$$f_3(q) = \sum_{n=0}^{\infty} a_3(n)q^n. \tag{3.5}$$

*Then for any nonnegative integer $n$ and any prime $p$ we have*

$$a_j(pn) = a_j(p)a_j(n) - \chi(p)a_j(n/p) \qquad \text{for } j \in \{1,2,3\} \tag{3.6}$$

*where $\chi$ is the completely multiplicative function defined on the positive integers by*

$$\chi(r) = r^2\left(\frac{-8}{r}\right) \tag{3.7}$$

*and $\left(\frac{-8}{r}\right)$ is the Kronecker symbol, and $a_j(x)$ is defined to be zero if $x$ is not an integer.*

*Proof.* The results for $j$=1 or 2 follow from work of Alaca et al. [1, pp. 291–292]. The result for $j$=3 was given by Martin [8, pp. 4828–4833]. □

We may note that

$$\varphi^5(q)\varphi(q^2) = \frac{32}{3}f_1(q) - \frac{2}{3}f_2(q), \tag{3.8}$$
$$\varphi^3(q)\varphi^3(q^2) = \frac{16}{3}f_1(q) - \frac{2}{3}f_2(q) + \frac{4}{3}f_3(q) \tag{3.9}$$

and

$$\varphi(q)\varphi^5(q^2) = \frac{8}{3}f_1(q) - \frac{2}{3}f_2(q). \tag{3.10}$$

Let $A_j(n)$ be defined by

$$\sum_{n=0}^{\infty} A_j(n)q^n = \left(\sum_{n=0}^{\infty} a_j(n)q^n\right)^2 \qquad \text{for } j \in \{1,2,3\}. \tag{3.11}$$

The next result is due to Hurwitz.

**Lemma 3.4.** *Suppose that $a(n)$ is a function, defined for all non-negative integers $n$, that satisfies the property*

$$a(pn) = a(p)a(n) - \chi(p)a\left(\frac{n}{p}\right)$$

*for all primes $p$, where $\chi$ is a completely multiplicative function. Then the coefficient of $q^{n^2}$ in*

$$\left(\sum_{j=-\infty}^{\infty} q^{j^2}\right) \times \left(\sum_{k=0}^{\infty} a(k)q^k\right)$$

*is equal to*

$$\sum_{r=1}^{\infty} A\left(\frac{2n}{r}\right)\chi(r)\mu(r)$$

*where $\mu$ is the Möbius function, $A(n)$ is defined by*

$$\sum_{n=0}^{\infty} A(n)q^n = \left(\sum_{k=0}^{\infty} a(k)q^k\right)^2$$

*and $A(x)$ is defined to be 0 if $x$ is not a non-negative integer.*

*Proof.* See the work of Sandham [9, Section 2]. □

Before starting the next lemma, let us define $[q^k]f(q)$ to be the coefficient of $q^k$ in the Taylor expansion of $f(q)$.

**Lemma 3.5.** *Let $m$ be a positive odd number and let its prime factorization be given by*

$$m = \prod_{p \geq 3} p^{\lambda_p}.$$

*Let $c_1(m)$, $c_2(m)$ and $c_3(m)$ be the coefficients of $q^{2m}$ in*

$$\frac{32}{3}f_1^2(q) - \frac{2}{3}f_2^2(q), \qquad \frac{16}{3}f_1^2(q) - \frac{2}{3}f_2^2(q) + \frac{4}{3}f_3^2(q) \quad \textit{and} \quad \frac{8}{3}f_1^2(q) - \frac{2}{3}f_2^2(q),$$

*respectively. Let $b(m)$ be defined by (2.3). Then*

$$c_1(m) = \frac{1}{22}[q^{2m}]\varphi^{12}(q), \tag{3.12}$$

$$c_2(m) = \frac{1}{44}[q^{2m}]\varphi^{12}(q) + 2b(m) \tag{3.13}$$

*and*

$$c_3(m) = \frac{1}{88}[q^{2m}]\varphi^{12}(q) + b(m). \tag{3.14}$$

*And thus*

$$c_1(m) = 12\sum_{d|m} d^5 = 12\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1}, \tag{3.15}$$

$$c_2(m) = 6\sum_{d|m} d^5 + 2b(m) = 6\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1} + 2b(m) \tag{3.16}$$

*and*

$$c_3(m) = 3\sum_{d|m} d^5 + b(m) = 3\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1} + b(m). \tag{3.17}$$

*Proof.* (3.12)–(3.14) may be deduced by the methods in [3, Section 4], and (3.15)–(3.17) follows from the result of $[q^{2m}]\varphi^{12}(q)$ given in [7]. □

We are now ready for

*Proof of Proposition 3.2.* We will deal with (3.1) first. The proofs of (3.2) and (3.3) will be similar.

