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ON FINITE SUMS AND INTEGRAL REPRESENTATIONS

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Abstract

In this paper we give an alternative proof for a reciprocal binomial coefficient sum given by Sun: $\sum_{k=1}^{n} \frac{1}{k^2\binom{k+n}{k}}$. We then generalize the result to $\sum_{k=1}^{n} \frac{1}{k^m\binom{k+n}{n}}$, for m, n positive integers, and give its integral representation.

1. Introduction and Preliminaries

In the paper [15], Sun states and proves the following new lemma.

Lemma 1. *For each $n = 1, 2, 3, \ldots$, we have:*

$$\sum_{k=1}^{n} \frac{1}{k^2 \binom{k+n}{k}} = 3 \sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}} - \sum_{k=1}^{n} \frac{1}{k^2}. \tag{1}$$

Sun's proof relies on a known identity and induction. The induction proof is perfectly valid; however, it does not shed light on how and if (1) may be generalized. In this paper we give an integral representation for the left-hand side of (1) and then generalize the result in various directions. For the sake of completeness we first highlight the induction proof as given in [15].

Proof. Observe that, for $n \in \mathbb{N}$ (where $\mathbb{N} := \{1, 2, 3...\}$),

$$\sum_{k=1}^{n}\left(\frac{1}{k^2\binom{k+n}{k}}-\frac{1}{k^2\binom{k+n+1}{k}}\right)=\sum_{k=1}^{n}\frac{\binom{k+n}{k-1}}{k^2\binom{k+n}{k}\binom{k+n+1}{k}}$$

$$=\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n-1}\frac{1}{\binom{k+n+2}{n+2}}.$$

There is an identity [4] that states: for b an integer greater than 1 :

$$\sum_{k=0}^{m}\frac{1}{\binom{x+k}{b}}=\frac{b}{b-1}\left(\frac{1}{\binom{x-1}{b-1}}-\frac{1}{\binom{x+m}{b-1}}\right);$$

and for $b = 1$,

$$\sum_{k=0}^{m}\frac{1}{x+k} = H_{m+x}-H_{x-1}.$$

Now

$$\sum_{k=1}^{n}\left(\frac{1}{k^2\binom{k+n}{k}}-\frac{1}{k^2\binom{k+n+1}{k}}\right) = \frac{1}{(n+1)^2}\left(1-\frac{1}{\binom{2n+1}{n+1}}\right)$$

and hence

$$\sum_{k=1}^{n+1}\frac{1}{k^2\binom{k+n+1}{k}}-\sum_{k=1}^{n}\frac{1}{k^2\binom{k+n}{k}} = \frac{3}{(n+1)^2\binom{2n+2}{n+1}}-\frac{1}{(n+1)^2}.$$

Therefore (1) follows by induction. □

In this paper we shall give a different proof of (1), then generalize the result, and also give integral representations of the specified sums. These results will in turn allow for stronger results on the congruences of similar sums. First we give some definitions which will be useful throughout this paper.

The generalized harmonic numbers in power α are defined as:

$$H_n^{(\alpha)} = \sum_{r=1}^{n}\frac{1}{r^\alpha},$$

and the n^{th} harmonic number for $\alpha = 1$ is:

$$H_n^{(1)} = H_n = \int_0^1 \frac{1-t^n}{1-t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi\left(n+1\right),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log\left(n\right) \right) = -\psi\left(1\right) \approx 0.5772156649.......$$

