EDGE GROWTH IN GRAPH SQUARES

Michael Goff
Department of Mathematics, Vanderbilt University, Nashville, Tennessee
michael.goff@vanderbilt.edu

Received: 3/4/12, Revised: 5/13/13, Accepted: 6/7/13, Published: 6/24/13

Abstract

We resolve a conjecture of Hegarty regarding the number of edges in the square of a regular graph. If G is a connected d-regular graph with n vertices, the graph square of G is not complete, and G is not a member of two narrow families of graphs, then the square of G has at least $(2 - o_d(1))n$ more edges than G.

1. Introduction

In this paper, we consider the following problem. Let G be a d-regular graph, and let G^2 be the graph with the same vertex set as G and an edge uv whenever u and v are within distance 2 in G. Then find a lower bound on the number of edges of G^2, or $e(G^2)$. With the assumptions that G is connected and that G^2 is not a complete graph, this question was posed by Hegarty [1, Conjecture 1.8].

In his work, Hegarty discussed general graph powers. Let G^k be the graph with an edge uv whenever u and v are within distance k in G. Several authors have considered lower bounds on $e(G^k)$. Hegarty found that $e(G^3) \geq (1 + c)dn/2$ if G is a d-regular graph with diameter at least three, with $c = 0.087$. Pokrovskiy [5] found a value of $c = 1/6$, and DeVos and Thomassé [3] improved the value of c to $3/4$ and provided examples to demonstrate that no higher value of c is possible. The latter authors also weakened the d-regular condition to a minimum degree of d.

DeVos, McDonald, and Scheide [2] considered higher powers of G. They found that if G has a minimum degree of $d \geq 2$ and G has at least $\frac{8}{3}d$ vertices, then G^4 has an average degree of at least $\frac{7}{3}d$. Examples demonstrate that neither the $8/3$ nor the $7/3$ constants may be improved. They also found that when the diameter is at least $3k + 3$, then the average degree of G^{3k+2} is at least $(2k + 1)(d + 1) - k(k + 1)(d + 1)^2/n - 1$, and examples show that this cannot be improved.

Inspiration for Hegarty's work comes from the Cauchy-Davenport theorem, which states that if A is a subset of $\mathbb{Z}_p$ for a prime p, and kA denotes the set of sums of collections of k elements of A, then $|kA| \geq \min(p, k|A| - (k - 1))$. Kneser [4] generalized the Cauchy-Davenport theorem to an abelian group H. Now suppose that $A = -A$ and that A contains the identity element. The connection to graph theory comes through the Cayley graph. The Cayley graph $G(H, A)$ has vertex set H and an edge $g_1 g_2$ whenever $g_1 - g_2 \in A$. Then $(G(H, A))^k = G(H, kA)$, and the growth in kA is equivalent to the growth in the vertex degrees of $(G(H, A))^k$.

For large values of d and $k > 2$, $e(G^k)$ exceeds $e(G)$ by at least a constant factor, so long as G^k is not a complete graph and G is d-regular and connected. This is not true for $k = 2$, as examples demonstrate. In that case, Hegarty conjectures the following [1, Conjecture 1.8].

Conjecture 1.1. Let G be a connected d-regular graph with n vertices such that G^2 is not a complete graph. Then $e(G^2) - e(G) \geq (2 - o_d(1))n$.

This conjecture is not true as stated, and we shall see some counterexamples below, but the counterexamples are confined to narrow families of graphs known as snake graphs and peanut graphs. The modifcation of Conjecture 1.1 that $e(G^2) - e(G) \geq (3/2 - o_d(1))n$ is true. However, our main theorem is as follows.

Theorem 1.2. *Let G be a connected d-regular graph with n vertices such that G^2 is not a complete graph and $d > 6$. Also suppose that G is not a snake graph or a peanut graph. Then*

$$e(G^2) - e(G) > 2n\left(1 - \frac{2}{d+1} - \frac{3}{d-3}\right).$$

Our approach is as follows. We define basic terms in Section 2. We rephase the problem by counting ordered pairs of vertices (u, v) such that u and v are at distance 2. In Section 3, we divide G into what we call regions, and into superregions in Section 4, and for each superregion R, we associate a collection of pairs of vertices S_R such that $|S_R| > 4|R|\left(1 - \frac{2}{d+1} - \frac{3}{d-3}\right)$. A particularly important type of superregion is the class of tails, which we discuss in Section 5. It is necessary to show that the superregions are a partition of the vertices of G, which we do in Section 6. In Section 7, we discuss the snake graph and peanut graph in detail. We complete the proof in Section 8 by showing that $S_R \cap S_{R'} = \emptyset$ for distinct superregions S_R and $S_{R'}$.

2. Definitions

This section contains basic definitions that are used for the rest of the paper.

Let G be a graph without multiple edges or loops. $V(G)$ denotes the vertex set of G, and $e(G)$ is the edge set of G. If X is a subset of vertices of G, then $G[X]$ is the *induced subgraph* on X, or the maximal subgraph of G with vertex set X.

The *distance* between two vertices u and v, denoted by $d(u,v)$, is the number of edges in a shortest path between u and v. Thus $d(u,u)=0$, $d(u,v)=1$ if there is an edge uv, and so on. For each $i \geq 1$ and vertex v, let $N_i(v)$ be the set of vertices that are distance i from v. We also say that $N(v) := N_1(v)$, and $\deg_i(v) := |N_i(v)|$. Also, $N_2'(v)$ is the set of vertices $u \in N_2(v)$ such that $u \in N(w)$ for some $w \in N_3(v)$. A d-regular graph is a graph such that every vertex v satisfies $\deg(v) := \deg_1(v) = d$.

Let the *graph power* G^k be the graph with $V(G^k) = V(G)$ and an edge uv whenever $d(u,v) \leq k$ in G.

A *low degree vertex* v is a vertex v satisfying $\deg_2(v) \leq 3$. Note that if v is a low degree vertex and $N_2'(v) = \emptyset$, then G contains at most $d+4$ vertices and thus G^2 is complete when G is d-regular with $d > 6$.

Lemma 2.1. *Let $v \in V(G)$ and let $u \in N_2'(v)$. Then* $\deg_2(u) \geq d - \deg_2(v) + 1$. *In particular, if v is low degree, then* $\deg_2(u) \geq d - 2$.

Proof. $G - N_2'(v)$ is disconnected, with one component of $G - N_2'(v)$ being $G_v := \{v\} \cup N(v) \cup (N_2(v) - N_2'(v))$ and another component G' containing a vertex $x' \in N(u)$ that is distance 3 from v. Since $|N_2'(v)| \leq \deg_2(v)$, x' has degree at least $d - \deg_2(v)$ in G', and let $N'(x')$ be the sets of neighbors of x' in G'. Also, choose $x \in G_v \cap N(u)$. Since $N(x) \subset \{v\} \cup N(v) \cup N_2(v)$, the sets $N(x) \cup \{x\}$ and $N'(x') \cup \{x'\}$ are disjoint, and thus $|N(x) \cup N'(x') \cup \{x, x'\}| \geq 2d - \deg_2(v) + 2$. Every vertex of $N(x) \cup N'(x') \cup \{x, x'\}$ is within distance 2 of u, and since at most $d+1$ of them are within distance 1 of u, we have that $\deg_2(u) \geq d - \deg_2(v) + 1$. $\square$

For the remainder of this paper, we assume that G is a connected d-regular graph such that $d > 6$ and G^2 is not complete.