By (3.4) and (3.8),

$$\begin{aligned} r_{(1,1,1,1,1,1,2)}(m^2) &= [q^{m^2}]\left(\varphi(q)\left(\frac{32}{3}f_1(q) - \frac{2}{3}f_2(q)\right)\right) \\ &= \frac{32}{3}[q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty}a_1(j)q^j\right) - \frac{2}{3}[q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty}a_2(j)q^j\right). \end{aligned}$$

By Lemmas 3.3 and 3.4 and (3.11) this is equivalent to

$$r_{(1,1,1,1,1,1,2)}(m^2) = \frac{32}{3}\sum_{r=1}^{\infty}A_1\left(\frac{2m}{r}\right)\chi(r)\mu(r) - \frac{2}{3}\sum_{r=1}^{\infty}A_2\left(\frac{2m}{r}\right)\chi(r)\mu(r)$$

where $\chi(r)$ is the completely multiplicative function defined by (3.7). Since $\chi(r) = 0$ if $r$ is even, the last sum in the above is over odd values of $r$ only. Moreover, since $m$ is odd, we may apply Lemma 3.5 to deduce that

$$r_{(1,1,1,1,1,1,2)}(m^2) = \sum_{r=1}^{\infty} c_1(m/r)\chi(r)\mu(r)$$

$$
\begin{aligned}
&= c_1(m)\sum_{r|m}\frac{c_1(m/r)}{c_1(m)}\chi(r)\mu(r)\\
&= c_1(m)\prod_{p\geq 3}\left(1-\chi(p)\frac{c_1(m/p)}{c_1(m)}\right)\\
&= \left(12\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1}\right)\times\left(\prod_{p\geq 3}\left(1-p^2\left(\frac{-8}{p}\right)\frac{p^{5\lambda_p}-1}{p^{5\lambda_p+5}-1}\right)\right)\\
&= 12\prod_{p\geq 3}\left(\frac{p^{5\lambda_p+5}-1}{p^5-1}-p^2\left(\frac{-8}{p}\right)\frac{p^{5\lambda_p}-1}{p^5-1}\right)\\
&= 12s(m).
\end{aligned}
$$

Similarly, we can deduce:

$$
\begin{aligned}
r_{(1,1,1,1,2,2,2)}(m^2) &= \sum_{r=1}^{\infty} c_2(m/r)\chi(r)\mu(r)\\
&= \sum_{r=1}^{\infty}\left(\frac{1}{44}[q^{2m/r}]\varphi(q)^{12}+2b(m/r)\right)\chi(r)\mu(r)\\
&= \left(6\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1}\right)\times\left(\prod_{p\geq 3}\left(1-p^2\left(\frac{-8}{p}\right)\frac{p^{5\lambda_p}-1}{p^{5\lambda_p+5}-1}\right)\right)\\
&\quad + 2b(m)\prod_{p\geq 3}\left(1-p^2\left(\frac{-8}{p}\right)\frac{b(m/p)}{b(m)}\right)\\
&= 6\prod_{p\geq 3}\left(\frac{p^{5\lambda_p+5}-1}{p^5-1}-p^2\left(\frac{-8}{p}\right)\frac{p^{5\lambda_p}-1}{p^5-1}\right)\\
&\quad + 2b(m)\prod_{p\geq 3}\left(1-p^2\left(\frac{-1}{p}\right)\frac{b(m/p)}{b(m)}\right)\\
&= 6s(m)+2h(m)
\end{aligned}
$$