Let $\mathbb{C}$ be the set of complex numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{N}$ the set of natural numbers. Then for $w \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $(w)_n$ is Pochhammer's symbol defined by

$$(w)_n = \begin{cases} \frac{\Gamma(w+n)}{\Gamma(n)} = w(w+1)....(w+n-1), \text{ for } n \in \mathbb{N} \\ \qquad\qquad 1, \qquad \text{for } n = 0 \end{cases}, \tag{2}$$

where $\mathbb{Z}_0^-$ denotes the set of non-positive integers, and the Gamma and beta functions are defined respectively as

$$\Gamma\left(z\right) = \int_0^\infty w^{z-1} e^{-w} dw, \text{ for } \mathrm{Re}\left(z\right) > 0,$$

and

$$B\left(s, z\right) = B\left(z, s\right) = \int_0^1 w^{s-1}\left(1-w\right)^{z-1} dw = \frac{\Gamma\left(s\right)\Gamma\left(z\right)}{\Gamma\left(s+z\right)}$$

for $\mathrm{Re}\left(s\right) > 0$ and $\mathrm{Re}\left(z\right) > 0$. The binomial coefficient is defined as

$$\binom{z}{w} = \frac{\Gamma\left(z+1\right)}{\Gamma\left(w+1\right)\Gamma\left(z-w+1\right)}$$

for z and w non-negative integers, where $\Gamma\left(x\right)$ is the Gamma function. The polygamma functions $\psi^{(k)}\left(z\right)$, $k \in \mathbb{N}$ are defined by:

$$\begin{aligned} \psi^{(k)}\left(z\right) &:= \frac{d^{k+1}}{dz^{k+1}} \log\Gamma\left(z\right) = \frac{d^k}{dz^k}\left(\frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)}\right) \\ &= -\int_0^1 \frac{\left[\log(t)\right]^k \; t^{z-1}}{1-t} dt, \;\; k \in \mathbb{N}, \end{aligned} \tag{3}$$

and $\psi^{(0)}\left(z\right) = \psi\left(z\right)$ denotes the psi, or digamma function, defined by:

$$\psi\left(z\right) = \frac{d}{dz} \log\Gamma\left(z\right) = \frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)}.$$

We also recall the relation for $m=1,2,3,...,$

$$H_{z-1}^{(m+1)}=\zeta\left(m+1\right)+\frac{\left(-1\right)^{m}}{m!}\psi^{(m)}\left(z\right). \tag{4}$$

There are many results of the type (1) for finite, infinite and alternating sums. For example see [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [16], [18] and references therein.

2. Generalizations

The next lemma deals with an alternative proof of (1) from which more generalized identities can be ascertained.

Lemma 2. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n}\frac{1}{k^{2}\binom{k+n}{n}}=\int_{0}^{1}\int_{0}^{1}\frac{\left(1-x\right)^{n}\left(1-\left(xy\right)^{n}\right)}{1-xy}dydx \tag{5}$$

$$=\quad -\left(H_{n}\right)^{2}+\sum_{k=1}^{n}\frac{\left(-1\right)^{k+1}\binom{n}{k}H_{n+k}}{k}. \tag{6}$$

Proof.

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k^{2}\binom{k+n}{n}} &= \sum_{k=1}^{n}\frac{n!}{k^{2}\prod\limits_{j=1}^{n}\left(k+j\right)}=\sum_{k=1}^{n}\frac{n!}{k^{2}\left(k+1\right)_{n+1}}\\ &= \sum_{k=1}^{n}\frac{n!}{k^{2}}\sum_{j=1}^{n}\frac{A_{j}}{k+j}\ ,\end{aligned}$$

where

$$A_{j}=\left(-1\right)^{j+1}\ \frac{j}{n!}\binom{n}{j}$$

which can be easily shown by multiplying the previous equation by $k+i$ and then taking the limit $k\to\left(-i\right)$.