3. Regions

In this section, we discuss regions, a key tool in the proof of our main theorem. Let X be the set of vertices $x \in G$ that satisfy $\deg_2(x) < 4$. Define an equivalence relation $\sim$ on X by saying that $u \sim v$ if there exists a sequence of vertices $(u = v_0, v_1, \ldots, v_t = v)$ such that for $0 \leq i \leq t-1$, $v_i \in X$ and $d(v_i, v_{i+1}) \leq 2$.

Definition 3.1. Let W be the union of an equivalence class $X' \subset X$ under $\sim$ and all neighbors of vertices in X'. Then $G[W]$ is a *region.*

Note that some vertices might not be contained in any region. We prove some basic properties of regions.

Lemma 3.2. *Let X be the set of vertices x of G that satisfy* $\deg_2(x) < 4$, *and let R be a region that contains $v \in X$. Then*
1) R contains at least $d+1$ vertices.

2) $R \subset \{v\} \cup N(v) \cup N_2(v)$.
3) Let $t := \min\{\deg_2(x)|x \in R\}$. Then R contains at most $d + t + 1$ vertices.
4) R contains at most $d + 4$ vertices.
5) R is disjoint from any other region R'.

Proof. Part 1 follows from the fact that, by definition, R contains v and all neighbors of v.

Note that $G - N_2'(v)$ is disconnected, and the component of $G - N_2'(v)$ that contains v is $G_v := \{v\} \cup N(v) \cup (N_2(v) - N_2'(v))$. Let G' be a different component, if there is one. By Lemma 2.1 and $d > 6$, no vertex in $N_2'(v)$ is in X. Consider $u \in N_2'(v)$. If some $w \in N(u) \cap G_v$ is in X, then no $w' \in N(u) \cap G'$ is also in X. To see this, observe that $N_2(w)$ contains $N(u) - G_v$, and so $|N(u) - G_v| < 4$ and $|N(u) \cap G_v| > d - 4$. Since $N_2(w')$ contains $N(u) \cap G_v$, $\deg_2(w') > d - 4 \geq 3$ and $w' \notin X$.

Suppose by way of contradiction that there exists $v' \in R \cap X$ outside of G_v. By definition of a region, there is a sequence of vertices $\{v = v_0, v_1, \ldots, v_k = v'\}$ such that each $v_i \in X$ and $d(v_i, v_{i+1}) \leq 2$ for all $0 \leq i \leq k - 1$. Let v_j be the first vertex in the sequence that is not in G_v. Since $G - N_2'(v)$ is disconnected and no vertex in $N_2'(v)$ is also in X, $d(v_{j-1}, v_j) = 2$, and there exists a vertex $u \in N_2'(v)$ adjacent to both v_{j-1} and v_j. This contradicts the previous paragraph. Part 2 follows. There exists $v \in R \cap X$ with $\deg_2(v) = t$, and Part 3 follows. Part 4 follows by $t < 4$.

Now consider $x \in R \cap R'$ for some region R'. By definition of a region, there exist vertices $v \in R \cap X$ and $v' \in R' \cap X$ such that $d(v, x) \leq 1$ and $d(v', x) \leq 1$. Then $d(v, v') \leq 2$ and thus $R = R'$. This proves Part 5. □

We define several classes of regions. Let R be a region with a vertex v with $\deg_2(v) = 1$ and $N_2(v) = \{u\}$. Let G_v be the component of $G - u$ that contains v. Then G_v contains $d+1$ vertices, namely $\{v\} \cup N(v)$. In G_v, all neighbors of u have degree $d-1$, and all other vertices have degree d. Hence the complement of G_v is a matching on the neighbors of u. Let $t := |G_v \cap N(u)|$, and consider $w \in G_v \cap N(u)$. Since u contains $d - t$ neighbors outside of G_v, $\deg_2(w) = d - t + 1$, and w is low degree if and only if $t \geq d - 2$. Then R is either G_v or $G_v \cup \{u\}$, and the latter holds if and only if $t \geq d-2$. If $t \neq d-1$, then we say that R is an *A region*, and if $t = d - 1$, we say that R is a *B region*. Since a B region contains the complement of a matching on $d - 1$ vertices, a B region can exist only when d is odd.

Next suppose that R is not an A or B region, but R contains a vertex v with $\deg_2(v) = 2$. If R contains a vertex v' with $\deg_2(v') = 2$ and $|N_2'(v')| = 1$, then we say that R is a *C region*. Otherwise, we say that R is a *D region*.

Now suppose that R is a region such that $\deg_2(v) = 3$ for all low degree vertices v in R. Let k be the minimum size of $N_2'(v)$ for low degree vertices in R. In the respective cases that $k = 1, 2$, or 3, we say that R is an *E region*, an *F region*, or a *G region*.

4. Superregions

We now define *superregions*. We show that the superregions of G are a partition of $V(G)$ in Section 6. Before we specify the superregions, we first define sets of vertices associated with G and the superregions.

Let $\mathcal{U}$ be the set of vertices u in G that satisfy $\deg_2(u) \geq d-2$. By Lemma 2.1, if v is a low degree vertex, then $N_2'(v) \subset \mathcal{U}$. In defining superregions, we will also designate special sets $\mathcal{W}$ and $\mathcal{N}$ such that if R is a superregion, then $|R \cap (\mathcal{W} \cup \mathcal{N} - \mathcal{U})| \leq \frac{2}{d+1}|R|$. Let $\mathcal{V} := V(G) - \mathcal{U} - \mathcal{W} - \mathcal{N}$.

Lemma 4.1. *Theorem 1.2 holds for G if $|\mathcal{U}| \geq \frac{3}{d-3}n$.*

Proof. If $v \notin \mathcal{U}$, then $\deg_2(v) \geq 1$. We then have that

$$\sum_{v \in V(G)} \deg_2(v) \geq \frac{3}{d-3}n(d-2) + \frac{d-6}{d-3}n(1) \geq 4n,$$

which implies Theorem 1.2. □

We therefore assume that

$$|\mathcal{U}| < |V(G)| \frac{3}{d-3},$$

and then, since superregions partition $V(G)$,

$$|\mathcal{V}| = |V(G)| - |\mathcal{W}| - |\mathcal{N}| - |\mathcal{U}| > |V(G)| \left(1 - \frac{2}{d+1} - \frac{3}{d-3}\right).$$

For every superregion R, we associate a collection S_R of at least $4|R \cap \mathcal{V}|$ ordered pairs of vertices of the form (x, y) such that $d(x, y) = 2$. Since the superregions partition $V(G)$, we have that $\sum_R |S_R| \geq 4|\mathcal{V}|$. This proves Theorem 1.2 as long as the S_R are disjoint, a matter that is partially addressed below and addressed more fully in Section 8. By the following constructions, S_R may be partitioned into subsets S_{Ri}, for $1 \leq i \leq 4$, as follows. For each pair $(x, y) \in S_R$,

- if x and y are both in R, then $(x, y) \in S_{R1}$;
- if $x \in \mathcal{V} \cap R$ and $y \notin R$, then $(x, y) \in S_{R2}$;
- if $y \in \mathcal{V} \cap R$, y is low degree, and $x \in N_2'(y) - R$, then $(x, y) \in S_{R3}$;
- if $x \in \mathcal{W} \cap R$ and $y \notin R$, then $(x, y) \in S_{R4}$.