and

$$
\begin{aligned}
r_{(1,1,2,2,2,2,2)}(m^2) &= \sum_{r=1}^{\infty} c_3(m/r)\chi(r)\mu(r)\\
&= \sum_{r=1}^{\infty}\left(\frac{1}{88}[q^{2m/r}]\varphi(q)^{12}+b(m/r)\right)\chi(r)\mu(r)\\
&= \left(3\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1}\right)\times\left(\prod_{p\geq 3}\left(1-p^2\left(\frac{-8}{p}\right)\frac{p^{5\lambda_p}-1}{p^{5\lambda_p+5}-1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ b(m)\prod_{p\geq 3}\left(1-p^2\left(\frac{-8}{p}\right)\frac{b(m/p)}{b(m)}\right)\\
&= 3\prod_{p\geq 3}\left(\frac{p^{5\lambda_p+5}-1}{p^5-1}-p^2\left(\frac{-8}{p}\right)\frac{p^{5\lambda_p}-1}{p^5-1}\right)\\
&\quad + b(m)\prod_{p\geq 3}\left(1-p^2\left(\frac{-8}{p}\right)\frac{b(m/p)}{b(m)}\right)\\
&= 3s(m)+h(m).
\end{aligned}
$$

□

## 4. Proof of Theorem 2.2

In this section, we will outline proofs of results in Theorem 2.2. The proof of (2.10) depends on:

**Lemma 4.1.** *Fix an odd integer $m$. For any nonnegative integer $k$ let*

$$u(k) = r_{(1,2,2,2,2,2,2,2)}(2^{2k}m^2).$$

*Then*

$$u(k+3) = 33u(k+2) - 32u(k+1),$$

$$u(1) = 31u(0), \qquad u(2) = 1055u(0)$$

*and*

$$u(0) = \frac{1}{7}r_{(1,1,1,1,1,1,1,1)}(m^2).$$

*Hence, on solving the recurrence relation, we have*

$$r_{(1,2,2,2,2,2,2,2)}(2^{2k}m^2) = \frac{1}{7}\left|\frac{2^{5k+5}-63}{2^5-1}\right| r_{(1,1,1,1,1,1,1,1)}(m^2).$$

*Proof.* These may be deduced by the methods in [3, Section 4]. □

We are now ready for:

*Proof of* (2.10). By Lemma 4.1 and Sandham's result (1.3), we can immediately deduce:

$$
\begin{aligned}
r_{(1,2,2,2,2,2,2,2)}(n^2) &= 2\left|\frac{2^{5\lambda_2+5}-63}{2^5-1}\right|\prod_{p\geq 3}\left(\frac{p^{5\lambda_p+5}-1}{p^5-1}-p^2\left(\frac{-1}{p}\right)\frac{p^{5\lambda_p}-1}{p^5-1}\right)\\
&= 2\left|\frac{2^{5\lambda_2+5}-63}{2^5-1}\right| t(n).
\end{aligned}
$$

This proves (2.10). □

Now we will outline the proof of (2.11). This will be achieved in two steps according to whether $n$ is even or odd. We begin with the case when $n$ is even.

**Lemma 4.2.** *Fix an odd integer $m$. For any nonnegative integer $k$ let*

$$u(k) = r_{(1,1,1,1,1,2,2)}(2^{2k}m^2).$$

*Then*

$$u(k+3) = 33u(k+2) - 32u(k+1),$$
$$u(1) = \frac{127}{7}r_{(1,1,1,1,1,1,1)}(m^2) \quad \text{and} \quad u(2) = \frac{4127}{7}r_{(1,1,1,1,1,1,1)}(m^2).$$

*Hence, on solving the recurrence relation, we have: for $k \geq 1$*

$$r_{(1,1,1,1,1,2,2)}(2^{2k}m^2) = \frac{125 \times 2^{5k} - 63}{(2^5-1) \times 7}r_{(1,1,1,1,1,1,1)}(m^2)$$

*and thus*

$$r_{(1,1,1,1,1,2,2)}(n^2) = \frac{250 \times 2^{5\lambda_2} - 126}{2^5 - 1}t(n) \tag{4.1}$$

*where $n$ is even.*

*Proof.* These may be deduced by the methods in [3, Section 4], and (4.1) follows from (2.6). □

It remains to deal with the case when $n$ is odd. We will need two lemmas.

