PROOFS OF RUEHR'S IDENTITIES

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Abstract

Naoki Kimura proposed a question concerning the "change of variable formula for definite integration" in the *American Mathematical Monthly* (Problem E2765). In this paper, we give proofs of two new combinatorial identities discovered by Otto G. Ruehr in his solution to Problem E2765.

1. Introduction

In 1979, Naoki Kimura proposed (in the *American Mathematical Monthly* Problem E2765) to establish the following two equations: For all continuous functions f on the interval $1/2 \leq x \leq 3/2$,

$$\int_{-1/2}^{3/2} f(3x^2 - 2x^3)\, dx = 2\int_0^1 f(3x^2 - 2x^3)\, dx$$

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and

$$\int_{-1/2}^{3/2} xf(3x^2 - 2x^3)\,dx = 2\int_0^1 xf(3x^2 - 2x^3)\,dx.$$

Otto G. Ruehr [3] proposed two solution methods, one using a trigonometric substitution and the other that involved the Weierstrass Approximation Theorem. The latter led to two combinatorial identities, namely

$$\sum_{j=0}^{2n}(-4)^j\binom{3n+1}{n+j+1} = \sum_{j=0}^{n}2^j\binom{3n+1}{2n+j+1} \tag{1}$$

and

$$\sum_{j=0}^{2n}(-3)^j\binom{3n-j}{n} = \sum_{j=0}^{n}3^j\binom{3n-j}{2n}. \tag{2}$$

Ruehr concluded that these identities are "apparently new and no easier to prove than the original equations [from which they are obtained]." Note that equation (2) is the corrected version of the identity that appeared in Ruehr's work (where he used 4^j instead of 3^j in the summand of the right-hand side).

The purpose of this paper is to provide proofs for these identities. We will also show that identities (1) and (2) are equivalent. Each sum gives a new formula for the sequence $(1, 6, 39, 258, 1719, 11496, 77052, \ldots)$. This is sequence **A006256** in the On-Line Encyclopedia of Integer Sequences [2]. The paper is organized as follows. In section 2, we state the main results. In section 3, we present a combinatorial proof for Theorem 1 and computer-generated proofs for Theorems 2 and 3 using the Wilf-Zeilberger method.

2. Main Results

Theorem 1. *For integers $n \geq 0$,*

$$\sum_{j=0}^{n}3^j\binom{3n-j}{2n} = \sum_{j=0}^{n}2^j\binom{3n+1}{n-j}.$$

Theorem 2. *For integers $n \geq 0$,*

$$\sum_{j=0}^{2n}(-3)^j\binom{3n-j}{n} = \sum_{j=0}^{n}3^j\binom{3n-j}{2n}.$$

Theorem 3. *For integers $n \geq 0$,*

$$\sum_{j=0}^{2n}(-4)^j\binom{3n+1}{n+j+1} = \sum_{j=0}^{n}2^j\binom{3n+1}{2n+j+1}.$$

From Theorem 1, it follows that Theorems 2 and 3 are equivalent.

3. Proofs of the Main Results

3.1. Combinatorial Proof of Theorem 1

Proof. For non-negative integers n, let

$$A(n) = \sum_{j=0}^{n} 3^j \binom{3n-j}{2n} \text{ and } B(n) = \sum_{j=0}^{n} 2^j \binom{3n+1}{n-j}.$$

Let Γ_1 be the set of words over the alphabet $\{1, 2, 3\}$ and Γ_2 be the set of words over the alphabet $\{1, 2\}$. For a word w, we denote its length by $|w|$. Let S be the set of pairs of words (w_1, w_2) with the following conditions:

1. $w_1 \in \Gamma_1$ and $w_2 \in \Gamma_2$,
2. $|w_1| + |w_2| = 3n$, and
3. $|w_1| + (\#\text{ of 1s in } w_2) = n$.

We show that both $A(n)$ and $B(n)$ count the size of S. First, we select elements for S depending upon the length of the first word w_1 in the ordered pair. From condition 3, we observe that $0 \le |w_1| \le n$. Therefore, for $|w_1| = j$, there are 3^j possible ways to select w_1. In this case, condition 2 implies that $|w_2| = 3n - j$. From condition 3, we infer that the number of 1's in w_2 is $n - j$. Thus, there are $3^j\binom{3n-j}{n-j}$ ways to select ordered pairs (w_1, w_2). Hence there are $\sum_{j=0}^{n} 3^j\binom{3n-j}{n-j}$ ways to select elements for S. Since $\binom{3n-j}{n-j} = \binom{3n-j}{2n}$, this implies $|S| = A(n)$.

Another way to choose elements for S would be to consider the number of 2's and 3's in w_1 of the ordered pair (w_1, w_2). Let j be the number of 2's and 3's in w_1 and let $|w_1| = k$. So, there are $2^j\binom{k}{j}$ ways to select such a w_1. In this case, condition 2 implies that $|w_2| = 3n - k$. From condition 3, we infer that the number of 1's in w_2 is $n - k$. Thus, there are $2^j\binom{k}{j}\binom{3n-k}{n-k}$ ways to select ordered pairs (w_1, w_2). Consequently there are $\sum_{j=0}^{n}\sum_{k=j}^{n} 2^j\binom{k}{j}\binom{3n-k}{n-k}$ ways to choose elements for S. Note that

$$\sum_{j=0}^{n}\sum_{k=j}^{n} 2^j \binom{k}{j}\binom{3n-k}{n-k} = \sum_{j=0}^{n} 2^j \sum_{k=j}^{n} \binom{k}{j}\binom{3n-k}{n-k}.$$