Lemma 4.2. *Suppose that $(x, y) \in S_R \cap S_{R'}$ for distinct superregions R and R'. Then either $(x, y) \in S_{R4} \cap S_{R'3}$ or $(x, y) \in S_{R3} \cap S_{R'4}$.*

Proof. By definition of the S_{Ri}, one of x or y is in R and the other is in R'. Without loss of generality, assume that $x \in R$ and $y \in R'$. Then $(x, y) \in S_{R'3}$. Then $x \notin \mathcal{V}$, and so $(x, y) \in S_{R4}$. $\square$

As we specify the superregions and the sets S_R over the rest of this section and Section 5, observe that the following holds by construction.

Lemma 4.3. *Let v be a low degree vertex of a superregion R. Then there are at most $4 - \deg_2(v)$ vertices u such that $(u, v) \in S_{R3}$.*

4.1. Single Vertex Superregions

If v is a vertex that is not contained in any region, then say that $\{v\}$ is a superregion R. Since v is not in a region, $\deg_2(v) \geq 4$. If $v \notin \mathcal{V}$, then $S_R = \emptyset$. Otherwise, let $S_R = S_{R2}$ be $\{(v, u) : u \in N_2(v)\}$.

4.2. D, E, F, G Regions

If R is a region that is not contained in any other superregion, then R is a superregion. If R is a D, E, F, or G region, for all $v \in R \cap \mathcal{V}$, $\deg_2(v) + |N_2'(v)| \geq 4$. Choose $A_v \subseteq N_2'(v)$ so that $A_v = N_2'(v)$ if $\deg_2(v) = 2$, $|A_v| = 1$ if $\deg_2(v) = 3$, and otherwise $A_v = \emptyset$. Set

$$S_v := \{(v, a)\}_{a \in N_2(v)} \cup \{(a, v)\}_{a \in A_v}.$$

Then set $S_R := \cup_{v \in R \cap \mathcal{V}} S_v$. The pairs (v, a) are either in S_{R1} or S_{R2}, and the pairs (a, v) are either in S_{R1} or S_{R3}. Since $A_v \cap \mathcal{V} = \emptyset$, the S_v are disjoint, and thus $|S_R| \geq 4|R \cap \mathcal{V}|$. The sets A_v are not chosen arbitrarily, but the choice will be made strategically in order to insure that $S_R \cap S_{R'} = \emptyset$ for all superregions $R' \neq R$. This matter is discussed more fully below.

4.3. A Regions

Next, suppose that R is an A region and a superregion. Let V be the set of vertices $v \in R$ with $\deg_2(v) = 1$, let u be the unique vertex in $N_2(v)$ for each vertex in $v \in V$, and let X be the set of remaining vertices of R. Choose distinct $w_1, w_2 \in X$ and set $R \cap \mathcal{W} = \{w_1, w_2\}$. By definition of an A region, $|V| \geq 3$. Note that $\deg_2(x) = |V|$ for $x \in X$. Let S_R be the the union of the following sets of pairs:

- if $|V| < d - 2$ (so that $X \cap \mathcal{U} = \emptyset$), all $(|X| - 2)(|V|)$ pairs of the form (x, y) for $x \in X - \mathcal{W}$ and $y \in N_2(x)$ (these pairs are either in S_{R1} or S_{R2});
- $(|X| - 2) \max(0, 4 - |V|)$ pairs of the form (y, x) for $x \in X - \mathcal{W}$ and $y \in N_2(x) - R$ (call this set S_R');
- all $|V|$ pairs (v, u) for $v \in V$ (these pairs are either in S_{R1} or S_{R2});

- all $|V|$ pairs (u, v) for $v \in V$ (these pairs are either in S_{R1} or S_{R3});
- and all $2|V|$ pairs of the form (w, y) for $w \in \{w_1, w_2\}$ and $y \in N_2(w)$ (these pairs are in S_{R4} if $y \notin R$ and in S_{R1} otherwise).

Then $|S_R| \geq 4(|V| + |X| - 2) \geq 4|R \cap \mathcal{V}|$.

Observe that $S'_R \subseteq S_{R3}$. To see this, if $(|X| - 2)\max(0, 4 - |V|) \neq 0$, which happens only if $|V| = 3$, then for $x \in X - \mathcal{W}$, $\deg_2(x) = 3$. Let $N_2(x) = \{x', y, y'\}$ for $x' \in R$ and $y, y' \notin R$. The only possible neighbors of y in $N_1(x) \cup N_2(x)$ are u and y', and thus y is adjacent to some vertex that is distance 3 from x. The same is true for y'. Thus $\{y, y'\} \subset N'_2(x)$, showing that $(y, x), (y', x) \in S_{R3}$.

4.4. C Regions

Suppose that R is a C region. Let v be a vertex in R with $\deg_2(v) = 2$ and $N'_2(v) = \{u\}$. Let G_v be the component of $G - \{u\}$ that contains v and $V := N_2(u) \cap G_v$. Choose a vertex w in $R \cap N(u)$ and set $R \cap \mathcal{W} = \{w\}$, and let X be the set of remaining vertices in $R \cap N(u)$. We prove several lemmas on the structure of C regions.

Lemma 4.4. *With all quantities as above, $V(R) \subseteq V \cup X \cup \{w, u\}$.*

Proof. By Lemma 3.2, $V(R) \subset G_v \cup \{u\}$. It suffices to show that if $v' \in R$, then $d(v', u) \leq 2$. Since $G_v = \{v\} \cup N(v) \cup N_2(v) - \{u\}$, $|G_v| = d + 2$. Suppose by way of contradiction that $d(v', u) \geq 3$. Then $\{v'\} \cup N(v') \cup N_2(v') \subseteq G_v$, and thus $\deg_2(v') \leq 1$, a contradiction to the definition of a C region. □

Lemma 4.5. *Let all quantities be as above. Then $G_v = R$.*

Proof. First we show that $G_v \subset R$. As in the proof of Lemma 4.4, G_v has $d + 2$ vertices. Since $N(v) \subset G_v$, there exists one vertex $z \in G_v$ that is not adjacent to v. By definition of a region, all vertices of $G_v - \{z\}$ are in R. Each $v' \in V$ is in R since v' is within distance 2 of v and is low degree. Since u must have at least one neighbor outside of G_v, u has at most $d - 1$ neighbors in G_v. Some vertex $z' \in \{z\} \cup N(z)$ is not in $\{u\} \cup N(u)$ and thus $z' \in V$, and therefore $z \in R$ by definition of a region.

