SOME GENERAL RESULTS AND OPEN QUESTIONS ON PALINTIPLE NUMBERS

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Abstract

Palintiples are natural numbers which are integer multiples of their digit reversals. The most well-known base-10 example is $87912 = 4 \cdot 21978$. Using only elementary number theory we elucidate some general properties of palintiples of an arbitrary base. Palintiples naturally fall into three mutually exclusive and exhaustive classes based upon the structure of their carries. We apply our results to finding all palintiples of the class whose carries exhibit a "shifted-symmetric" structure. Moreover, we find all 5-digit palintiples whose carries are "symmetric" (the same when read forward or backward). We go on to answer some open questions posed in a paper of Pudwell while leaving several questions of our own: is divisibility of the base by the multiplier plus 1 enough to determine all symmetric palintiples? Which bases (among these is base-10) only allow for the existence of symmetric palintiples? Are there infinitely many such bases? Finally, we reveal some connections between palintiples and complex roots of unity.

1. Introduction

A *palintiple* (short for palindromic multiple) is a natural number with the property of being an integral multiple of the number represented by the reversal of its base-b expansion. The most well-known examples of base-10 palintiples include 87912 and 98901 since $87912 = 4 \cdot 21978$ and $98901 = 9 \cdot 10989$. More examples which include bases other than 10 may be found in Table 1.

At first glance it may seem that such numbers are merely curiosities that only make for cute puzzle problems. Such was the belief of G. H. Hardy who, in his classic essay *A Mathematician's Apology* [1], cited the fact that "8712 and 9801 are the only four-figure numbers which are integral multiples of their 'reversals' " as an example of a theorem that is not "serious." Furthermore, "[this fact is] very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals much to a mathematician" and is "not capable of any significant general-

ization." Sutcliffe [8], Pudwell [7], and Young [9] demonstrate otherwise; in spite of Hardy's comments the palintiple problem generalizes quite naturally. The title of Pudwell's paper, *Digit Reversal Without Apology*, is a clever acknowledgement of this fact (among other clever acknowledgements in Pudwell's paper).

Sutcliffe is the first to give a serious mathematical treatment of the palintiple problem. After determining all 2-digit palintiples, Sutcliffe shows that the existence of a 2-digit palintiple guarantees the existence of a 3-digit palintiple by constructing it directly from the 2-digit example. However, Sutcliffe leaves the question open as to whether or not a 3-digit palintiple in hand guarantees the existence of a 2-digit palintiple. The work of Kaczynski [3] fills this gap by proving that the converse to Sutcliffe's result does indeed hold. Moreover, Kaczynski also shows that the 2-digit palintiple may be constructed from the 3-digit palintiple by simply removing the middle digit.

The tidy correspondence between 2 and 3-digit palintiples led Pudwell to ask whether or not such a correspondence might exist between 4 and 5-digit palintiples. Under the condition of Kaczynski's paper which assures that middle-digit truncation always works in the 3-digit case, Pudwell demonstrates with several counterexamples that there are 5-digit palintiples for which there is no corresponding 4-digit palintiple. However, she does provide a partial converse by showing that the result does extend to a large family of 5-digit palintiples (which includes examples noted by Klosinski and Smolarski [4]) and in doing so shows that for every base larger than 2, there is a 5-digit palintiple for which middle-digit truncation results in a 4-digit palintiple.

The work of Sutcliffe, Kaczynski, and Pudwell establishes results for palintiples of five digits or less. Young [9] takes a different approach by showing how to construct all palintiples of a particular base and multiplier within a graph-theoretical framework. Her methods are used to find all base-10 palintiples whose multiplier is 4 (see the above example).

The approach taken here is somewhat different than the above by considering palintiples having a fixed, but arbitrary, number of digits. As with Young [9], the methods outlined here pay particular attention to the carries which, as will be seen, play as critical a role as the digits themselves. Relationships between the carries naturally partition all palintiples into three classes. The first of these classes includes the most well-known examples already listed above and the family of palintiples described in Pudwell's *Theorem 1*. Using the results developed here we will find all 5-digit palintiples belonging to the first class as well as find all palintiples belonging to the second.

We also apply these techniques to answering some open questions posed by Pudwell. Among these is a necessary and sufficient condition that tells us exactly when removing the middle digit of a palintiple with an odd number of digits results in yet another palintiple. Finally, after posing a few open questions of our own, it will

be shown that palintiples are related to complex roots of unity.

