IDENTITIES BETWEEN POLYNOMIALS RELATED TO STIRLING AND HARMONIC NUMBERS

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Abstract

We consider two types of polynomials $\mathbf{F}_n(x) = \sum_{\nu=1}^n \nu!\, \mathbf{S}_2(n,\nu)\, x^\nu$ and $\widehat{\mathbf{F}}_n(x) = \sum_{\nu=1}^n \nu!\, \mathbf{S}_2(n,\nu)\mathbf{H}_\nu\, x^\nu$, where $\mathbf{S}_2(n,\nu)$ are the Stirling numbers of the second kind and $\mathbf{H}_\nu$ are the harmonic numbers. We show some properties and relations between these polynomials. Especially, the identity $\widehat{\mathbf{F}}_n(-\frac{1}{2}) = -(n-1)/2 \cdot \mathbf{F}_{n-1}(-\frac{1}{2})$ is established for even n, where the values are connected with Genocchi numbers. For odd n the value of $\widehat{\mathbf{F}}_n(-\frac{1}{2})$ is given by a convolution of these numbers. Subsequently, we discuss some of these convolutions, which are connected with Miki type convolutions of Bernoulli and Genocchi numbers, and derive some 2-adic valuations of them.

1. Introduction

The purpose of this paper is to show some relations between the polynomials

$$\mathbf{F}_n(x) = \sum_{\nu=1}^{n} \left\langle {n \atop \nu} \right\rangle x^\nu, \quad \widehat{\mathbf{F}}_n(x) = \sum_{\nu=1}^{n} \left\langle {n \atop \nu} \right\rangle \mathbf{H}_\nu x^\nu \quad (n \geq 1).$$

These polynomials are composed of harmonic numbers

$$\mathbf{H}_n = \sum_{\nu=1}^{n} \frac{1}{\nu}$$

and Stirling numbers of the second kind $\mathbf{S}_2(n,k)$ where we use the related numbers

$$\left\langle {n \atop k} \right\rangle = k!\, \mathbf{S}_2(n,k) \tag{1.1}$$

obeying the recurrence

$$\left\langle {n+1 \atop k} \right\rangle = k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle \right). \tag{1.2}$$

Note that $\mathbf{S}_2(n,1) = \mathbf{S}_2(n,n) = 1$ for $n \geq 1$ and $\mathbf{S}_2(n,0) = \left\langle {n \atop 0} \right\rangle = \delta_{n,0}$ for $n \geq 0$ using Kronecker's delta. For properties of Stirling and harmonic numbers we refer to [10]. The polynomials $\mathbf{F}_n$ are related to the Eulerian polynomials, see [6, pp. 243–245] and [16], [19] for a survey. The numbers $\mathbf{F}_n(1)$ are called ordered Bell numbers or Fubini numbers, cf. [6, p. 228]. For a discussion and further generalizations of the polynomials $\mathbf{F}_n$ and $\widehat{\mathbf{F}}_n$ see [3] and [7], respectively. Note that the notation $\left\langle {n \atop k} \right\rangle$ is frequently used also for the Eulerian numbers, which we denote by $\mathbf{A}(n,k)$ as in [6]; the notation $\mathbf{F}_n$ is used as in [16], [19].

Lemma 1.1. *We have for $n \geq 1$:*

$$\begin{aligned}
\mathbf{F}_n(-1) &= (-1)^n, \\
\widehat{\mathbf{F}}_n(-1) &= (-1)^n n, \\
\mathbf{F}_{n+1}(x) &= (x^2 + x)\mathbf{F}'_n(x) + x\mathbf{F}_n(x), \\
\widehat{\mathbf{F}}_{n+1}(x) &= (x^2 + x)\widehat{\mathbf{F}}'_n(x) + x\widehat{\mathbf{F}}_n(x) + x\mathbf{F}_n(x).
\end{aligned}$$