Finally, we must show that $u \notin R$. Note that $|V| = 3$ is impossible, since then G_v would contain $d - 1$ vertices of degree $d - 1$ and 3 vertices of degree d, and the sum of the degrees of all vertices would be odd. Thus $|V| \geq 4$, and since u has $d + 2 - |V|$ neighbors in G_v, u has $|V| - 2 \geq 2$ neighbors outside of G_v. If $x \in R \cap N(u)$, then $N_2(x)$ contains 2 vertices in G_v and $|V| - 2 \geq 2$ outside of G_v, and thus x is not low degree. By construction of a region, $u \notin R$. □

Lemma 4.6. *With all quantities as above, $|R \cap N(u)| \geq 2$.*

Proof. Suppose by way of contradiction that $R \cap N(u) = \{w\}$. By Lemma 4.5, $R = \{v\} \cup N(v) \cup N_2(v) - N_2'(v)$. Thus some vertex x of R is not adjacent to w, and $d(x, u) \geq 3$. This contradicts Lemma 4.4. □

Let y_1 and y_2 be two distinct neighbors of u outside of R, which exist as in the proof of Lemma 4.5. Also observe that $u \in \mathcal{U}$. Let S_R be the union of the following sets:

- the $|V|+2|X|$ pairs of the form (x, y) for $x \in V \cup X$ and $y \in R \cap N_2(x)$. These pairs are each in S_{R1}, and they exist since every vertex in V and X have, respectively 1 and 2 non-neighbors in G_v. Furthermore, $(G_v)^2$ is complete since in G_v, every vertex has degree at least $d-1$ and there are $d+2$ vertices in total;
- the $|V|$ pairs (v', u) for $v' \in V$ (these pairs are in S_{R2});
- the $|V|$ pairs (u, v') for $v' \in V$ (these pairs are in S_{R3});
- the $|V|$ pairs of the form (w, y) for $y \in N_2(w)$ (these pairs are either in S_{R1} or S_{R4});
- and if $|V| < d-2$, the $2|X|$ pairs $(x, y_1), (x, y_2)$ for each $x \in X$ (call this set of pairs S_R').

As in the proof of Lemma 4.5, $\deg_2(x) = |V|$ for all $x \in R \cap N(u)$, and the condition that $|V| < d-2$ is equivalent to $x \notin \mathcal{U}$ for each $x \in R \cap N(u)$, and so $S_R' \subset S_{R2}$. We have that $|S_R| \geq 4|R \cap \mathcal{V}|$.

B regions are discussed in the context of tails in the next section.

5. Tails and Superregions

In this section we define several types of superregions that are based on a tail subgraph. Every superregion described in this section contains a B region, and thus these superregions exist only when d is odd.

5.1. Tails

The following construction, a tail, is a subgraph of all superregions defined in this section. A tail itself is a superregion unless it is contained in a larger superregion.

Definition 5.1. Let R_1 be a B region, and for $2 \leq i \leq k$, let R_i be a clique on $d+1$ vertices with the edge $v_i v_i'$ removed. Let v be the degree $d-1$ vertex of R_1. Suppose that there are edges vv_2 and $v_i' v_{i+1}$ for $2 \leq i \leq k-1$. Then $G[R_1 \cup \ldots \cup R_k]$ is a *tail*. See Figure 1.

The R_i are the *segments* of T. We may have $k = 1$, in which case the tail is a B region. An *improper tail* is contained in a larger tail, and otherwise a tail is *proper*. If T is a tail, let u_T be the unique vertex that is adjacent to a vertex in T but not itself in T, and let w_T be the unique vertex in T that is adjacent to u_T.

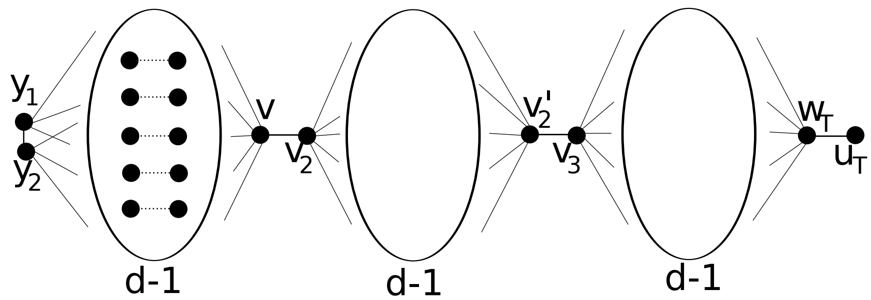

Figure 1: Tail. Note that u_T is not part of the tail.

Let T be a tail. Let $\mathcal{N} \cap T$ be $\{y_1, y_2\} = R_1 \cap N_2(v)$. Associate the following sets of pairs of vertices with T:

- S_{T1} is the set of $(4k-3)(d-1)$ pairs of vertices (x, y) and (y, x) such that $x, y \in T$, $d(x, y) = 2$, and $\deg_2(x) = 2$;
- S_{T2} (resp. S_{T3}) is the set of $d-1$ pairs of vertices of the form (x, u_T) (resp. (u_T, x)) such that $x \in T$ and $d(x, u_T) = 2$.

In the event that T is a superregion, let Y be the set of $d-1$ neighbors of u_T that are not in T and set $T \cap \mathcal{W} = \{w_T\}$. Then say that

- S_{T4} is the set of $d-1$ pairs (w_T, y) for $y \in Y$.

Then $T \cap \mathcal{V}$ contains $d-1$ vertices in each segment of T, and $|S_T| = |S_{T1}| + |S_{T2}| + |S_{T3}| + |S_{T4}| = 4|T \cap \mathcal{V}|$.

We now define a snake graph, our first exception to Conjecture 1.1. See Figure 2 for an illustration.

Definition 5.2. A *snake graph G* is the union of tails $T \cup T'$ with $u_T = w_{T'}$.

The two superregions of G above are $G - R_1$ and $G - R'_1$, where R_1 and R'_1 are the two B regions. These superregions intersect if either T or T' contain more than 1 segment.

We prove an important lemma on tails.

Lemma 5.3. *Let T and T' be tails with nonempty intersection. Then either G is a snake graph or one of T or T' is contained in the other.*

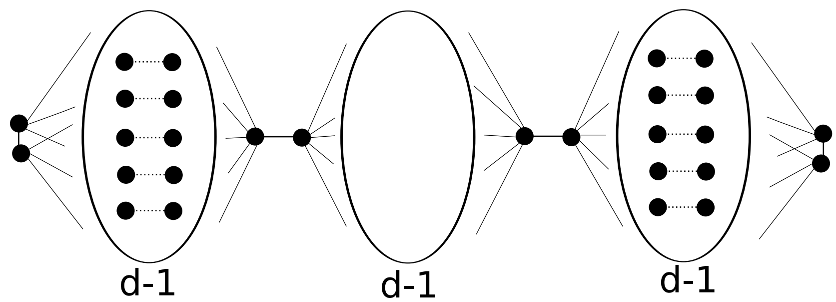

Figure 2: Snake graph

Proof. Let the segments of T and T' be $R_1, R_2, \ldots, R_k$ and $R'_1, \ldots, R'_j$ respectively, and without loss of generality, $k \geq j$. Also suppose that R_1 and R'_1 are B regions, and that R_i and R_{i+1} are joined by an edge for $1 \leq i \leq k-1$, as are R'_i and R'_{i+1} for $1 \leq i \leq j-1$. Since a tail is a union of regions, $T \cap T'$ is also a union of regions.

First suppose that $R_1 = R'_1$, and we show by induction that $T' \subset T$. By construction of a B region, there is only one vertex outside of R_1 that is adjacent to a vertex in R_1, and thus R_2 and R'_2 intersect and are thus equal. Now suppose that $R_{i-1} = R'_{i-1}$ for some $3 \leq i \leq j$. By construction, there are two vertices outside of R_{i-1} that are adjacent to a vertex in R_{i-1}, one of which is in $R_{i-2} = R'_{i-2}$. Thus the other must be in both R_i and R'_i, and we conclude that $R_i = R'_i$. Thus if $R_1 = R'_1$, then $T' \subseteq T$.