**Lemma 4.3.** *Let $g_1$, $g_2$ and $g_3$ be defined by*

$$g_1(q) = q\varphi^2(q)\psi^4(q^2), \;\; g_2(q) = -\frac{1}{4}\varphi^2(q)\varphi^4(-q), \;\; \text{and} \;\; g_3(q) = q\psi^2(q^4)\varphi^4(-q^2);$$

*and let their series expansions be given by*

$$g_1(q) = \sum_{n=0}^{\infty} a_1(n)q^n, \quad g_2(q) = \sum_{n=0}^{\infty} a_2(n)q^n \tag{4.2}$$

*and*

$$g_3(q) = \sum_{n=0}^{\infty} a_3(n)q^n. \tag{4.3}$$

*Then for any nonnegative integer $n$ and any prime $p$ we have*

$$a_j(pn) = a_j(p)a_j(n) - \chi(p)a_j(n/p) \qquad \text{for } j \in \{1,2,3\}$$

*where $\chi$ is the completely multiplicative function defined on the positive integers by*

$$\chi(r) = \begin{cases} r^2\left(\frac{-1}{r}\right) & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} \tag{4.4}$$

*and $\left(\frac{-1}{r}\right)$ is the Kronecker symbol, and $a_j(x)$ is defined to be zero if $x$ is not an integer.*

*Proof.* This follows from [2, Theorem 2.4]. □

We note that

$$\varphi^4(q)\varphi^2(q^2) = 8g_1(q) - 4g_2(q) + 4g_3(q), \tag{4.5}$$

and we let $A_j(n)$ be defined by

$$\sum_{n=0}^{\infty} A_j(n)q^n = \left(\sum_{n=0}^{\infty} a_j(n)q^n\right)^2 \qquad \text{for } j \in \{1,2,3\}. \tag{4.6}$$

**Lemma 4.4.** *Let $m$ be an positive odd number and $c(m)$ be the coefficient of $q^{2m}$ in $8g_1^2(q) - 4g_2^2(q) + 4g_3^2(q)$. Let $b(m)$ be defined by (2.3). Then*

$$c(m) = \frac{1}{33}[q^{2m}]\varphi^{12}(q) + 2b(m). \tag{4.7}$$

*And thus*

$$c(m) = 8\sum_{d|m} d^5 + 2b(m) = 8\prod_{p\geq 3} \frac{p^{5\lambda_p+5}-1}{p^5-1} + 2b(m). \tag{4.8}$$

*Proof.* Equation (4.7) may be deduced by the methods in [3, Section 4], and (4.8) follows from the result of $[q^{2m}]\varphi^{12}(q)$ given in [7]. □

**Proposition 4.5.** *Let $m$ be a positive odd number and let its prime factorization be given by $m = \prod_{p\geq 3} p^{\lambda_p}$. Let $t(m)$ and $k(m)$ be defined by (2.2) and (2.5) respectively. Then $r_{(1,1,1,1,1,2,2)}(m^2) = 8t(m) + 2k(m)$.*

*Proof.* By (4.2), (4.3) and (4.5),

$$\begin{aligned} r_{(1,1,1,1,1,2,2)}(m^2) &= [q^{m^2}]\left(\varphi(q)\left(8g_1(q) - 4g_2(q) + 4g_3(q)\right)\right) \\ &= 8[q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty} a_1(j)q^j\right) - 4[q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty} a_2(j)q^j\right) \\ &\quad + 4[q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty} a_3(j)q^j\right). \end{aligned}$$

By Lemmas 3.4 and 4.3, and (4.6), this is equivalent to

$$\begin{aligned} r_{(1,1,1,1,1,2,2)}(m^2) = 8\sum_{r=1}^{\infty} A_1\left(\frac{2m}{r}\right)\chi(r)\mu(r) - 4\sum_{r=1}^{\infty} A_2\left(\frac{2m}{r}\right)\chi(r)\mu(r) \\ + 4\sum_{r=1}^{\infty} A_3\left(\frac{2m}{r}\right)\chi(r)\mu(r), \end{aligned}$$

where $\chi(r)$ is the completely multiplicative function defined by (4.4). Since $\chi(r)=0$ if $r$ is even, the last sum in the above is over odd values of $r$ only.