Proof. The recurrences follow easily by (1.2) and the values at $x = -1$ by induction. $\square$

The Bernoulli numbers $\mathbf{B}_n$ and the Genocchi numbers $\mathbf{G}_n$ may be defined by

$$\mathbf{B}(t) = \frac{t}{e^t - 1} = \sum_{n \geq 0} \mathbf{B}_n \frac{t^n}{n!} \quad (|t| < 2\pi) \tag{1.3}$$

and

$$\mathbf{G}(t) = \frac{2t}{e^t + 1} = \sum_{n \geq 0} \mathbf{G}_n \frac{t^n}{n!} \quad (|t| < \pi), \tag{1.4}$$

where $\mathbf{G}_0 = 0$ and $\mathbf{B}_n = \mathbf{G}_n = 0$ for odd $n > 1$, cf. [6, pp. 48–49]. Note that

$$\mathbf{G}_n = 2(1 - 2^n)\mathbf{B}_n \quad (n \geq 0). \tag{1.5}$$

The numbers $\mathbf{B}_n$ are rational, whereas the numbers $\mathbf{G}_n$ are integers.

Define the semiring $\mathcal{S} \subset \mathbb{Z}[x]$, which consists of polynomials having nonnegative integer coefficients. Further define the set

$$\mathfrak{S}_\alpha = \{f \in \mathcal{S} : f(\alpha + x) = (-1)^{\deg f} f(\alpha - x) \text{ for } x \in \mathbb{R}\}, \tag{1.6}$$

where such polynomials have a reflection relation around $x = \alpha$. For $f(x) = 0$ we declare $\deg f = 0$, such that $0 \in \mathfrak{S}_\alpha$ is well defined.

Theorem 1.2. *We have the following relations for $n \geq 1$:*
(a)

$$\int_{-1}^{0} \mathbf{F}_n(x) dx = \mathbf{B}_n.$$

(b)

$$\mathbf{F}_n(-\tfrac{1}{2}) = \frac{\mathbf{G}_{n+1}}{n+1}.$$

(c)

$$\mathbf{F}_n(x)/x,\ (x+1)\mathbf{F}_n(x) \in \mathfrak{S}_{-1/2}.$$

This theorem can be deduced from known results, which we will give later. By $\mathbf{F}_n \in \mathcal{S}$ and the symmetry property (c) above, we conclude that $\mathbf{F}_n(x) > 0$ and $(-1)^n\mathbf{F}_n(-1-x) > 0$ for $x > 0$; both expressions strictly increasing as $x \to \infty$. Except for a simple zero at $x = 0$, all real zeros of $\mathbf{F}_n$ symmetrically lie around $x = -\frac{1}{2}$ in the interval $(-1, 0)$. For an illustration see Figure A.1. The value of $\mathbf{F}_n(-\frac{1}{2})$ can be seen as a central value. Note also that $x^2 + x \in \mathfrak{S}_{-1/2}$ occurs in the recurrences of $\mathbf{F}_n$ and $\widehat{\mathbf{F}}_n$ given in Lemma 1.1. Interestingly, the integral over the interval $[-1, 0]$ and the central value are mainly connected with Bernoulli numbers. Similar properties also exist for the polynomials $\widehat{\mathbf{F}}_n$ as follows.

Theorem 1.3. *We have the following relations:*

(a)

$$\int_{-1}^{0} \widehat{\mathbf{F}}_n(x)dx = -\frac{n}{2}\mathbf{B}_{n-1} \quad (n \geq 1).$$

(b)

$$\widehat{\mathbf{F}}_n(-\tfrac{1}{2}) = (-1)^{\delta_{n,1}} \frac{1}{2} \sum_{\nu=1}^{n} \binom{n}{\nu} \frac{\mathbf{G}_\nu}{\nu} \frac{\mathbf{G}_{n+1-\nu}}{n+1-\nu} \quad (\textit{odd } n \geq 1).$$