Now

$$\sum_{k=1}^{n}\frac{1}{k^{2}\binom{k+n}{n}}\quad=\quad\sum_{j=1}^{n}\left(-1\right)^{j+1}\ j\binom{n}{j}\sum_{k=1}^{n}\frac{1}{k^{2}\left(k+j\right)}$$

$$= \sum_{j=1}^{n} (-1)^{j+1}\, j \binom{n}{j} \sum_{k=1}^{n} \left[\frac{1}{k^2 j} - \frac{1}{k j^2} + \frac{1}{j^2 (k+j)} \right]$$

$$= \sum_{j=1}^{n} (-1)^{j+1}\, j \binom{n}{j} \left[\frac{H_n^{(2)}}{j} - \frac{H_n}{j^2} + \frac{H_{n+j}}{j^2} - \frac{H_j}{j^2} \right]$$

$$= H_n^{(2)} - H_n H_n - H_n^{(2)} + \sum_{j=1}^{n} \frac{(-1)^{j+1} \binom{n}{j} H_{n+j}}{j},$$

which is the result (6). The identity (6) is an alternate representation to (1) and shows that:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1} \binom{n}{k} H_{n+k}}{k} = 3 \sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}} + (H_n)^2 - H_n^{(2)}.$$

From (6), (1) and some manipulations we can ascertain

$$-(H_n)^2 + \sum_{k=1}^{n} \frac{(-1)^{k+1} \binom{n}{k} H_{n+k}}{k} = 3 \sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}} - H_n^{(2)};$$

$$\sum_{k=1}^{n} \left(\frac{(-1)^{k+1} \binom{n}{k} H_{n+k}}{k} - \frac{3}{k^2 \binom{2k}{k}} \right) = (H_n)^2 - H_n^{(2)} = \frac{2}{n!} \left[\begin{matrix} n+1 \\ 3 \end{matrix} \right]$$

$$= 2 \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left(\frac{H_k}{k} - \frac{1}{k^2} \right) \qquad (7)$$

where $\left[\begin{matrix} \cdot \\ \cdot \end{matrix} \right]$ are the unsigned Stirling numbers of the first kind (see [17]). Also from (7), after some algebraic manipulations, we recover the alternating sum identity [14], given by:

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{H_k}{k} = H_n^{(2)}.$$

The integral representation (5) is obtained as follows. Let $m \in \mathbb{R} \backslash \{-1, -2, -3, ..\}$ and expand,

$$\sum_{k=1}^{n} \frac{1}{k^2 \binom{k+n}{n} \binom{k+m}{m}} = \sum_{k=1}^{n} \frac{\Gamma(k)\Gamma(n+1)\Gamma(k)\Gamma(m+1)}{\Gamma(k+n+1)\Gamma(k+m+1)}$$

$$= \sum_{k=1}^{n} B(k, n+1) \ B(k, m+1) = \int_0^1 \int_0^1 \frac{(1-x)^n (1-y)^m}{xy} \sum_{k=1}^{n} (xy)^k \, dydx$$

$$= \int_0^1 \int_0^1 (1-x)^n (1-y)^m \left(\frac{1-(xy)^n}{1-xy} \right) dydx,$$

and putting $m = 0$ we arrive at (5). $\square$

Remark 3. As a matter of interest, in hypergeometric form (5) is written as:

$$\sum_{k=1}^{n} \frac{1}{k^2 \binom{k+n}{n}} = \zeta(2) - H_n^{(2)} - \frac{{}_3F_2 \left[\begin{matrix} 1, n+1, n+1 \\ 2+n, 2+2n \end{matrix} \middle| 1 \right]}{(n+1)^2 \binom{2n+1}{n}}$$

from which we see:

$$\frac{{}_3F_2 \left[\begin{matrix} 1, n+1, n+1 \\ 2+n, 2+2n \end{matrix} \middle| 1 \right]}{(n+1)^2 \binom{2n+1}{n}} = \zeta(2) - 3 \sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}}$$

and

$$\frac{1}{(n+1)} \ {}_3F_2 \left[\begin{matrix} 1, 1, 1 \\ 2, 2+n \end{matrix} \middle| 1 \right] = \zeta(2) - H_n^{(2)} = \sum_{k=n+1}^{\infty} \frac{1}{k^2},$$

where ${}_pF_q \left[\begin{matrix} \cdot, \cdot, \cdot, ... \\ \cdot, \cdot, \cdot, ... \end{matrix} \middle| 1 \right]$ is the hypergeometric function. Also from (5), we obtain the very useful integral identity:

$$\sum_{k=1}^{n} \frac{H_k}{k^2} = -\int_0^1 \int_0^1 \frac{\ln(1-x)(1-(xy)^n)}{1-xy} dydx.$$