Now consider the case that $R_1 \neq R'_1$. Choose k' and j' so that $R_{k'} = R'_{j'}$ and the sum $j' + k'$ is minimized. Let T_1 be $R_1 \cup \ldots \cup R_{k'-1}$, and let T_2 be $R'_1 \cup \ldots \cup R'_{j'-1}$. Let v and v' be the two vertices of $R_{k'} = R'_{j'}$ that are adjacent to vertices outside of $R_{k'}$. Since T_1 and T_2 are disjoint, $w_{T_1} \neq w_{T_2}$. Then either v or v' is adjacent to w_{T_1}; assume without loss of generality an edge vw_{T_1}. Either vw_{T_2} or $v'w_{T_2}$ is an edge, and since v is adjacent to only one vertex outside of R_k, there is an edge $v'w_{T_2}$. There can be no other vertices or edges in G, as the above establishes a d-regular graph. Thus G is a snake graph. □

5.2. Multitails

Let $\{T_1, \ldots, T_m\}$ be a maximal collection of tails such that $u_{T_1} = \cdots = u_{T_m}$. If $m \geq 2$, then $R = T_1 \cup \cdots \cup T_m$ is an *m-multitail*, and multitails are superregions. Let S_R, each S_{Ri} for $1 \leq i \leq 4$, $R \cap \mathcal{N}$, and $R \cap \mathcal{W}$ be the unions of the corresponding sets over the T_j. Then $|S_R| = 4|R \cap \mathcal{V}|$.

5.3. α-Tails

We define our next superregion R, an α-*tail*, as follows. See Figure 3 for an illustration.

Definition 5.4. Let T be a tail with k segments. Let H' be a subgraph of G consisting of the following vertices:

- u_T,
- a vertex z,
- a set X with $|X| = d - 2$,
- a subset $X' \subset X$ of even cardinality less than $d - 3$, and
- vertices y_1 and y_2.

Let the edge set of H' be as follows:

- $u_T z$,
- $u_T x$ for all $x \in X$,
- a complete subgraph on X with a matching on X' removed,
- xy_1 and xy_2 for all $x \in X$,
- $y_1 y_2$,
- zy_1 and zy_2, and
- xz for all $x \in X'$.

All vertices $x \in X - X'$ are low degree and are all in the same region H. Then we say that $T \cup H$ is an α-tail.

Either $H = H'$ or $H = H' - \{z\}$; this follows from Lemma 3.2 and the observations that $N(x) = H' - \{z\}$ for $x \in X - X'$, and $N_2(x) = \{w_T, z\}$, whereas $w_T \notin H$ since $w_T \in T$. Let $R := T \cup H$. We call H the *head region* of R. Note that $G - \{z\}$ is disconnected with $R - \{z\}$ a component. Also note that if we were to allow $|X'| = d - 3$, then z would have d neighbors in R, and G would be a snake graph.

Take $R \cap \mathcal{N}$ to be the two vertices $v_1, v_2 \in T$ that satisfy $\deg_2(v_1) = \deg(v_2) = 1$, together with y_1 and y_2. We take $R \cap \mathcal{W} = \{u_T\}$. Note that $z \in \mathcal{U}$. Also, $X - X' \subset \mathcal{V}$, whereas X' might or might not be a subset of $\mathcal{V}$. Then $|R \cap \mathcal{V}| = k(d-1) + |X \cap \mathcal{V}|$. Let Z be the set of vertices adjacent to z and not in R; $|Z| = d - |X'| - 3 \geq 2$ by $|X'| < d - 3$, $|X'|$ even, and d odd. Let b_1 and b_2 be two distinct vertices of Z.

Now we let S_R be the following sets of pairs:

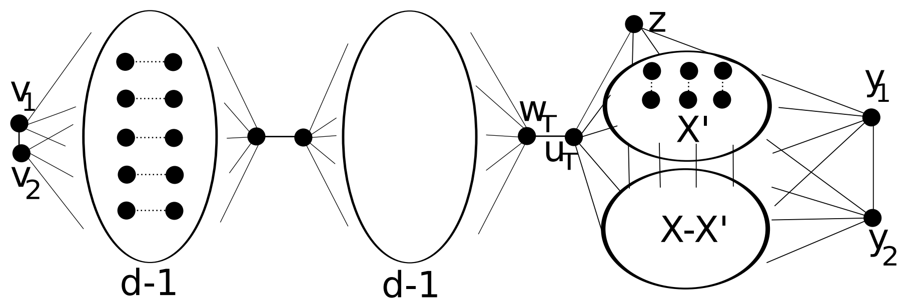

Figure 3: α-tail

- the $(4k-1)(d-1)$ elements of $S_{T1} \cup S_{T2} \cup S_{T3}$ (these are in S_{R1});
- the 2 pairs (u_T, y_1) and (u_T, y_2) (these are in S_{R1});
- the $|X - X'| - 1$ pairs (u_T, s) for all $s \in Z$ (these are in S_{R4});
- the $|X'|$ pairs (x, x') for $x, x' \in X'$ (these are in S_{R1});
- the $2|X - X'|$ pairs (x, w_T), (w_T, x) for $x \in X - X'$ (these are in S_{R1});
- the $|X - X'|$ pairs (x, z) for $x \in X - X'$ (these are in S_{R1} if $z \in R$ and otherwise in S_{R2});
- the $|X - X'|$ pairs (z, x) for $x \in X - X'$ (these are in S_{R1} if $z \in R$ and otherwise in S_{R3});
- the $2|X' \cap \mathcal{V}|$ pairs (x, w_T), (w_T, x) for $x \in X' \cap \mathcal{V}$ (these are in S_{R1});
- and the $2|X' \cap \mathcal{V}|$ pairs (x, b_1), (x, b_2) for $x \in X' \cap \mathcal{V}$ (these are in S_{R2}).

Then $|S_R| = 4|\mathcal{V} \cap R|$.

5.4. β-Tails

Our final superregion is a β-*tail*, defined as follows. See Figure 4 for an illustration.

Definition 5.5. Let T be a tail with k segments, and let H' be a subgraph of G with the following vertices:

- X with $|X| = d - 1$;
- $X' \subset X$ such that $|X'|$ is even and not equal to 0 or $d - 1$;
- a vertex w;

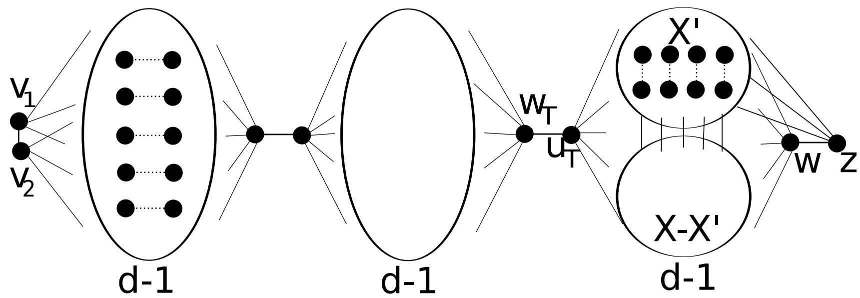

Figure 4: β-tail

- a vertex z;
- and u_T.

Suppose that H' consists of the following edges:

- a complete graph on X with a matching on X' removed;
- all edges $u_T x$ for $x \in X$;
- all edges wx for $x \in X$;
- all edges zx for $x \in X'$;
- and wz.

Each $x \in X - X'$ is low degree and is contained in a common region H. Then $T \cup H$ is a β-tail.