(c)

$$\widehat{\mathbf{F}}_n(-\tfrac{1}{2}) = -\frac{n-1}{2}\,\mathbf{F}_{n-1}(-\tfrac{1}{2}) \quad (\textit{even } n \geq 2).$$

(d)

$$\operatorname{ord}_2 \widehat{\mathbf{F}}_n(-\tfrac{1}{2}) = -1 - \begin{cases} \operatorname{ord}_2 n, & \textit{if even } n \geq 2, \\ 2(r-1), & \textit{if } n = 2^r - 1 \ (r \geq 1), \\ \operatorname{ord}_2(n+1) + [\log_2(n+1)], & \textit{otherwise}, \end{cases}$$

where ord_2 *is the* 2*-adic valuation and* $[\,\cdot\,]$ *gives the integer part.*

(e)

$$\widehat{\mathbf{F}}_n(x) = \mathbf{F}_n(x) + (n-1)x\mathbf{F}_{n-1}(x) + \sum_{\nu=1}^{n-2} \lambda_{n,\nu}(x)\mathbf{F}_\nu(x) \quad (n \geq 2),$$

where $\lambda_{n,\nu} \in \mathfrak{S}_{-1/2}$ *and* $\deg \lambda_{n,\nu} = n - \nu$ *for* $\nu = 1, \dots, n-2$.

(f)

$$\widehat{\mathbf{F}}_n(x)/x - (n-1)\mathbf{F}_{n-1}(x) \in \mathfrak{S}_{-1/2} \quad (n \geq 2),$$

where the resulting polynomial has degree $n - 1$.

The polynomials $\lambda_{n,\nu}$ will be recursively defined later in Proposition 3.4. The symmetry of $\widehat{\mathbf{F}}_n(x)/x-(n-1)\mathbf{F}_{n-1}(x)$ is shown in Figure A.2. The first relations between the polynomials $\widehat{\mathbf{F}}_n$ and $\mathbf{F}_n$ are given as follows.

Table 1.4.

$$\begin{aligned}
\widehat{\mathbf{F}}_1(x) &= \mathbf{F}_1(x),\\
\widehat{\mathbf{F}}_2(x) &= \mathbf{F}_2(x)+x\mathbf{F}_1(x),\\
\widehat{\mathbf{F}}_3(x) &= \mathbf{F}_3(x)+2x\mathbf{F}_2(x)+(x^2+x)\mathbf{F}_1(x),\\
\widehat{\mathbf{F}}_4(x) &= \mathbf{F}_4(x)+3x\mathbf{F}_3(x)+3(x^2+x)\mathbf{F}_2(x)+(2x^3+3x^2+x)\mathbf{F}_1(x).
\end{aligned}$$

In the following theorem a different relation is given by derivatives of $\mathbf{F}_n$.

Theorem 1.5. *We have*

$$\widehat{\mathbf{F}}_n(x)=\sum_{\nu=1}^{n}(-1)^{\nu+1}\frac{\mathbf{F}_n^{(\nu)}(x)}{\nu!}\frac{x^\nu}{\nu}\quad (n\geq 1).$$

Remark. The identity (c) of Theorem 1.3 occurred in [13] as an important key step in proofs. We shall use different approaches to prove the theorems in a comprehensive way.

2. Bernoulli and Stirling Numbers

Define

$$S_n(m)=\sum_{\nu=0}^{m-1}\nu^n\quad (n\geq 0).$$

It is well known that

$$S_n(x)=\frac{1}{n+1}(\mathbf{B}_{n+1}(x)-\mathbf{B}_{n+1}), \tag{2.1}$$

where $\mathbf{B}_n(x)$ is the nth Bernoulli polynomial, cf. [10, p. 367], with the properties

$$\mathbf{B}_{n+1}'(x)=(n+1)\mathbf{B}_n(x),\quad \mathbf{B}_n(0)=\mathbf{B}_n. \tag{2.2}$$