We note that the nomenclature used by this article differs from that of Sutcliffe, Pudwell, and Young who never actually use the term "palintiple." However, this term is adopted as a precise, convenient, and descriptive label for natural numbers which fit the rather convoluted description given above. The term seems to have been coined in an online article by Hoey [2] in which he finds all base-10 palintiples using finite state machines.

Before proceeding it will be necessary to define some additional terminology and notation that will be used throughout this article. The examples already given motivate the following definition.

Definition. Let b be a natural number greater than 1 and suppose $0 \leq d_j < b$ for all $0 \leq j \leq k$. The natural number $\sum_{j=0}^{k} d_j b^j$ is called an (n, b)-*palintiple* provided

$$\sum_{j=0}^{k} d_j b^j = n \sum_{j=0}^{k} d_{k-j} b^j$$

for some natural number n.

Using the language established above, 87912 is a (4,10)-palintiple and 98901 is a (9,10)-palintiple.

As with other works cited, this article does not consider examples such as $1010 = 10 \cdot 0101$ since the leading digit of the reversal is zero and does not qualify as a valid base-b representation. Consequently, it is assumed that $n < b$. Additionally, every base-b palindrome is a $(1, b)$-palintiple. Such trivial examples will be ignored so that $n > 1$. Furthermore, $b = 2$ implies that $n = 1$ (in which case our palintiple is merely a binary palindrome). Therefore, an additional restriction $b \neq 2$ is imposed. Thus, hereafter we assume that n and b are natural numbers such that $1 < n < b$ and $b > 2$.

For notational convenience the convention used by Sutcliffe, Kaczynski, and Pudwell is observed so that $(d_k, d_{k-1}, \ldots, d_0)_b$ represents the natural number $\sum_{j=0}^{k} d_j b^j$.

2. Some General Results

We begin our investigation by considering single-digit multiplication in general. Letting p_j denote the jth digit of the product, c_j the jth carry, and q_j the jth digit of the number being multiplied by n, the iterative algorithm for single-digit multiplication is

$$c_0 = 0$$
$$p_j = \lambda(nq_j + c_j)$$
$$c_{j+1} = (nq_j + c_j - \lambda(nq_j + c_j)) \div b$$

where λ is a function giving the least non-negative residue modulo b. $(p_k, p_{k-1}, \ldots, p_0)_b$ is a $k+1$-digit number so that $c_{k+1} = 0$. Since $d_j = p_j = \lambda(nd_{k-j}+c_j)$ and $q_j = d_{k-j}$ for any (n,b)-palintiple $(d_k, d_{k-1}, \ldots, d_0)_b$, we have

Theorem 1. *Let $(d_k, d_{k-1}, \ldots d_0)_b$ be an (n,b)-palintiple and let c_j be the jth carry. Then*

$$bc_{j+1} - c_j = nd_{k-j} - d_j$$

for all $0 \leq j \leq k$.

Manipulation of these equations gives the following important corollary which allows us to state the value of each digit in terms of the carries.

Corollary 2. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n,b)-palintiple and let c_j be the jth carry. Then*

$$d_j = \frac{nbc_{k-j+1} - nc_{k-j} + bc_{j+1} - c_j}{n^2 - 1}$$

for all $0 \leq j \leq k$.

The above also implies that the carries must satisfy the following system of congruences for all $0 \leq j \leq k$:

$$nbc_{k-j+1} + bc_{j+1} \equiv nc_{k-j} + c_j \mod (n^2 - 1). \tag{1}$$

Of course, not every solution to the above system of congruences will yield the digits of a palintiple, but since the carries necessarily satisfy (1), every possible $(k+1)$-digit (n,b)-palintiple may be found by finding all solutions to (1). The next theorem narrows down the possibilities for solutions to (1) that yield palintiples.

Theorem 3. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n,b)-palintiple and let c_j be the jth carry. Then $c_j \leq n - 1$ for all $0 \leq j \leq k$.*

Proof. The proof will proceed by induction. First, $c_0 = 0 \leq n - 1$. Now suppose that $c_j \leq n - 1$. For a contradiction, suppose that $c_{j+1} \geq n$. Then Theorem 1 implies $bc_{j+1} - c_j + d_j = nd_{k-j}$. By our inductive hypothesis and Theorem 1 we have $bn - (n-1) = (b-1)n + 1 \leq nd_{k-j}$. Therefore, $d_{k-j} > b - 1$ which is a contradiction. $\square$

Since $c_j \leq n - 1$ for all $0 \leq j \leq k$, and since $c_1 > 0$ (otherwise, by Corollary 2, we would have $d_k \leq 0$), $(c_1, c_2, \ldots, c_k)_b$ is the base-b representation of the number $\sum_{j=0}^{k-1} c_{k-j} b^j$. From this, the number whose digits are reversed when multiplied by n (yielding a palintiple) may be stated in terms of the base-b representation determined by the carries.