To the authors knowledge no identity, in terms of special numbers, exists for $\sum\limits_{k=1}^{n} \frac{H_k}{k^2}$, but we have the known identities:

$$\sum_{k=1}^{n}\left(\frac{H_k}{k^2}+\frac{H_k^{(2)}}{k}\right)=H_nH_n^{(2)}+H_n^{(3)} \text{ and } \sum_{k=1}^{n}\frac{H_k^{(2)}}{k^2}=\frac{H_n^{(4)}+\left(H_n^{(2)}\right)^2}{2}.$$

The next lemma deals with the derivatives of binomial coefficients.

Lemma 4. *Let q and p be positive integers. Also let $Q(p,x)=\binom{p+x}{x}^{-1}$ be an analytic function in x and*

$$Q^{(q)}(p,x)=\frac{d^q}{dx^q}\left(\binom{p+x}{x}^{-1}\right).$$

Then,

$$\begin{aligned}(-1)^{q-1}\,q!\sum_{r=0}^{p}(-1)^r\binom{p}{r}\frac{r}{(r+x)^{q+1}} &= Q^{(q)}(p,x)\\ =\frac{(-1)^q\,q!\;p}{(x+1)^{q+1}}\quad {}_{q+2}F_{q+1}&\left[\left.\begin{array}{c}\overbrace{x+1,\ldots\ldots,x+1}^{(q+1)-terms},1-p\\ \underbrace{x+2,\ldots\ldots,x+2}_{(q+1)-terms}\end{array}\right|1\right].\end{aligned}$$

For $x=0$,

$$\begin{aligned}Q^{(q)}(p,0) &= (-1)^{q-1}\,q!\sum_{r=1}^{p}(-1)^r\binom{p}{r}\frac{1}{r^q}\\ &= (-1)^q\,q!\;p\;{}_{q+2}F_{q+1}\left[\left.\begin{array}{c}\overbrace{1,\ldots\ldots,1}^{(q+1)-terms}\;,1-p\\ \underbrace{2,\ldots\ldots,2}_{(q+1)-terms}\end{array}\right|1\right].\end{aligned}$$

Proof. A proof and the following examples have been given in [12]. $\square$

Now we list some particular cases of Lemma 4:

$$\begin{aligned} Q^{(1)}(p,x) &= -\begin{pmatrix} p+x \\ x \end{pmatrix}^{-1} \sum_{r=1}^{p} \frac{1}{r+x}, \\ Q^{(2)}(p,x) &= \begin{pmatrix} p+x \\ x \end{pmatrix}^{-1} \left[\left(\sum_{r=1}^{p} \frac{1}{r+x} \right)^2 + \sum_{r=1}^{p} \frac{1}{(r+x)^2} \right] \\ &= \begin{pmatrix} p+x \\ x \end{pmatrix}^{-1} \left[\sum_{r=1}^{p} \sum_{s=1}^{r} \frac{2}{(r+x)(s+x)} \right], \end{aligned}$$

$$Q^{(3)}(p,x) = -\begin{pmatrix} p+x \\ x \end{pmatrix}^{-1} \left[\begin{array}{c} \left(\sum_{r=1}^{p} \frac{1}{r+x} \right)^3 + 2 \sum_{r=1}^{p} \frac{1}{(r+x)^3} \\ +3 \sum_{r=1}^{p} \frac{1}{(r+x)^2} \sum_{r=1}^{p} \frac{1}{r+x} \end{array} \right]$$

and

$$Q^{(4)}(p,x) = \begin{pmatrix} p+x \\ x \end{pmatrix}^{-1} \left[\begin{array}{c} 6 \sum_{r=1}^{p} \frac{1}{(r+x)^2} \left(\sum_{r=1}^{p} \frac{1}{r+x} \right)^2 \\ +8 \sum_{r=1}^{p} \frac{1}{(r+x)^3} \sum_{r=1}^{p} \frac{1}{r+x} + 3 \left(\sum_{r=1}^{p} \frac{1}{(r+x)^2} \right)^2 \\ + \left(\sum_{r=1}^{p} \frac{1}{r+x} \right)^4 + 6 \sum_{r=1}^{p} \frac{1}{(r+x)^4} \end{array} \right].$$