In order to get the result $|S| = B(n)$, it suffices to show

$$\sum_{k=j}^{n} \binom{k}{j}\binom{3n-k}{n-k} = \binom{3n+1}{n-j}. \tag{3}$$

The right-hand side of equation (3) counts the number of $(n-j)$-element subsets of the ordered set $[3n+1] = \{1, 2, \ldots, 3n+1\}$. Since $\binom{3n+1}{n-j} = \binom{3n+1}{2n+1+j}$, another way of counting the number of $(n-j)$-element subsets of the ordered set $[3n+1]$ would be to analyze the $(j+1)$-st element of a $(2n+j+1)$-element subset $\{a_1, a_2, \ldots, a_{2n+j+1}\}$. To this end, we consider the number of possibilities where $a_{j+1} = k+1$. We have to choose j elements from $\{1, 2, \ldots, k\}$ and the remaining $2n$ elements from $\{k+2, \ldots, 3n+1\}$. This can be done in $\binom{k}{j}\binom{3n-k}{2n}$ ways. Since $\binom{3n-k}{2n} = \binom{3n-k}{n-k}$, we have

$$\sum_{k=j}^{n} \binom{k}{j}\binom{3n-k}{n-k} = \binom{3n+1}{n-j}.$$

Therefore $|S| = B(n)$. Consequently, $A(n) = B(n)$ for all integers $n \geq 0$. □

3.2. Computerized Proofs of Theorems 2 and 3

3.2.1. Proof of Theorem 2

Proof. Let $F_A(n,j)$ and $F_C(n,j)$ be the summands of

$$A(n) = \sum_{j=0}^{n} 3^j \binom{3n-j}{2n} \text{ and } C(n) = \sum_{j=0}^{2n} (-3)^j \binom{3n-j}{n},$$

respectively.

Applying the Wilf-Zeilberger (WZ)-algorithm on F_A, we get the WZ-equation:

$$-27F_A(n,j) + 4F_A(n+1,j) = G_A(n,j+1) - G_A(n,j) \tag{4}$$

where

$$G_A(n,j) = \frac{(3n+3-8j-9jn+j^2)3^j}{(n+1)(3n+2-j)}\binom{3n+2-j}{2n+1}.$$

Summing both sides of equation (4) for all values of j, we get

$$-27A(n) + 4A(n+1) = -\frac{3}{3n+2}\binom{3n+2}{n+1}.$$

Applying the WZ-algorithm on F_C, we get the WZ-equation:

$$-27F_C(n,j) + 4F_C(n+1,j) = G_C(n,j+1) - G_C(n,j) \tag{5}$$

where

$$G_C(n,j) = \frac{(6+6n-j-j^2)(-3)^j}{(3n+3-j)(3n+2-j)}\binom{3n+3-j}{n+1}.$$

Summing both sides of equation (5) for all values of j, we get

$$-27C(n) + 4C(n+1) = -\frac{3}{3n+2}\binom{3n+2}{n+1}.$$

Therefore, $A(n)$ and $C(n)$ satisfy the same recurrence equation. Moreover, $A(0) = C(0) = 1$. Hence $A(n) = C(n)$ for all $n \geq 0$. □

Remarks:

1. The recurrence equations (4) and (5) are automatically generated by the Maple package `EKHAD` [1], which is freely available from `http://www.math.rutgers.edu/~zeilberg/`. Alternatively, one can also use the built-in `SumTools` package in Maple® that implements Zeilberger's algorithm.

2. For an exposition of the Wilf-Zeilberger algorithm see, among others, the book, $A = B$ [4], which is devoted to this and other methods.

3.2.2. Proof of Theorem 3

Proof. Let $F_B(n,j)$ and $F_D(n,j)$ be the summands of

$$B(n) = \sum_{j=0}^{n} 2^j \binom{3n+1}{2n+j+1} \text{ and } D(n) = \sum_{j=0}^{2n} (-4)^j \binom{3n+1}{n+j+1},$$

respectively.

Applying the WZ-algorithm on F_B, we get the WZ-equation:

$$-27F_B(n,j) + 4F_B(n+1,j) = G_B(n,j+1) - G_B(n,j) \tag{6}$$

where

$$G_B(n,j) = \frac{(2n+2-7j-9jn-3j^2)2^j}{(n+1)(3n+2)} \binom{3n+3}{2n+j+2}.$$

Summing both sides of equation (6) for all values of j, we get

$$-27B(n) + 4B(n+1) = -\frac{2}{3n+2}\binom{3n+3}{n+1}.$$

Applying the WZ-algorithm on F_D, we get the WZ-equation:

$$-27F_D(n,j) + 4F_D(n+1,j) = G_D(n,j+1) - G_D(n,j) \tag{7}$$

where

$$G_D(n,j) = \frac{(2+2n+j-3j^2)(-4)^j}{(n+1)(3n+2)} \binom{3n+3}{n+1+j}.$$

Summing both sides of equation (7) for all values of j yields

$$-27D(n) + 4D(n+1) = -\frac{2}{3n+2}\binom{3n+3}{n+1}.$$

Thus, $B(n)$ and $D(n)$ satisfy the same recurrence equation and $B(0) = D(0) = 1$. Hence $B(n) = D(n)$ for all $n \geq 0$. □

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