Theorem 4. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n,b)-palintiple with carries c_k, c_{k-1},..., c_0. Then*

$$(d_0, d_1, \ldots, d_k)_b = \frac{b^2 - 1}{n^2 - 1}(c_1, c_2, \ldots, c_k)_b.$$

Proof. Using Corollary 2, a straightforward calculation reveals that $\sum_{j=0}^{k} d_{k-j} b^j = \sum_{j=0}^{k} \frac{nbc_{j+1} - nc_j + bc_{k-j+1} - c_{k-j}}{n^2 - 1} b^j = \frac{b^2 - 1}{n^2 - 1} \sum_{j=0}^{k-1} c_{k-j} b^j.$ □

If it is palintiples of a particular base we seek, there may be several cases of n which we may exclude from our search. The next theorem helps us to eliminate such cases.

Theorem 5. *If $\frac{b}{\gcd(b,n)} < n + 1$, then no (n,b)-palintiples exist.*

Proof. The theorem shall be established by the contrapositive. Suppose that an (n,b)-palintiple $(d_k, d_{k-1}, \ldots d_0)_b$ exists where c_j is the jth carry. For $j = 0$, Corollary 2 gives $(n^2 - 1)d_0 = bc_1 - nc_k$. Therefore, $\gcd(b,n)$ divides $(n^2 - 1)d_0$. But since n and $n^2 - 1$ are relatively prime, $\gcd(b,n)$ divides d_0 so that $\gcd(b,n) \le d_0$. By Corollary 2 and Theorem 3, we have that $d_0 = \frac{bc_1 - nc_k}{n^2 - 1} \le \frac{bc_1}{n^2 - 1} \le \frac{b}{n+1}$ so that $\gcd(b,n) \le \frac{b}{n+1}$. □

Considering base-10 palintiples as an example, we see that there are no $(5,10)$, $(6,10)$, or $(8,10)$-palintiples.

On the other hand, are there conditions under which the existence of (n,b)-palintiples is assured? The following gives one such condition.

Theorem 6. *Suppose $n + 1$ divides b. Then there exists an (n,b)-palintiple. Furthermore, for every palintiple such that $(n+1)|b$ we have $c_j = c_{k-j}$ for all $0 \le j \le k$ where c_j is the jth carry.*

Proof. Let $n + 1$ divide b with quotient q. The base-b digits defined by $d_k = nq$, $d_{k-1} = nq - 1$, $d_j = b - 1$ for all $2 \le j \le k - 2$, $d_1 = q - 1$, and $d_0 = q$ are those of an (n,b)-palintiple since $\sum_{j=0}^{k} d_j b^j = n \sum_{j=0}^{k} d_{k-j} b^j$.

Since $n+1$ divides b, Theorem 1 implies that $c_j \equiv d_{k-j} + d_j \equiv c_{k-j} \mod (n+1)$ for all $0 \le j \le k$. Since Theorem 3 guarantees that c_j and c_{k-j} are less than $n + 1$, we have $c_j = c_{k-j}$. □

The most well-known examples, namely the $(4,10)$ and $(9,10)$-palintiples 87912 and 98901, fall into a class of palintiples for which the order of the carries is the same both forward and backward (since $n + 1$ divides b in both cases). It is precisely this structure which motivates the following definition which partitions all palintiples into three mutually exclusive and exhaustive classes based upon the pattern, or lack thereof, among the carries.

$(d_k, d_{k-1}, \ldots, d_0)_b$	n	$(c_k, c_{k-1}, \ldots, c_0)$	**Class**
$(8, 7, 9, 1, 2)_{10}$	4	$(0, 3, 3, 3, 0)$	symmetric
$(9, 8, 9, 0, 1)_{10}$	9	$(0, 8, 8, 8, 0)$	symmetric
$(5, 4, 0, 1, 5, 4, 0, 1)_6$	5	$(0, 4, 4, 0, 0, 4, 4, 0)$	symmetric
$(26, 2, 0, 26, 2)_{29}$	9	$(8, 0, 0, 8, 0)$	shifted-symmetric
$(26, 15, 14, 27, 2)_{29}$	9	$(8, 4, 4, 8, 0)$	shifted-symmetric
$(26, 28, 28, 28, 2)_{29}$	9	$(8, 8, 8, 8, 0)$	shifted-symmetric
$(18, 13, 29, 15, 20, 4)_{34}$	4	$(2, 1, 3, 1, 2, 0)$	shifted-symmetric
$(11, 9, 1, 4, 1)_{14}$	9	$(2, 1, 6, 7, 0)$	asymmetric
$(16, 13, 3, 8, 2)_{22}$	7	$(2, 1, 4, 5, 0)$	asymmetric
$(8, 9, 10, 2, 1)_{11}$	7	$(1, 6, 6, 5, 0)$	asymmetric
$(34, 1, 30, 24, 2)_{40}$	13	$(8, 9, 0, 11, 0)$	asymmetric

Table 1: Examples of palintiples sorted by type.