For $x \in X - X'$, $N_2(x) = N_2'(x) = \{z, w_T\}$. Since $w_T \in T$, $w_T \notin H$. Also, since d is odd, $|X'| \leq d - 3$ and z has $|X - X'| \geq 2$ neighbors outside of R. Therefore, no vertex of X' is low degree, and w is low degree if and only if $|X'| = d - 3$. We conclude that $H = H'$ exactly when $|X'| = d - 3$, and otherwise $H = H' - \{z\}$. We say that H is the *head region* of R. Note that $G - \{z\}$ is disconnected with $R - \{z\}$ a component. Also note that if we were to allow $|X'| = d - 1$, then R would be a snake graph, whereas if $|X'| = 0$, then R would be an ordinary tail.

We take $R \cap \mathcal{W} = \{w\}$ and $R \cap \mathcal{N}$ to be the two vertices v_1 and v_2 of T such that $\deg_2(v_1) = \deg_2(v_2) = 1$. Also, $X - X' \subset \mathcal{V}$, whereas X' might or might not be a subset of $\mathcal{V}$. Note that $|R \cap \mathcal{V}| = k(d-1) + |X \cap \mathcal{V}|$. Also, z has $|X - X'|$ neighbors outside of R, and since $|X - X'| \geq 2$, consider distinct $b_1, b_2 \in N(z) - R$. Then we define S_R as the union of the following sets:

- the $(4k-1)(d-1)$ elements of $S_{T1} \cup S_{T2} \cup S_{T3}$ (these are in S_{R1});

- the $|X - X'|$ pairs (w, s) for $s \in N(z) - R$ (these are in S_{R4});
- the $|X'|$ pairs (x, x') for $x, x' \in X'$ and xx' not an edge (these are in S_{R1});
- the $2|X|$ pairs $(x, w_T), (w_T, x)$ for $x \in X$ (these are in S_{R1});
- the $|X - X'|$ pairs (x, z) for each $x \in X - X'$ (these are in S_{R1} if $z \in H$ and otherwise in S_{R2});
- the $|X - X'|$ pairs (z, x) for each $x \in X - X'$ (these are in S_{R1} if $z \in H$ and otherwise in S_{R3});
- the $2|X' \cap \mathcal{V}|$ pairs (x, b_1), (x, b_2) for $x \in X' \cap \mathcal{V}$ (these are in S_{R2}).

Then $|S_R| = 4k(d-1) + 4|X - X'| + 2|X'| + 2|X' \cap \mathcal{V}| \geq 4|\mathcal{V} \cap R|$.

5.5. Identifying α- and β-Tails

In this section we prove an important lemma on the structure of α- and β-tails.

Lemma 5.6. *A region H is the head region of at most one α- or β-tail.*

Proof. We show that an α- or β-tail $R = T \cup H$ containing H is uniquely determined, given H. By construction, H contains a set of vertices M, $|M| \geq 2$, such that each $m \in M$ is adjacent to some vertex outside of H. First consider the case that $M = \{a, b\}$. Then, by construction, one of a or b, say a, is adjacent to exactly one vertex outside of H, and b is adjacent to at least 2 vertices outside of H. Then a must be u_T, and T and R are uniquely determined since w_T is the only vertex outside of H adjacent to a and by Lemma 5.3.

Now suppose $|M| \geq 3$. By construction, R is an α-tail, and there exists a unique vertex $w \notin H$ such that w is adjacent to exactly one vertex in H. Thus $w = w_T$, and by Lemma 5.3, T and R are uniquely determined. □

6. Superregions as a Partition

In this section, we show that the superregions of G partition $V(G)$ unless G is a snake graph.

Theorem 6.1. *Suppose that G is not a snake graph. Then $V(G)$ is partitioned by the superregions of G.*

Proof. First, every vertex is contained in a superregion, since singleton sets are superregions if not contained in any larger superregion. We need only to show that $R \cap R' = \emptyset$ if R and R' are distinct superregions. If either R or R', say R, is a single vertex, then either $R \subset R'$ or $R \cap R' = \emptyset$. The former is impossible by definition. Now we assume that R and R' each consist of multiple vertices.

By construction, all superregions with multiple vertices are unions of regions. Therefore, if R is both a region and a superregion, then either $R \subset R'$ or $R \cap R' = \emptyset$. Again, the former is impossible by definition, and now we assume that R and R' each consist of multiple regions. By construction, both R and R' contain tails.

Next, suppose that R and R' are both either a tail or a multitail, and suppose that $R \cap R' \neq \emptyset$. By Lemma 5.3, $R \cap R'$ is a collection of proper tails, and let T be a proper tail in $R \cap R'$. By $T \subset R \cap R'$, for all proper tails $T' \subset R \cup R'$, $u_T = u_{T'}$. Thus $R \cup R' = R = R'$.

Now let R be an α- or β-tail with proper tail T and head H, and let R' be either a tail or multitail. Write $R' = T'_1 \cup \cdots \cup T'_k$ as a union of disjoint tails. The head region of an α- or β-tail is not isomorphic to any segment of a tail. Thus, if $R \cap R' \neq \emptyset$, then $H \cap R' = \emptyset$ and $T \cap R' \neq \emptyset$. By Lemma 5.3, $T = T_i$ for some $1 \leq i \leq k$. Furthermore, $H \cup T'_j$ is also an α- or β-tail for each $1 \leq j \leq k$ by $u_{T_1} = u_T = u_{T_j}$, which implies that $k = 1$ by Lemma 5.6. Then $T = R'$ and R' is not a superregion, a contradiction.

Finally, if R and R' are both α- or β-tails such that $R \cap R' \neq \emptyset$, then we show that $R = R'$. Let T and T' be the respective tails of R and R', and let H and H' be the respective head regions. By Lemma 5.3, if $T \cap T' \neq \emptyset$, then $T = T'$. Since only one vertex outside of T is adjacent to T, then $H = H'$ and thus $R = R'$. Next, we have that $H \cap T' = H' \cap T = \emptyset$ by the fact that H and H' are not isomorphic to any segment of a tail. Finally, if $H = H'$, then $R = R'$ by Lemma 5.6.

This establishes that superregions partition $V(G)$. $\square$

7. Exceptions

There are two families of graphs that are exceptions to Conjecture 1.1. In this section we discuss these exceptions in more detail.

7.1. Snake Graphs

A snake graph G consisting of $k \geq 2$ regions, as described above, has $n = k(d+1)+2$ vertices. Since a snake graph contains a B region, a snake graph exists only if d is odd. A snake graph is determined, to isomorphism, by d and k.

By construction, we may calculate that

$$\sum_{v \in V(G)} \deg_2(v) = (4k-2)(d-1) + 8 = \frac{(4k-2)(d-1)+8}{k(d+1)+2} n.$$

For large d, this quantity is approximately $(4 - 2/k)n$.

7.2. Peanut Graphs

A *peanut graph* G is defined as follows. Partition $V(G)$ into sets R_1 and R_2 with $d+1$ and $d+2$ vertices respectively. The only edge in the complement of $G[R_1]$ is w_1w_2. The only edges in the complement of $G[R_2]$ are uv_1, uv_2, uv_3 and a matching on the remaining vertices. The only edges between R_1 and R_2 are uw_1 and uw_2. Note that R_1 and R_2 are both A regions. A peanut graph exists only when d is even, due to the matching in the complement of R_2, and is determined up to isomorphism by d. See Figure 5 for an illustration.