In the special case when $x = 0$ we may write:

$$Q^{(1)}(p,0) = -H_p^{(1)}, \tag{8}$$

$$Q^{(2)}(p,0) = \left(H_p^{(1)} \right)^2 + H_p^{(2)}, \tag{9}$$

$$Q^{(3)}(p,0) = -\left(H_p^{(1)} \right)^3 - 3H_p^{(1)} H_p^{(2)} - 2H_p^{(3)} \tag{10}$$

and

$$Q^{(4)}(p,0) = \left(H_p^{(1)} \right)^4 + 6 \left(H_p^{(1)} \right)^2 H_p^{(2)} + 8H_p^{(1)} H_p^{(3)} + 3 \left(H_p^{(2)} \right)^2 + 6H_p^{(4)}. \tag{11}$$

We can generalize (5) as follows.

Theorem 5. *Let m and n be positive integers. Then:*

$$\sum_{k=1}^{n} \frac{1}{k^m \begin{pmatrix} k+n \\ n \end{pmatrix}} = \underbrace{\int_0^1 \cdots \cdot \int_0^1}_{m-fold} \frac{(1-x_1)^n \left[1 - \left(\prod_{j=1}^{m} x_j \right)^n \right]}{1 - \prod_{j=1}^{m} x_j} dx_j \tag{12}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{m+k+1} \binom{n}{k} (H_{n+k} - H_k)}{k^{m-1}} + \sum_{s=1}^{m} \frac{H_n^{(s)}\ Q^{(m-s)}(n,0)}{(m-s)!} \tag{13}$$

where $Q^{(0)}(n,0) = 1$. *For* $m \geq 3$,

$$\sum_{k=1}^{n} \frac{1}{k^m \binom{k+n}{n}} = \frac{1}{n+1}\ {}_{m+1}F_m \left[\left. \begin{array}{c} \overbrace{1, ..., 1}^{(m+1)-terms} \\ \underbrace{2, ..., 2}_{(m-1)-terms}\ , 2+n \end{array} \right| 1 \right] \tag{14}$$

$$- \frac{1}{(n+1)^m \binom{2n+1}{n}}\ {}_{m+1}F_m \left[\left. \begin{array}{c} 1, \overbrace{1+n, ..., 1+n}^{(m)-terms} \\ \underbrace{2+n, ..., 2+n}_{(m-1)-terms}, 2+2n \end{array} \right| 1 \right].$$

Proof. For m and n positive integers we have:

$$\sum_{k=1}^{n} \frac{1}{k^m \binom{k+n}{n}} = \sum_{k=1}^{n} \frac{n!}{k^m \prod_{r=1}^{n} (k+r)} = \sum_{k=1}^{n} \frac{n!}{k^m (k+1)_{n+1}} = \sum_{k=1}^{n} \frac{n!}{k^m} \sum_{r=1}^{n} \frac{A_r}{k+r},$$

where A_r is given in Lemma 2. Now

$$\sum_{k=1}^{n} \frac{1}{k^m \binom{k+n}{n}} = \sum_{r=1}^{n} (-1)^{r+1}\ r \binom{n}{r} \sum_{k=1}^{n} \frac{1}{k^m (k+r)}$$

$$= \sum_{r=1}^{n} (-1)^{r+1}\ r \binom{n}{r} \sum_{k=1}^{n} \left(\frac{(-1)^m}{r^m (k+r)} + \sum_{s=1}^{m} \frac{(-1)^{m-s}}{r^{m+1-s} k^s} \right)$$