The Gregory-Newton expansion of x^n reads

$$x^n=\sum_{k=0}^{n}\left\langle {n \atop k}\right\rangle\binom{x}{k}, \tag{2.3}$$

which follows by (1.1) and the usual definition of the numbers $\mathbf{S}_2(n,k)$ by

$$x^n = \sum_{k=0}^{n} \mathbf{S}_2(n,k)(x)_k$$

with falling factorials $(x)_k$. The summation of (2.3) yields another familiar formula

$$S_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x}{k+1}. \tag{2.4}$$

Note that $\left\langle {n \atop 0} \right\rangle = 0$ for $n \geq 1$ and $\binom{-1}{k} = (-1)^k$. The following formula is a classical result which is due to Worpitzky. We give a short proof.

Proposition 2.1 (Worpitzky [20, (36), p. 215]). *We have*

$$\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle \frac{(-1)^k}{k+1} = \mathbf{B}_n \quad (n \geq 1).$$

Proof. Since $S_n(0) = 0$, we conclude by (2.1), (2.2), and (2.4) that

$$\mathbf{B}_n = S_n'(0) = \lim_{x \to 0} S_n(x)/x = \lim_{x \to 0} \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \frac{1}{k+1} \binom{x-1}{k} = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle \frac{(-1)^k}{k+1}. \qquad \square$$

A similar result with harmonic numbers is the following.

Proposition 2.2. *We have*

$$\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle \mathbf{H}_k \frac{(-1)^k}{k+1} = -\frac{n}{2} \mathbf{B}_{n-1} \quad (n \geq 1).$$

Proof. The derivative of (2.4) provides that

$$S_n'(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x}{k+1} \sum_{j=0}^{k} \frac{1}{x-j} = S_n(x)/x - x V_n(x)$$

where

$$V_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \frac{1}{k+1} \binom{x-1}{k} \sum_{j=1}^{k} \frac{1}{j-x}.$$

Since $|V_n(0)| < \infty$ and $S_n(x) - x S_n'(x) \to 0$ as $x \to 0$, we obtain by L'Hôpital's rule that

$$V_n(0) = \lim_{x \to 0} \frac{S_n(x) - x S_n'(x)}{x^2} = \lim_{x \to 0} \frac{-x S_n''(x)}{2x} = -\frac{1}{2} S_n''(0).$$

Using (2.1) and (2.2) we then derive that

$$S_n''(x) = n \mathbf{B}_{n-1}(x) \quad \text{and} \quad V_n(0) = -\frac{n}{2} \mathbf{B}_{n-1}.$$

This shows the claimed identity. $\square$

3. Symmetry Properties

We shall give some symmetry relations of the polynomials $\mathbf{F}_n$ and $\widehat{\mathbf{F}}_n$. Note that the Eulerian numbers, as used in [6], [19], are symmetric such that $\mathbf{A}(n,k) = \mathbf{A}(n, n-k)$. Recall the definition of $\mathfrak{S}_\alpha$ in (1.6).

Lemma 3.1. *The set $\mathfrak{S}_\alpha$ has pseudo semiring properties. If $f, g \in \mathfrak{S}_\alpha$, then*

$$\begin{aligned} f \cdot g &\in \mathfrak{S}_\alpha, \\ f + g &\in \mathfrak{S}_\alpha, \quad (*) \\ f' &\in \mathfrak{S}_\alpha, \end{aligned}$$

where in case of addition and $f \cdot g \neq 0$ a parity condition must hold such that

$$(*) \quad \deg f \equiv \deg g \pmod 2.$$

If f has odd degree, then $f(\alpha) = 0$.