Moreover, since $m$ is odd, we may apply Lemma 4.4 to deduce that

$$\begin{aligned}
r_{(1,1,1,1,1,2,2)}(m^2) &= \sum_{r=1}^{\infty} c(m/r)\chi(r)\mu(r)\\
&= \sum_{r=1}^{\infty}\left(\frac{1}{33}[q^{2m/r}]\varphi(q)^{12}+2b(m/r)\right)\chi(r)\mu(r)\\
&= \left(8\prod_{p\geq 3}\frac{p^{5\lambda_p+5}-1}{p^5-1}\right)\times\left(\prod_{p\geq 3}\left(1-p^2\left(\frac{-1}{p}\right)\frac{p^{5\lambda_p}-1}{p^{5\lambda_p+5}-1}\right)\right)\\
&\quad + 2b(m)\prod_{p\geq 3}\left(1-p^2\left(\frac{-1}{p}\right)\frac{b(m/p)}{b(m)}\right)\\
&= 8\prod_{p\geq 3}\left(\frac{p^{5\lambda_p+5}-1}{p^5-1}-p^2\left(\frac{-1}{p}\right)\frac{p^{5\lambda_p}-1}{p^5-1}\right)\\
&\quad + 2b(m)\prod_{p\geq 3}\left(1-p^2\left(\frac{-1}{p}\right)\frac{b(m/p)}{b(m)}\right)\\
&= 8t(m)+2k(m).
\end{aligned}$$

□

## 5. Proof of Theorem 2.3

In this section, we will give the proof of (2.12) and regard it as an illustration. Proofs of the remaining results are all similar: for $n$ is even, the value can be deduced from the value of $r_{(1,1,1,1,1,1,1,1)}(n^2)$ and for $n$ is odd, the value can be deduced from Theorem 2.2.

**Lemma 5.1.** *Fix an odd integer $m$. For any nonnegative integer $k$ let*

$$u(k) = r_{(1,1,1,2,2,2,2)}(2^{2k}m^2).$$

*Then $u(k+3) = 33u(k+2) - 32u(k+1)$,*

$$u(1) = 9r_{(1,1,1,1,1,1,1,1)}(m^2) \quad \textit{and} \quad u(2) = 297r_{(1,1,1,1,1,1,1,1)}(m^2).$$

*Hence, on solving the recurrence relation, we have: for $k\geq 1$*

$$r_{(1,1,1,2,2,2,2)}(2^{2k}m^2) = \frac{9\times 2^{5k}-9}{2^5-1}r_{(1,1,1,1,1,1,1,1)}(m^2)$$

*and thus*

$$r_{(1,1,1,2,2,2,2)}(n^2) = \frac{126 \times 2^{5\lambda_2} - 126}{2^5 - 1} t(n) \tag{5.1}$$

*where $n$ is even.*

*Proof.* These may be deduced by the methods in [3, Section 4], and (5.1) follows from (2.6). □

It remains to deal with the case when $n$ is odd.

**Lemma 5.2.** *Let $n$ be a positive integer and let its prime factorization be given by*

$$n = 2^{\lambda_2} m \quad \textit{where} \quad m = \prod_{p \geq 3} p^{\lambda_p}.$$

*Let $t(m)$ and $k(m)$ be defined by (2.2) and (2.5). Then $t(m) = \frac{1}{2} r_{(1,2,2,2,2,2,2)}(m^2)$ and $k(m) = \frac{1}{2} r_{(1,1,1,1,1,2,2)}(m^2) - 2r_{(1,2,2,2,2,2,2)}(m^2)$.*

*Proof.* This follows from Lemma 4.1 and Proposition 4.5. □

**Lemma 5.3.** *Let $n$ be a positive integer and let its prime factorization be given by*

$$n = 2^{\lambda_2} m \quad \textit{where} \quad m = \prod_{p \geq 3} p^{\lambda_p}.$$

*Then*

$$r_{(1,1,1,2,2,2,2)}(m^2) = r_{(1,1,1,1,1,2,2)}(m^2) - 2r_{(1,2,2,2,2,2,2)}(m^2). \tag{5.2}$$

*Hence,*

$$r_{(1,1,1,2,2,2,2)}(m^2) = 4t(m) + 2k(m). \tag{5.3}$$

*Proof.* (5.2) can be deduced by the method in [3, Section 4], and (5.3) immediately follows from Lemma 5.2 and (5.2). □

By Lemma 5.1 and Lemma 5.3, we immediately deduce (2.12).