Definition. Let $p = (d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b)-palintiple with carries c_k, $c_{k-1}, \ldots, c_0$. We say that p is *symmetric* if $c_j = c_{k-j}$ for all $0 \le j \le k$ and that p is *shifted-symmetric* if $c_j = c_{k-j+1}$ for all $0 \le j \le k$. A palintiple that is neither symmetric nor shifted-symmetric is called an *asymmetric* palintiple.

Table 1 gives several examples of each palintiple type for a variety of bases. (All examples used in the body of the text are also included in this table for convenient reference.) The family of (n, b)-palintiples for which $n+1$ divides b found by Pudwell [7] given by $(\frac{nb}{n+1}, \frac{nb}{n+1} - 1, b - 1, \frac{b}{n+1} - 1, \frac{b}{n+1})_b$ with carries $(c_4, c_3, c_2, c_1, c_0) = (0, n-1, n-1, n-1, 0)$ provide even more examples of symmetric palintiples.

By Theorem 6, every (n, b)-palintiple for which $n + 1$ divides b is symmetric. It is natural to ask whether or not the converse is true: does the existence of a symmetric (n, b)-palintiple guarantee that $n + 1$ divides b? Computer generated evidence suggests that the answer might be yes as a single counterexample has not yet been found. We leave this as an open question. The next theorem does, however, establish a partial converse to Theorem 6.

Theorem 7. *If a symmetric (n, b)-palintiple exists for which b and $n - 1$ are relatively prime, then $n + 1$ divides b.*

Proof. Let $p = (d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b)-palintiple with carries c_k, $c_{k-1}, \ldots, c_0$. Since p is symmetric, $c_k = c_0 = 0$. Thus, by Corollary 2, $(n^2 - 1)d_0 = bc_1$. But since b and $n - 1$ are relatively prime, it must be that $n - 1$ divides c_1. But Theorem 3 implies that $c_1 \le n - 1$ so that $c_1 = n - 1$. The result follows. $\square$

As promised in the introduction, we now give a characterization of symmetric palintiples which sheds more light upon how $n + 1$ and b are related.

Theorem 8. *Let* $p = (d_k, d_{k-1}, \ldots, d_0)_b$ *be an* (n, b)-*palintiple with carries* c_k, $c_{k-1}, \ldots, c_0$. *Then the following are equivalent:*

(1) *p is symmetric,*

(2) *$bc_j \equiv 0 \mod (n+1)$ for all $0 \le j \le k$,*

(3) *$(n+1)d_j \equiv (n+1)d_{k-j} \mod b$ for all $0 \le j \le k$.*

Proof. We will show (1) $\iff$ (2) and then (1) $\iff$ (3).

Let p be symmetric. Then the congruence given by (1) implies that $nbc_{j+1} + bc_{k-j+1} \equiv (n+1)c_j \mod (n^2 - 1)$ so that $bc_{j+1} \equiv bc_{k-j+1} \mod (n+1)$. Now $c_0 = 0$ and $bc_1 \equiv 0 \mod (n+1)$. A simple induction argument establishes the desired conclusion. Suppose, then, that $bc_j \equiv 0 \mod (n+1)$. It then follows by (1) that $c_j - c_{k-j} \equiv bc_{j+1} - bc_{k-j+1} \equiv 0 \mod (n+1)$. Since $c_j \le n - 1$ for all $0 \le j \le k$, it must be that $c_j = c_{k-j}$.

For the second equivalence, suppose p is symmetric. Then Theorem 1 implies both $bc_{j+1} - c_j = nd_{k-j} - d_j$ and $bc_{k-j+1} - c_j = nd_j - d_{k-j}$. Thus $b(c_{j+1} - c_{k-j+1}) = (n+1)(d_{k-j} - d_j)$ from which the desired conclusion follows. Suppose, then, that $(n+1)d_j \equiv (n+1)d_{k-j} \mod b$. Another use of Theorem 1 shows that $b(c_{j+1} - c_{k-j+1}) - c_j + c_{k-j} = (n+1)(d_{k-j} - d_j)$. Reducing modulo b we have that $c_j \equiv c_{k-j} \mod b$. $\square$

The following theorem determines all shifted-symmetric palintiples.