Proof. The cases, where $f = 0$ or $g = 0$, are trivial. Since $\mathfrak{S}_\alpha \subset \mathcal{S}$, the pseudo semiring properties follow by the parity of $(-1)^{\deg f}$, resp., $(-1)^{\deg g}$. If $\deg f$ is odd, then $f(\alpha) = -f(\alpha)$, which implies that $f(\alpha) = 0$. □

Proposition 3.2 (Tanny [19, (16), p. 737]). *We have*

$$\mathbf{F}_n(x) = \sum_{k=1}^{n} \mathbf{A}(n,k) x^{n-k+1} (x+1)^{k-1} \quad (n \geq 1).$$

Corollary 3.3. *We have*

$$\mathbf{F}_n(x)/x, \, (x+1)\mathbf{F}_n(x) \in \mathfrak{S}_{-1/2} \quad (n \geq 1).$$

Proof. Using the symmetry of $\mathbf{A}(n,k)$, we obtain that

$$f_n(x) := \mathbf{F}_n(x)/x = \sum_{k=1}^{n} \mathbf{A}(n,k) x^{n-k} (x+1)^{k-1} = (-1)^{n-1} f_n(-(x+1)).$$

Hence, $f_n(-\frac{1}{2} + x) = (-1)^{n-1} f_n(-\frac{1}{2} - x)$ and $\deg f_n = n - 1$ show that $\mathbf{F}_n(x)/x \in \mathfrak{S}_{-1/2}$. Since $x^2 + x \in \mathfrak{S}_{-1/2}$, it also follows that

$$(x^2 + x) \cdot \mathbf{F}_n(x)/x = (x+1)\mathbf{F}_n(x) \in \mathfrak{S}_{-1/2}. \qquad \square$$

Proposition 3.4. *We have for $n \geq 1$:*

$$\widehat{\mathbf{F}}_n(x) = \sum_{\nu=1}^{n} \lambda_{n,\nu}(x) \mathbf{F}_\nu(x)$$

where

$$\lambda_{n,\nu}(x)=\begin{cases}1, & \text{if } n=\nu=1,\\ 0, & \text{if } \nu\notin\{1,\ldots,n\},\end{cases}$$

otherwise recursively defined by

$$\lambda_{n+1,\nu}(x)=(x^2+x)\lambda'_{n,\nu}(x)+\lambda_{n,\nu-1}(x)+\delta_{n,\nu}x.$$

Furthermore $\lambda_{n,\nu}\in\mathcal{S}$ *and* $\deg\lambda_{n,\nu}=n-\nu$ *for* $\nu=1,\ldots,n$. *Especially*

$$\lambda_{n,n-1}(x)=(n-1)x.$$

Proof. We use induction on n. For $n=1$ we have

$$\widehat{\mathbf{F}}_1(x)=\mathbf{F}_1(x)\quad\text{and}\quad\lambda_{1,\nu}(x)=\delta_{1,\nu}.$$

Now assume the result is true for n. By assumption we have

$$\begin{aligned}\widehat{\mathbf{F}}_n(x)&=\sum_{\nu=1}^{n}\lambda_{n,\nu}(x)\mathbf{F}_\nu(x),\\ \widehat{\mathbf{F}}'_n(x)&=\sum_{\nu=1}^{n}\lambda'_{n,\nu}(x)\mathbf{F}_\nu(x)+\lambda_{n,\nu}(x)\mathbf{F}'_\nu(x),\end{aligned}$$

and by Lemma 1.1 that

$$\begin{aligned}\widehat{\mathbf{F}}_{n+1}(x)&=(x^2+x)\widehat{\mathbf{F}}'_n(x)+x\widehat{\mathbf{F}}_n(x)+x\mathbf{F}_n(x),\\ (x^2+x)\mathbf{F}'_n(x)&=\mathbf{F}_{n+1}(x)-x\mathbf{F}_n(x).\end{aligned}$$