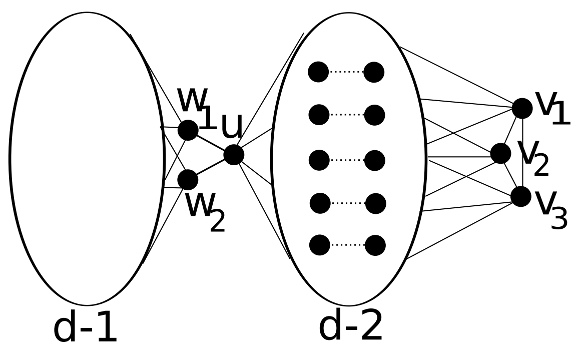

Figure 5: Peanut graph

One may check that $n = 2d+3$ and

$$\sum_{v \in V(G)} \deg_2(v) = 7d - 4 = \frac{7d-4}{2d+3} n.$$

For large d, this quantity is approximately $(7/2)n$.

8. Proof of Theorem 1.2

In this section, we conclude the proof of Theorem 1.2. We need to show that the S_R are disjoint over all superregions R. By Lemma 4.2, we need only to consider $(x, y) \in S_{R4}$ for a superregion R and show that, for each superregion R', that $(x, y) \notin S_{R'3}$, assuming that $S_{R'}$ is properly chosen. Now suppose that $(x, y) \in S_{R'3}$. Note that $x \in R \cap \mathcal{W}$, and y is a low degree vertex of R' with $x \in N_2'(y)$. We consider several cases on R.

8.1. *R* is a Single Vertex, or a D, E, F, or G Region

This case is trivial, since by construction, $\mathcal{W} \cap R = \emptyset$ and $S_{R4} = \emptyset$.

8.2. *R* is an A Region

Let $X = R \cap N_2(y)$. Since the complement of $G[X]$ is a matching, $|X|$ is even, and since y is low degree, $|X| = 2$. We must have $\deg_2(y) = 3$; otherwise, let u be the one vertex adjacent to both x and y, and let $Y := N(y) - \{u\}$ so that $|Y| = d - 1$. No vertex of Y may have a neighbor outside of $Y \cup \{y, u\}$ if $\deg_2(y) = 2$, and thus for $y' \in Y$, $N(y') = Y - \{y'\} \cup \{u, y\}$. But then $X \cup Y \subset N(u)$ is a contradiction to $\deg(u) = d$.

Let z be the unique element of $N_2(y) - X$. By construction, since y is a low-degree vertex with $|N_2(y) - R'| \geq 2$ and $S_{R'3}$ contains a pair of the form $(-, y)$, R' must be a A, D, E, F, or G region.

Consider the case that $z \in N_2(y) - N_2'(y)$. Then $V(G) = V(R) \cup Y \cup \{u, y, z\}$. Counting degrees on $G[Y \cup \{u, y, z\}]$, which has $d + 2$ vertices, we see that u has degree $d - 2$ and all other vertices have degree d. Let a, b, c be the three vertices of $G[Y \cup \{u, y, z\}]$ that are not adjacent to u. Then by degree considerations, the complement of $G[Y \cup \{u, y, z\}]$ contains edges ua, ub, uc and a matching on all other vertices. Thus G is a peanut graph. If R' is an A region, then $z \in N_2(y) - N_2'(y)$ and G is a peanut graph. Then suppose that R' is a D, E, F, or G region.

In $S_{R'}$, we may replace (x, y) with (z, y), unless, by Lemma 4.2, $(z, y) \in S_{R''4}$ for some superregion R''. Then $z \in \mathcal{W}$, and since z is the only vertex in $R'' \cap N_2(y)$, by the choice of $\mathcal{W}$ for the various classes of superregions, R'' must be a tail with $z = w_{R''}$. It must be that $u_{R''} \neq u$; otherwise, since $X \subset N(u)$, at most $d - 2$ vertices of $Y = N(y) - \{u\}$ are adjacent to u, and so let $y' \in Y$ be not adjacent to u. Then $N(y') \subseteq Y - \{y'\} \cup \{y\}$ by $\deg_2(y) = 3$, but then y' has degree at most $d-1$, a contradiction. Now, $H := G[\{y\} \cup N(y)]$ has $d+1$ vertices. By $\deg_2(y) = 3$, all vertices of H have degree d in H, except that u has degree $d - 2$ and $u_{R''}$ has degree $d - 1$. This would imply an odd degree sum on H, which is impossible.

8.3. *R* is an *m*-Multitail

First suppose that $m \geq 3$. Then y is distance 2 from each vertex in $R \cap \mathcal{W}$, and since y is a low degree vertex, $m = 3$. Let u be the vertex adjacent to both x and y, and let $Y := N(y) - \{u\}$. Since u is adjacent to y and 3 vertices in R, u has at most $d - 4$ neighbors in Y. Choose $y' \in Y - N(u)$. Since $\deg_2(y) = 3$, $N(y') \subset Y - \{y'\} \cup \{y\}$. Then y' has degree at most $d - 1$, a contradiction. We conclude that $m = 2$.

With $m = 2$, we may perform the same analysis as in Section 8.2 and conclude that R' is the complement of a graph with $d + 2$ vertices containing edges ua, ub, uc and a matching on all other vertices. This implies that d is even, and the existence of a tail implies that d is odd, and so R cannot be a multitail.

8.4. R is a C Region

Let u be the unique vertex that is adjacent to both x and y. By Lemma 4.6, R contains a vertex $x' \neq x$ in $N(u)$, and $\{x, x'\} \subseteq N_2(y)$. Suppose, by way of contradiction, that $\deg_2(y) = 2$. Let $Y := N(y) - \{u\}$. For each $y' \in Y$, $N(y') \subset Y - \{y'\} \cup \{u, y\}$, and since $\deg_2(y') = d$, then $N(y') = Y - \{y'\} \cup \{u, y\}$. Then $Y \cup \{x, x'\} \subset N(u)$, a contradiction to $\deg_2(u) = d$. We conclude that $\deg_2(y) = 3$.

Note that $x' \notin \mathcal{W}$. By Lemma 4.3 and the fact that $\deg_2(y) = 3$, if $(\tilde{x}, y) \in S_{R3}$, then $\tilde{x} = x$. Then in $S_{R'3}$, we may replace (x, y) by (x', y), and since $x' \notin \mathcal{W}$, $(x, y) \notin S_{R''4}$ for any superregion R''.

8.5. R is an α-Tail or a β-Tail

Let u be the unique vertex that is adjacent to both x and y. By construction, there are at least three vertices x, x', x'' in $R \cap N_2(y)$. Let $Y := N(y) - \{u\}$. Since y is low degree, for each $y' \in Y$, $N(y') \subseteq Y - \{y'\} \cup \{u, y\}$. Since $\deg(y') = d$, $N(y') = Y - \{y'\} \cup \{u, y\}$. Then $Y \cup \{x, x', x''\} \subset N(u)$, a contradiction to $\deg_2(u) = d$.

8.6. R is a Tail

For the final case, that R is a tail, we consider several cases on R'. Let u be the one vertex that is adjacent to both x and y.

8.6.1. R' is an A Region or a Tail

If R' is an A region, then let $Y := N(u) \cap R'$. By definition of an A region, $|Y| < d-1$. Also $|Y|$ must be even, and d must be odd by the existence of a tail R, and so $|Y| \leq d - 3$. Then $N(u)$ consists of at least 3 vertices outside of R'. Also, $N(u) - R' \subset N_2(y)$, and $N_2(y)$ consists of at least one vertex in R', contradicting the definition of a low degree vertex. Thus R' is not an A region. If R' is a tail, then G is a snake graph.

