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AN UPPER BOUND FOR RAMANUJAN PRIMES

Anitha Srinivasan
Dept. of Mathematics, Saint Louis University- Madrid Campus, Madrid, Spain

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Abstract

For $n \geq 1$, the n^{th} Ramanujan prime is defined as the least positive integer R_n such that for all $x \geq R_n$, the interval $(\frac{x}{2}, x]$ has at least n primes. Let p_i be the i^{th} prime. Laishram showed that $R_n < p_{3n}$ for all n. Sondow improved this result to $R_n < \frac{41}{47}p_{3n}$ for all n. Our main result states that for each $\epsilon > 0$, there exists an N such that $R_n < p_{[2n(1+\epsilon)]}$ for all $n > N$. This allows us to give upper bounds such as $R_n \leq p_{[2.6n]}$ for all n or $R_n \leq p_{[2.4n]}$ for all $n > 43$.

1. Introduction

For $n \geq 1$, the n^{th} Ramanujan prime is defined as the least positive integer R_n, such that for all $x \geq R_n$ the interval $(\frac{x}{2}, x]$ has at least n primes. Note that by the minimality condition, R_n is prime and the interval $(\frac{R_n}{2}, R_n]$ contains exactly n primes. Let p_n denote the n^{th} prime. Sondow [5] showed that $p_{2n} < R_n < p_{4n}$ for all n and conjectured that $R_n < p_{3n}$ for all n. This conjecture was proved by Laishram [4] and subsequently Sondow, Nicholson and Noe [6] improved Laishram's result by showing that $R_n < \frac{41}{47}p_{3n}$. We show that $R_n \leq p_{[2.6n]}$ for all n, which for large n, is a better bound than the ones mentioned above. We also obtain results that do not hold for all n, such as $R_n \leq p_{[2.4n]}$ for all $n > 43$. Our results are particular cases of the following theorem, where $[x]$ denote the integer part of x.

Theorem 1.1 *For every $\epsilon > 0$, there exists an integer N such that if $\alpha = [2n(1+\epsilon)]$, then $R_n < p_\alpha$ for all $n > N$.*

For $\epsilon = .3$, we have $N = 249$ in the above theorem, so that on verifying the result for the first 249 Ramanujan primes, we obtain that $R_n \leq p_{[2.6n]}$ for all n. When $\epsilon = .2$, similarly we obtain that $R_n \leq p_{[2.4n]}$ for all $n > 43$. In the case of $\epsilon = .5$, we obtain Laishram's result, with only $N = 30$ values to check. The results of Laishram, and Sondow, Nicholson and Noe mentioned above use the following result of Sondow.

Theorem 1.2 (Sondow [5]) *For every $\epsilon > 0$, there exists an integer N such that $R_n < (2+\epsilon)n \log n$ for all $n > N$.*

As a consequence of the above result, Sondow was able to show that $R_n < p_{4n}$. Laishram gave specific values of N for each ϵ in Theorem 1.2, that enabled him to arrive at $R_n < p_{3n}$. The proof of Theorem 1.2 uses the Prime Number Theorem and hence the values of N are large. For the same reason, the explicit values of N in Theorem 1.2 provided by Laishram also tend to be large, making it harder to obtain better upper bounds for R_n. The proof of Theorem 1.1 is based on the simple fact that if $R_n = p_s$, then $p_{s-n} < \frac{p_s}{2}$. This follows because the interval $(\frac{p_s}{2}, p_s]$ contains exactly n primes. Then, using known upper and lower bounds for the i^{th} prime, a decreasing function $F(x)$ is defined (for each fixed n) that satisfies $F(s) > 0$, so that each time $F(x) < 0$ for some x, we have $s < x$, hence obtaining an upper bound for s and thus for R_n.

2. Proof of Main Theorem

Our proof is based on the following lemma that is a direct consequence of the definition of a Ramanujan prime.

Lemma 2.1 *Let $R_n = p_s$ be the n^{th} Ramanujan prime where p_s is the s^{th} prime. Then $p_{s-n} < \frac{p_s}{2}$ for all $n \geq 2$.*

Proof. By the minimality of R_n, the interval $(\frac{p_s}{2}, p_s]$ contains exactly n primes and hence $p_{s-n} < \frac{p_s}{2}$. □

The following lemma gives well-known bounds for the n^{th} prime.

Lemma 2.2 ([3, 2]) *For all $n \geq 2$ we have*

$$n(\log n + \log\log n - 1) < p_n < n(\log n + \log\log n).$$

Proof of Theorem 1.1. Let $R_n = p_s$. We assume that $n, s \geq 2$. Then by Lemmas 2.1 and 2.2, we have $2(s-n)(\log(s-n) + \log\log(s-n) - 1) < s(\log s + \log\log s)$. For $x \geq 2n$, consider the function

$$F(x) = x(\log x + \log\log x) - 2(x-n)(\log(x-n) + \log\log(x-n) - 1).$$

Note that $F(s) > 0$. We have

$$F'(x) = 1 + \frac{1}{\log x} + A - \frac{2}{\log(x-n)} - 2\log\log(x-n),$$

where $A = \log x + \log\log x - 2\log(x-n)$. We will show that $F'(x) < 0$ for $x \geq 2n$. It is easy to verify that $1 + \frac{1}{\log x} - 2\log\log(x-n) < 0$ when $x \geq 2n > 16$. As $x \geq 2n$, we have $\frac{n}{x} < \frac{1}{2}$. Also, $\frac{\log x}{x} < \frac{1}{4}$ and hence $\frac{n}{x} + \sqrt{\frac{\log x}{x}} < 1$. It follows that

$(1-\frac{n}{x})^2 > \frac{\log x}{x}$ and therefore $(x-n)^2 > x\log x$, that is $A < 0$. Therefore $F(x)$ is a decreasing function for $x \geq 2n$.

Now let $\alpha = 2n(1+\epsilon)$. Denoting $\log\log n$ by $\log_2 n$, we have

$$F(\alpha) = -2\epsilon\log n + (2+2\epsilon)\log_2(2n+2n\epsilon) - (2+4\epsilon)\log_2(n+2n\epsilon) + a(\epsilon),$$

where $a(\epsilon)$ is a constant that depends on ϵ. Thus, there exists N such that for $n > N$, we have $F(\alpha) < 0$. As F is a decreasing function and $F(s) > 0$, we have $s \leq 2n(1+\epsilon)$ for $n > N$. Hence, we have $R_n = p_s \leq p_{[2n(1+\epsilon)]}$ for all $n > N$. ☐

Corollary 2.1 *$R_n \leq p_{[2.6n]}$ for all n.*

Proof. Let $R_n = p_s$. We take $\epsilon = .3$. Then $F(2.6n) < 0$ for $n > 249$. Hence $s < 2.6n$ for $n > 249$. The result follows on verification that it holds for the first 249 Ramanujan numbers. ☐

Remark 2.1 Observe that to obtain Laishram's result that $R_n < p_{3n}$, we use $\epsilon = .5$ in Theorem 1.1. It is easy to verify that $F(3n) < 0$ for all $n > 30$. It follows that $s < 3n$, that is $R_n < p_{3n}$ when $n > 30$. We may check that the first thirty Ramanujan numbers satisfy $R_n < p_{3n}$. Theorem 1.1 may be used to give other (better) bounds for R_n that do not hold for all n. For example, for $\epsilon = .2$, we obtain $N = 3400$ and on checking these N values, we obtain the result that $R_n < p_{[2.4n]}$ for all $n > 43$.

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