It follows that

$$\begin{aligned}\widehat{\mathbf{F}}_{n+1}(x)&=(x^2+x)\sum_{\nu=1}^{n}\big(\lambda'_{n,\nu}(x)\mathbf{F}_\nu(x)+\lambda_{n,\nu}(x)\mathbf{F}'_\nu(x)\big)+x\widehat{\mathbf{F}}_n(x)+x\mathbf{F}_n(x)\\ &=(x^2+x)\sum_{\nu=1}^{n}\lambda'_{n,\nu}(x)\mathbf{F}_\nu(x)+\sum_{\nu=1}^{n}\lambda_{n,\nu}(x)\left(\mathbf{F}_{\nu+1}(x)-x\mathbf{F}_\nu(x)\right)\\ &\quad+x\widehat{\mathbf{F}}_n(x)+x\mathbf{F}_n(x)\\ &=(x^2+x)\sum_{\nu=1}^{n}\lambda'_{n,\nu}(x)\mathbf{F}_\nu(x)+\sum_{\nu=1}^{n}\lambda_{n,\nu}(x)\mathbf{F}_{\nu+1}(x)+x\mathbf{F}_n(x)\\ &=\sum_{\nu=1}^{n+1}\lambda_{n+1,\nu}(x)\mathbf{F}_\nu(x).\end{aligned}$$

Thus

$$\lambda_{n+1,\nu}(x)=(x^2+x)\lambda'_{n,\nu}(x)+\lambda_{n,\nu-1}(x)+\delta_{n,\nu}x.$$

In particular, we have

$$\lambda_{n+1,n+1}(x) = \lambda_{n,n}(x) = 1 \tag{3.1}$$

and

$$\lambda_{n+1,n}(x) = \lambda_{n,n-1}(x) + x = (n-1)x + x = nx. \tag{3.2}$$

The recurrence shows that $\lambda_{n+1,\nu} \in \mathcal{S}$ for $\nu = 1, \ldots, n+1$. Therefore we conclude for $1 \leq \nu < n$ that

$$\deg \lambda_{n+1,\nu} = \max(2 + \deg \lambda'_{n,\nu}, \lambda_{n,\nu-1}) = n - \nu + 1.$$

Along with (3.1) and (3.2) this shows the claimed properties for $n+1$. □

Proposition 3.5. *We have $\lambda_{n,\nu} \in \mathfrak{S}_{-1/2}$ for $n-2 \geq \nu \geq 1$.*

Proof. We make use of Proposition 3.4 and Lemma 3.1. For $n-1 \geq \nu \geq 1$ we have

$$\lambda_{n+1,\nu}(x) = (x^2 + x)\lambda'_{n,\nu}(x) + \lambda_{n,\nu-1}(x).$$

We use induction on n. For $n = 3$ we have

$$\lambda_{3,1}(x) = (x^2 + x)\lambda'_{2,1}(x) + \lambda_{2,0}(x) = x^2 + x \in \mathfrak{S}_{-1/2}.$$

Now assume the result holds for $n \geq 3$. For $\nu = 1, \ldots, n-2$ we have

$$\lambda_{n+1,\nu}(x) = (x^2 + x)\lambda'_{n,\nu}(x) + \lambda_{n,\nu-1}(x) \in \mathfrak{S}_{-1/2},$$

since $\lambda'_{n,\nu}, \lambda_{n,\nu-1} \in \mathfrak{S}_{-1/2}$ by assumption and $2 + \deg \lambda'_{n,\nu} = \deg \lambda_{n,\nu-1}$ if $\nu \neq 1$, otherwise $\lambda_{n,\nu-1} = 0$. It remains the case $\nu = n-1$:

$$\lambda_{n+1,n-1}(x) = (x^2 + x)(n-1) + \lambda_{n,n-2}(x) \in \mathfrak{S}_{-1/2},$$

since $\lambda'_{n,n-1}(x) = n-1$ and $\lambda_{n,n-2} \in \mathfrak{S}_{-1/2}$ with $\deg \lambda_{n,n-2} = 2$. □

Corollary 3.6. *We have*

$$\widehat{\mathbf{F}}_n(x)/x - (n-1)\mathbf{F}_{n-1}(x) \in \mathfrak{S}_{-1/2} \quad (n \geq 2),$$

where the resulting polynomial has degree $n-1$.