Theorem 9. *Let* $p = (d_k, d_{k-1}, \ldots, d_0)_b$ *be a shifted-symmetric* (n, b)-*palintiple with carries* c_k, $c_{k-1}, \ldots, c_0$. *Then $(b-n)c_j \equiv 0 \mod (n^2 - 1)$ for all $0 \le j \le k$. Furthermore, for all $0 \le j \le k$ suppose $\hat{c}_j$ is a solution to $(b-n)\hat{c}_j \equiv 0 \mod (n^2-1)$ where $\hat{c}_j = \hat{c}_{k-j+1}$, $\hat{c}_k = \hat{c}_1 \ne 0$, $\hat{c}_0 = 0$, and $0 \le \hat{c}_j \le n - 1$. Then*

$$n\frac{b^2 - 1}{n^2 - 1}(\hat{c}_k, \hat{c}_{k-1}, \ldots, \hat{c}_1)_b$$

is a shifted-symmetric (n, b)-*palintiple.*

Proof. Clearly $(b-n)c_0 \equiv 0 \mod (n^2 - 1)$. Suppose, then, that $(b-n)c_j \equiv 0 \mod (n^2 - 1)$. Multiplying by n gives $(nb-1)c_j \equiv 0 \mod (n^2 - 1)$. Our hypothesis and Corollary 2 imply that

$$d_j = \frac{(b-n)c_{j+1} + (nb-1)c_j}{n^2 - 1}.$$

Hence, $(b-n)c_{j+1} \equiv -(nb-1)c_j \equiv 0 \mod (n^2 - 1)$.

Now suppose that $(b-n)\hat{c}_j \equiv 0 \mod (n^2 - 1)$ for each $0 \le j \le k$. Defining d_j as

$$d_j = \frac{(b-n)\hat{c}_{j+1} + (nb-1)\hat{c}_j}{n^2 - 1},$$

the above congruence assures that d_j is an integer. Since $\hat{c}_j \leq n - 1$, it follows that $d_j \leq b - 1$ so that d_j is a base-b digit. The condition $\hat{c}_k = \hat{c}_1 \neq 0$ ensures that d_0 and d_k are nonzero. From here it is a simple exercise to show that $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b)-palintiple. A straightforward induction argument establishes that the carries of this palintiple are indeed $\hat{c}_k$, $\hat{c}_{k-1}, \ldots, \hat{c}_0$. Since $\hat{c}_j = \hat{c}_{k-j+1}$, Theorem 4 implies that

$$(d_k, d_{k-1}, \ldots, d_0)_b = n(d_0, d_1, \ldots, d_k)_b = n\frac{b^2 - 1}{n^2 - 1}(\hat{c}_k, \hat{c}_{k-1}, \ldots, \hat{c}_1)_b$$

is a shifted-symmetric palintiple and the proof is complete.   □

It is interesting to note that all of the examples mentioned by Pudwell [7] of 5-digit (n, b)-palintiples for which removing the middle digit does not yield a 4-digit (n, b)-palintiple are asymmetric. One such example is the $(7,22)$-palintiple $(16, 13, 3, 8, 2)_{22}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (2, 1, 4, 5, 0)$.

It is then tempting to ask whether or not palintiple symmetry extends Kaczynski's result (that is, removing the middle digit of any 3-digit palintiple always yields a 2-digit palintple). Although it is true that removing the middle digit of a 5-digit symmetric or shifted-symmetric (n, b)-palintiple results in a 4-digit (n, b)-palintiple (the arguments presented in the next section establish that this is indeed the case for symmetric palintiples), it turns out that palintiples for which middle-digit truncation yields another palintiple are not necessarily symmetric or shifted-symmetric. Consider the case of the $(7,11)$-palintiple $(8, 9, 10, 2, 1)_{11}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (1, 6, 6, 5, 0)$. This palintiple is asymmetric, but $(8, 9, 2, 1)_{11}$ is also a $(7,11)$-palintiple. We must then ask: is there a condition which tells us exactly when middle-digit truncation yields another palintiple? The next theorem provides such a condition for any palintiple having an odd number of digits. It also addresses Pudwell's question [7] if there are analogous results to her *Theorem 1 and Theorem 2* for palintiples having more than five digits. The condition is stated in terms of the carries.