$$= \sum_{r=1}^{n} \frac{(-1)^{m+r+1}\ \binom{n}{r} (H_{n+r} - H_r)}{r^{m-1}} + \sum_{s=1}^{m} \sum_{r=1}^{n} (-1)^{r+1}\ r \binom{n}{r} \sum_{k=1}^{n} \frac{(-1)^{m-s}}{r^{m+1-s} k^s}$$

$$= \sum_{r=1}^{n} \frac{(-1)^{m+r+1}\ \binom{n}{r} (H_{n+r} - H_r)}{r^{m-1}} + \sum_{s=1}^{m}\ (-1)^{m-s} H_n^{(s)} \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \frac{1}{r^{m-s}},$$

where the last sum is defined in Lemma 4, and hence equals

$$\sum_{r=1}^{n} \frac{(-1)^{m+r+1} \ \binom{n}{r} \left(H_{n+r} - H_r\right)}{r^{m-1}} + \sum_{s=1}^{m} \frac{H_n^{(s)} Q^{(m-s)}(n,0)}{(m-s)!},$$

arriving at the result (13). The integral (12) is obtained in the same way as was done in Lemma 2. □

Some illustrative cases follow.

Example 6. For $m = 0$ and 1, we have:

$$\sum_{k=1}^{n} \frac{1}{\binom{k+n}{n}} = \frac{1}{n-1}\left(1 - \frac{n+1}{\binom{2n}{n}}\right), \text{ for } n > 1;$$

$$\sum_{k=1}^{n} \frac{1}{k\binom{k+n}{n}} = \frac{1}{n}\left(1 - \frac{1}{\binom{2n}{n}}\right).$$

For $m = 2$, we recover the result (5), and for $m = 4$,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^4\binom{k+n}{n}} &= \sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n}{k}\left(H_{n+k} - H_k\right)}{k^3} \\
&\quad - \left(\frac{1}{6}H_n^4 + \frac{4}{3}H_n H_n^{(3)} - H_n^{(4)} - \frac{1}{2}\left(H_n^{(2)}\right)^2\right) \\
&= \frac{1}{n+1} \ {}_5F_4\left[\begin{array}{c} 1,1,1,1,1 \\ 2,2,2,2+n \end{array} \middle| 1\right] \\
&\quad - \frac{1}{(n+1)^4\binom{2n+1}{n}} \ {}_5F_4\left[\begin{array}{c} 1,1+n,1+n,1+n,1+n \\ 2+n,2+n,2+n,2+2n \end{array} \middle| 1\right],
\end{aligned}$$

and

$$\sum_{k=1}^{n} \frac{1}{k^3\binom{k+2}{2}} = \frac{n\left(13n+27\right)}{8\left(n+1\right)\left(n+2\right)} - \frac{3}{2}H_n^{(2)} + H_n^{(3)}.$$

Remark 7. Some infinite versions of (12) have been evaluated (see [13], for example):

$$\sum_{k\geq 1}\frac{1}{k^4\binom{k+n}{n}} = \zeta(4) - H_n\zeta(3) + \frac{\zeta(2)}{2}\left[(H_n)^2 + H_n^{(2)}\right] + \sum_{r=1}^{n}\frac{(-1)^r}{r^3}\binom{n}{r}H_r,$$

$$\sum_{k\geq 1}\frac{(-1)^k}{k^4\binom{k+n}{n}} = -\frac{7}{8}\zeta(4) + \frac{3}{4}H_n\zeta(3) - \frac{\zeta(2)}{4}\left[(H_n)^2 + H_n^{(2)}\right] + \sum_{r=1}^{n}\frac{(-1)^r}{r^3}\binom{n}{r}\left(H_{\frac{r}{2}} - H_r\right)$$

and

$$\sum_{k\geq 1}\frac{1}{k^2\binom{k+n}{n}} = \zeta(2) - H_n^{(2)}.$$

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