8.6.2. R' is a C Region

By construction, $\deg_2(y) = 2$ and $|N_2'(y)| = 1$. Let y' be the one vertex in $N_2(y) - N_2'(y)$. Since u has $d - 2$ neighbors outside of R, excluding y, and y has $d - 1$ neighbors excluding u, y has a neighbor z that is not adjacent to u. Then $N(z) \subseteq N(y) \cup \{y, y'\} - \{z, u\}$, and since $\deg(z) = d$, $N(z) = N(y) \cup \{y, y'\} - \{z, u\}$. The only vertices in the component of $G - \{u\}$ that contains y are $N(y) - \{u\} \cup \{y, y'\}$, and thus $N_2(z) = \{u\}$, which implies that R' is an A or B region, a contradiction.

8.6.3. $\deg_2(y) = 2$ and $|N_2'(y)| = 2$

Let $N_2'(y) = \{x, z\}$. We consider two cases: if z and u are neighbors, and if they are not neighbors.

If z and u are neighbors, then u has a set Y of $d-3$ neighbors outside of $\{x, y, z\}$, and y has $d-1$ neighbors, excluding u. Let w and w' be distinct vertices in $N(y) - N(u) - \{u\}$. Every vertex in Y is within distance 2 of y, and since $N_2(y) \cap Y = \emptyset$, $Y \subset N(y)$. Thus $Y \subset R'$. Also, w and w' are adjacent to each vertex in $Y \cup \{y, z\}$ as well as each other, since neither is adjacent to u, x, or any vertex of distance 3 or more from y. Similarly, every vertex in Y can only have neighbors among $Y \cup \{y, w, w', z, u\}$, a set of size $d+2$, and so $y' \in Y$ is adjacent to all but possibly one other vertex in Y. The complement of $G[Y]$ has no edges except a (possibly empty) matching, and let Y' be the set of such vertices that are in such a matching. Every vertex in Y' is adjacent to z. If $y' \in Y - Y'$, then y' is not adjacent to z since y' has been established to be adjacent to every other vertex in $Y \cup \{y, w, w', z, u\}$ besides itself. We conclude that $R \cup R'$ is an α-tail, a contradiction to the assumption that R is a superregion.

Now, if u and z are not neighbors, let $Y := N(y) \cap N(u)$. Since $\{x, y\} \subseteq N(u) - N(y)$, $|Y| \leq d-2$. Since $(N(u) - N(y) - \{y\}) \sqcup \{z\} \subseteq N_2(y)$, in fact $N(u) - N(y) = \{x, y\}$ and $|Y| = d-2$. Choose u' so that $N(y) = Y \cup \{u, u'\}$. By $\deg_2(y) = 2$, u' can have no neighbors outside of $Y \cup \{y, z\}$, and so $N(u') = Y \cup \{y, z\}$. For $y' \in Y$, the only possible neighbors of y' are in $Y \cup \{u, y, u', z\}$, since y' is not adjacent to x or any vertex of distance 3 or more from y. Of the vertices of $Y \cup \{u, y, u', z\} - \{y'\}$, y' is adjacent to all but 1. Let $Y' := Y \cap N(z)$. Each $y' \in Y'$ is adjacent to u, u', y, z and thus all but 1 other vertex of $Y - \{y'\}$. If $y' \in Y - Y'$, then y' is adjacent to every vertex of $Y - \{y'\}$, and so the complement of $G[Y]$ is a matching on Y'. If $Y' \neq \emptyset$, then $R \cup R'$ is a β-tail, a contradiction to the assumption that R is a superregion, and if $Y = \emptyset$, then $R \cup R'$ is a tail, also a contradiction.

8.6.4. $\deg_2(y) = 3$ and $|N_2'(y)| = 1$

Let G_y be the component of $G - x$ that contains y. Then G_y has $d+3$ vertices, namely y, all neighbors of y, and $N_2(y) - \{x\}$. Each vertex in G_y has degree d, except that u has degree $d-1$. This is impossible since the degree sum would be odd.

8.6.5. $\deg_2(y) = 3$ and $|N_2'(y)| = 2$

Let $N_2'(y) = \{x, z\}$. By construction, R' is not a tail, multitail, or α- or β-tail. Assume that R' is a D, E, F, or G region. By Lemma 4.3, in $S_{R'}$, we may replace (x, y) with (z, y), unless $(z, x) \in S_{R''4}$ for some superregion R''. In this case, since $z \in \mathcal{W}$, and R'' has no vertex besides z in $N_2(y)$, R'' is a tail. Let u' be the unique that is adjacent to both y and z. Then $u \neq u'$, since otherwise R is a multitail. Let G_y be the component of $G - \{x, z\}$ that contains y. Then G_y has $d + 2$ vertices, namely y, $N(y)$, and the one vertex of $N_2(y) - N'_2(y)$. In G_y, all vertices have degree d except for u and u', which each have degree $d - 1$. This requires d to be even, so that sum of the degrees of all vertices in G_y is even. However, the existence of a tail R requires d to be odd. We conclude that $(z, x) \notin S_{R''4}$ as desired.

8.6.6. $\mathbf{deg_2(y) = 3}$ and $\mathbf{|N'_2(y)| = 3}$

As above, we may assume that R' is a D, E, F, or G region. Let $N_2(y) = N'_2(y) = \{x, z, z'\}$. By Lemma 4.3, if $z \notin \mathcal{W}$, then in $S_{R'}$, we may replace (x, y) with (z, y). Likewise, if $z' \notin \mathcal{W}$, then in $S_{R'3}$, we may replace (x, y) with (z', y). Now suppose that both z and z' are in $\mathcal{W}$. If z and z' are in the same region R'', then R'' is either an A region or a multitail. In that case, let u' be the unique vertex adjacent to each of y, z, z'. Then in $G[\{y\} \cup N(y)]$, all vertices have degree d except for u and u'. If $u = u'$, then $\deg(u) = d - 3$ in $G[\{y\} \cup N(y)]$, while if $u \neq u'$, then u and u' have degrees $d - 1$ and $d - 2$ respectively in $G[\{y\} \cup N(y)]$. Both of these cases are impossible since the degree sum would be odd.

Now suppose that x, z, and z' are all in different regions. Since the regions that contain x, z, z' respectively each have exactly one vertex in $N_2(y)$, they must all be tails. Let u, u_1, u_2 be the vertices adjacent to y and respectively x, z, z'. Since each of x, z, z' are contained in tails and not multitails, u, u_1, u_2 are distinct. Then G_y, the induced subgraph consisting of y and its neighbors, has $d + 1$ vertices, and all vertices have degree d except for u, u_1, u_2, which each have degree $d - 1$. This is also impossible, since the sum of the degrees would be odd.

This enumerates all cases.

References

[1] P. Hegarty, *A Cauchy-Davenport type result for arbitrary regular graphs*, Integers **11** A19 (2011), 8 pp.

[2] M. DeVos, J. McDonald, D. Scheide, *Average degree in graph powers*, J. Graph Theory **72(1)**, (2013), 7-18.

[3] M. DeVos, S. Thomassé, *Edge growth in graph cubes*, arXiv: 1009.0343 (2010).

[4] M. Kneser, *Abschaetzung der asymptotischen Dichte von Summenmengen*, Math. Z. **58** (1953), 459-484.

[5] A. Pokrovskiy, *Growth of graph powers*, Electron. J. Combin. **18(1)**, P88 (2011), 7 pp.