Theorem 10. *Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b)-palintiple with an odd number of digits and that $c_k, c_{k-1}, \ldots, c_0$ are its carries. If d_M is the middle digit, then the number obtained by removing the middle digit, $(d_k, d_{k-1}, \ldots, d_{M+1}, d_{M-1}, \ldots, d_0)_b$, is also an (n, b)-palintiple if and only if $c_{M+1} = c_M$.*

Proof. For the truncated number to be an (n, b)-palintiple, it must be that $(d_k, d_{k-1}, \ldots, d_{M+1}, d_{M-1}, \ldots, d_0)_b - n(d_0, d_1, \ldots, d_{M-1}, d_{M+1}, \ldots, d_k)_b = 0$. Theorem 1 and a routine calculation involving summation signs show that

$$\sum_{j=0}^{M-1} d_j b^j + \sum_{j=M+1}^{k} d_j b^{j-1} - n\left(\sum_{j=0}^{M-1} d_{k-j} b^j + \sum_{j=M+1}^{k} d_{k-j} b^{j-1}\right) = b^M(c_{M+1} - c_M).$$

□

In the above, $M = \frac{k}{2}$. In this case it is clear that removing the middle digit of a shifted-symmetric (n, b)-palintiple results in another (n, b)-palintiple since $c_M = c_{k-M+1} = c_{M+1}$. However, it is not clear whether or not truncating symmetric palintiples always results in another palintiple. Empirical evidence suggests that $c_M = c_{M+1}$ in every case.

3. Palintiples of Five Digits or Less

Theorem 9 determines all shifted-symmetric palintiples including those having five digits or less. We now find all symmetric palintiples of five digits or less.

Clearly no 2-digit palintiple can be symmetric since this would require all the carries to be zero. A symmetric 3-digit (n, b)-palintiple would require a non-zero middle carry c_1 with $c_2 = c_0 = 0$. Corollary 2 then implies that the middle digit is negative which is a contradiction.

As for 4-digit symmetric palintiples, if $(d_3, d_2, d_1, d_0)_b$ is a symmetric (n, b)-palintiple with carries c_3, c_2, c_1, and $c_0 = 0$, then $c_3 = c_0 = 0$ and $c_2 = c_1 = c$. Equation (1) yields $(n + 1)c \equiv bc \equiv 0 \mod (n^2 - 1)$ from which we conclude $c \equiv 0 \mod (n - 1)$. If $c = 0$, then, by Corollary 2, all the digits equal zero. Hence, $c = n - 1$ from which it follows that $n + 1$ divides b and $(d_3, d_2, d_1, d_0)_b = (\frac{nb}{n+1}, \frac{nb}{n+1} - 1, \frac{b}{n+1} - 1, \frac{b}{n+1})_b$.

For the 5-digit case, let $(d_4, d_3, d_2, d_1, d_0)_b$ be a symmetric (n, b)-palintiple with carries c_4, c_3, c_2, c_1, and $c_0 = 0$. Then $c_4 = c_0 = 0$, and $c_3 = c_1$. Equation (1) implies that $bc_1 \equiv 0 \mod (n^2 - 1)$, $bc_2 \equiv (n+1)c_1 \mod (n^2 - 1)$, and $(n+1)bc_1 \equiv (n+1)c_2 \mod (n^2 - 1)$. The first and third congruence imply that $(n + 1)c_2 \equiv 0 \mod (n^2 - 1)$ from which we deduce that $c_2 \equiv 0 \mod (n - 1)$. Then either $c_2 = 0$ or $c_2 = n - 1$. But if $c_2 = 0$, then, by Corollary 2, it could only be that $c_1 = 0$ since otherwise d_1 and d_3 would be negative. But if c_2 and c_1 equal zero, then all the digits equal zero. Therefore, it must be that $c_2 = n - 1$. Hence, by the second of the congruences listed above, $(n+1)c_1 \equiv (n-1)b \mod (n^2 - 1)$. It follows that $2c_1 \equiv 0 \mod (n - 1)$. If n is even, then $c_1 = n - 1$. If n is odd, then either $c_1 = \frac{n-1}{2}$ or $c_1 = n - 1$. Since Corollary 2 guarantees that $d_0 = \frac{bc_1}{n^2-1}$, it follows in any of these cases that $n + 1$ divides b. However, this implies, by Corollary 2, that if $c_1 = \frac{n-1}{2}$, then d_1 is not an integer which is a contradiction. Hence, the only possibility is that $c_1 = n - 1$. Thus, the unique 5-digit symmetric (n, b)-palintiple is given by $(d_4, d_3, d_2, d_1, d_0)_b = (\frac{nb}{n+1}, \frac{nb}{n+1} - 1, b - 1, \frac{b}{n+1} - 1, \frac{b}{n+1})_b$.