Finally, let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$. Let $n$ be a positive integer and let its prime factorization be given by

$$n = 2^{\lambda_2} m \quad \text{where} \quad m = \prod_{p \geq 3} p^{\lambda_p}.$$

Then similar to (2.12), we can deduce that

$$r_{\mathbf{a}}(n^2) = \begin{cases} d_{\mathbf{a}} r_{(1,1,1,1,1,2,2)}(m^2) + e_{\mathbf{a}} r_{(1,2,2,2,2,2,2)}(m^2) & \text{if } n \text{ is odd,} \\ c_{\mathbf{a}}(\lambda_2) r_{(1,1,1,1,1,1,1)}(m^2) & \text{if } n \text{ is even.} \end{cases}$$

Hence,

$$r_{\mathbf{a}}(n^2) = \begin{cases} (8d_{\mathbf{a}} + 2e_{\mathbf{a}})t(n) + 2d_{\mathbf{a}}k(n) & \text{if } n \text{ is odd,} \\ 14c_{\mathbf{a}}(\lambda_2)t(n) & \text{if } n \text{ is even.} \end{cases}$$

for the values of $\mathbf{a}$, $c_{\mathbf{a}}(\lambda_2)$, $d_{\mathbf{a}}$ and $e_{\mathbf{a}}$ given in the Table 1, below.

| $\mathbf{a}$ | $c_{\mathbf{a}}$ | $d_{\mathbf{a}}$ | $e_{\mathbf{a}}$ |
|---|---|---|---|
| $(1,1,1,1,1,1,4)$ | $\frac{125\times 2^{5\lambda_2}-63}{(2^5-1)\times 7}$ | $3$ | $-9$ |
| $(1,1,1,1,1,4,4)$ | $\frac{95\times 2^{5\lambda_2-1}-63}{(2^5-1)\times 7}$ | $\frac{7}{2}$ | $-\frac{25}{2}$ |
| $(1,1,1,1,4,4,4)$ | $\frac{33\times 2^{5\lambda_2-1}-63}{(2^5-1)\times 7}$ | $3$ | $-11$ |
| $(1,1,1,4,4,4,4)$ | $\frac{5\times 2^{5\lambda_2-2}-9}{2^5-1}$ | $\frac{9}{4}$ | $-\frac{33}{4}$ |
| $(1,1,4,4,4,4,4)$ | $\frac{5\times 2^{5\lambda_2-2}-9}{2^5-1}$ | $\frac{3}{2}$ | $-\frac{11}{2}$ |
| $(1,4,4,4,4,4,4)$ | $\frac{5\times 2^{5\lambda_2-2}-9}{2^5-1}$ | $\frac{3}{4}$ | $-\frac{11}{4}$ |
| $(1,1,1,1,2,2,4)$ | $\frac{9\times 2^{5\lambda_2}-9}{2^5-1}$ | $2$ | $-4$ |
| $(1,1,1,2,2,4,4)$ | $\frac{32\times 2^{5\lambda_2}-63}{(2^5-1)\times 7}$ | $2$ | $-7$ |
| $(1,1,2,2,4,4,4)$ | $\frac{33\times 2^{5\lambda_2-1}-63}{(2^5-1)\times 7}$ | $\frac{3}{2}$ | $-\frac{11}{2}$ |
| $(1,1,2,2,2,2,4)$ | $\frac{32\times 2^{5\lambda_2}-63}{(2^5-1)\times 7}$ | $1$ | $-3$ |
| $(1,2,2,2,2,4,4)$ | $\frac{33\times 2^{5\lambda_2-1}-63}{(2^5-1)\times 7}$ | $\frac{1}{2}$ | $-\frac{3}{2}$ |
| $(1,2,2,4,4,4,4)$ | $\frac{5\times 2^{5\lambda_2-2}-9}{2^5-1}$ | $\frac{3}{4}$ | $-\frac{11}{4}$ |

Table 1: Data for (2.13)–(2.24)

## 6. Concluding Remarks

It is natural to ask if Theorem 2.1 can be extended by allowing some of the $a_j$ to be equal to 4, that is, to consider the case $1 = a_1 \leq a_2 \leq \cdots \leq a_7 = 4$ for which the product $a_1 a_2 \cdots a_7$ is an odd power of 2. For example, consider the case $r_{(1,1,1,1,1,2,4)}(n^2)$. The methods in [3, Section 4] can be used to find a formula in the case that $n$ is even. For odd values of $n$, it would be necessary to study the sextenary form $r_{(1,1,1,1,2,4)}(n)$ and be able to express $\varphi^4(q)\varphi(q^2)\varphi(q^4)$ as a linear combination of functions whose coefficients satisfy (3.6). Such a formula is not known. This could be the subject of further investigation.

**Acknowledgment.** We sincerely thank the anonymous referee for his/her valuable comments that improve the quality of this paper. The third author is highly grateful to the first author for his kind guidance and encouragement.

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