It has already been seen that removing the middle digit of any shifted-symmetric (n, b)-palintiple with an odd number of digits results in another shifted-symmetric (n, b)-palintiple. Hence, as claimed previously, removing the middle digit of any 5-digit symmetric or shifted-symmetric (n, b)-palintiple results in yet another (n, b)-palintiple.

We now address a few of the questions raised by Pudwell [7] for 5-digit palintiples. Suppose $b + 1$ is prime. Corollary 2 implies that $d_0 - d_1 + d_2 - d_3 + d_4 = \frac{(b+1)(c_1 - c_2 + c_3 - c_4)}{n-1}$. Thus, the value of f mentioned in Pudwell's paper may then be stated in terms of the carries: $f = \frac{c_1 - c_2 + c_3 - c_4}{n-1}$. It may come as no surprise that $f = 0$ for shifted-symmetric palintiples and $f = 1$ for symmetric palintiples. In fact, it is not difficult to show that the family of palintiples characterized by Pudwell's *Theorem 1* are all symmetric (in each case the multiplier plus 1 divides the base). Pudwell asked if there were palintiples outside of this family for which $f = 1$. The (13,40)-palintiple $(34, 1, 30, 24, 2)_{40}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (8, 9, 0, 11, 0)$ serves as an example since it is clearly not symmetric.

With regard to the open question as to whether or not a counterexample exists to Pudwell's *Question 1* (if $b+1$ is prime, when does digit truncation fail to produce another palintiple?) for which $f \neq 0$, we may look no further than the example just provided.

Finally, we show that there are no 5-digit palintiples for which $f = 2$. Since $c_j \leq n - 1$ for all $0 \leq j \leq 4$, the only way f could equal 2 is if $c_1 = c_3 = n - 1$ and $c_2 = c_4 = 0$. But this would mean that $d_2 = b$ by Corollary 2. Hence, the case $f = 2$ is impossible as computer-generated evidence has suggested.

4. Some Open Questions

In addition to the question already posed (if symmetric implies $n + 1$ divides b), there are still many unanswered questions.

Without exception, the carries of every symmetric palintiple we have observed, no matter the base, no matter the number of digits, always equal either $n - 1$ or 0 (as seen above for the four and five-digit case). If it could be shown that this indeed holds in general, then all symmetric palintiples would be completely determined (as well as guarantee that $n + 1$ divides b).

It is also unknown whether or not more than one type of palintiple can exist for a particular choice of n and b. So far, the evidence suggests that this cannot happen. If it could be shown that b being divisible by $n + 1$ is equivalent to being symmetric, then it would follow that symmetric and shifted-symmetric palintiples cannot simultaneously exist for the same n and b since $(n+1)|b$ implies symmetric.

Hoey [2] stated that $b = 10$ is a particularly "boring" base for palintiples. However, we disagree. Essentially, Hoey argued that every base-10 palintiple is symmetric, that is, that all base-10 palintples have a very nice structure. Additionally, it is easily seen that every base-3 palintiple is symmetric since $n = 2$ is the only suitable multiplier ($n+1$ divides b in every case). Every base-4 palintiple is symmetric since Theorem 5 eliminates the possibility that $n = 2$. Similar arguments establish that all base-6 palintiples are symmetric.

A cascade of questions regarding bases immediately come to mind:

(1) What other bases only allow for symmetric palintiples? Are there infinitely many such bases?

Since for every divisor $n + 1$ of b there is a symmetric (n, b)-palintiple, every base has at least one symmetric palintiple. However, we must then ask

(2) What bases exclude the possibility of asymmetric palintiples?

(3) If there are infinitely many such "symmetric bases," are there any other interesting properties shared by these numbers? Do the integer sequences determined by symmetric bases have any interesting properties?

5. Palintiples and Complex Roots of Unity

We shall conclude this article with some connections between palintiples and complex roots of unity.

Definition. Suppose $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b)-palintiple. We define the (n, b)-*palinomial induced by* p to be the polynomial $\mathrm{Pal}(x) = \sum_{j=0}^{k} (d_j - nd_{k-j}) x^j$.

Clearly the above definition was constructed so that $\mathrm{Pal}(b) = 0$. We now consider other roots of $\mathrm{Pal}(x)$. The following theorem sheds even more light upon the relationship between the digits and the carries.

Theorem 11. *Suppose* $(d_k, d_{k-1}, \ldots, d_0)_b$ *is an* (n, b)-*palintiple with carries* c_k, $c_{k-1}, \ldots, c_0$. *Then* $\mathrm{Pal}(x) = (x - b) \sum_{j=1}^{k} c_j x^{j-1}$.

Proof. The result follows directly from Theorem 1.   □

Thus, finding other roots of palinomials amounts to finding roots of the polynomial having the carries as its coefficients.

Corollary 12. *The only positive real root of an* (n, b)-*palinomial is* b.

It follows from Theorem 11 that roots of symmetric and shifted-symmetric palinomials (palinomials induced by symmetric and shifted-symmetric palintiples) besides b are roots of palindrome polynomials. If it is indeed true that the carries of a symmetric palintiple are either 0 or $n - 1$ as conjectured, then all the roots of its palinomial, besides b, are roots of palindromic polynomials having coefficients of 0 or 1. The work of Konvalina and Matache [6] would then imply that every symmetric palinomial has at least one root on the unit circle in the complex plane.

We now consider palinomials induced by the family of symmetric palintiples encountered in the proof of Theorem 6. Suppose $n + 1$ divides b with quotient q. Then substituting $c_k = c_0 = 0$ and $c_j = n - 1$ for all $0 < j < k$ into the equation of Corollary 2, we see that the digits are precisely those of the symmetric palintiple considered in Theorem 6. It follows from Theorem 11 that the palinomial induced by this palintiple is $\mathrm{Pal}(x) = (n-1)(x-b)\sum_{j=1}^{k-1} x^{j-1} = (n-1)(x-b)\frac{x^{k-1}-1}{x-1}$. Since this argument is valid for any $k \geq 3$, we have

Theorem 13. *Every complex root of unity is the root of some palinomial.*

Lifting the restrictions set forth in the introduction (that is, allowing examples such as $10 \cdot 01010101 = 10101010$), we leave the reader with an image of the collection of roots near the unit circle of all palinomials induced by all palintiples up to base-10 having up to 8 digits (generated in GNU Octave [5]). Note the concentration of roots on the unit circle.

Roots of palinomials induced by all palintiples up to base-10 having up to 8 digits
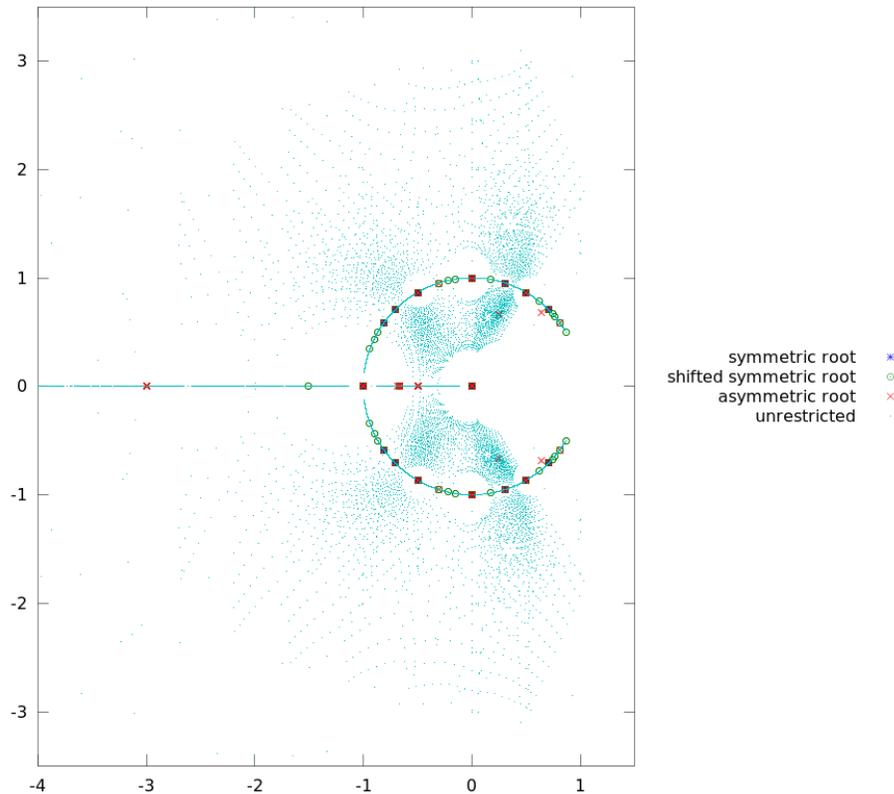
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