Proof. Propositions 3.4 and 3.5 show that

$$\widehat{\mathbf{F}}_n(x)/x - (n-1)\mathbf{F}_{n-1}(x) = \mathbf{F}_n(x)/x + \sum_{\nu=1}^{n-2} \lambda_{n,\nu}(x) \cdot \mathbf{F}_\nu(x)/x, \tag{3.3}$$

where $\lambda_{n,\nu} \in \mathfrak{S}_{-1/2}$ and $\deg \lambda_{n,\nu} = n - \nu$ for $\nu = 1, \ldots, n-2$. By Corollary 3.3 and Lemma 3.1 we conclude that $\lambda_{n,\nu}(x) \cdot \mathbf{F}_\nu(x)/x \in \mathfrak{S}_{-1/2}$. Since the first term and the products on the right-hand side of (3.3) lie in $\mathfrak{S}_{-1/2}$ having the same degree $n-1$, the left-hand side of (3.3) also lies in $\mathfrak{S}_{-1/2}$ and has degree $n-1$. □

While $\mathbf{F}_n(x)/x \in \mathfrak{S}_{-1/2}$ for $n \geq 1$, we have $\widehat{\mathbf{F}}_n(x)/x \notin \mathfrak{S}_{-1/2}$ for $n \geq 2$; compare values at $x = 0$ and $x = -1$ using Lemma 1.1. The latter function needs a correction term to lie in $\mathfrak{S}_{-1/2}$. For an illustration of the symmetry of $\mathbf{F}_n(x)/x$ and $\widehat{\mathbf{F}}_n(x)/x - (n-1)\mathbf{F}_{n-1}(x)$ see Figures A.1 and A.2.

4. Generating Functions

Let $[t^n]$ be the linear operator, that gives the coefficient of t^n of a formal power series, such that

$$f(t) = \sum_{n \geq 0} a_n t^n, \quad [t^n]\, f(t) = a_n,$$

see [10, p. 197]. Define $\langle t^n \rangle = n![t^n]$. Let $(h_n(x))_{n \geq 0}$ be a sequence of functions. Then we denote by

$$\mathcal{G}h(x,t) = \sum_{n \geq 0} h_n(x) \frac{t^n}{n!}$$

the two-variable exponential generating function, such that

$$\langle t^n \rangle\, \mathcal{G}h(x,t) = h_n(x).$$

Recall the definitions of $\mathbf{B}(t)$ and $\mathbf{G}(t)$ in (1.3) and (1.4), respectively. We further set

$$\widetilde{\mathbf{B}}(t) = \mathbf{B}(t)/t, \quad \widetilde{\mathbf{G}}(t) = \mathbf{G}(t)/t,$$

and

$$\widetilde{\mathbf{B}}_0 = \widetilde{\mathbf{G}}_0 = 0, \quad \widetilde{\mathbf{B}}_n = \mathbf{B}_n/n, \quad \widetilde{\mathbf{G}}_n = \mathbf{G}_n/n \quad (n \geq 1).$$

Lemma 4.1. *The function $y(t) = \widetilde{\mathbf{G}}(t)$ satisfies the Bernoulli differential equation*

$$y' + y = \frac{1}{2} y^2, \quad \textit{which is equivalent to} \quad (\log y)' = \frac{1}{2} y - 1.$$

Proof. Both equations are identical due to $(\log y)' = y'/y$ and are verified by $y(t) = 2/(e^t + 1)$. □

Corollary 4.2. *We have*

$$\widetilde{\mathbf{G}}(t) = \sum_{n \geq 0} \widetilde{\mathbf{G}}_{n+1} \frac{t^n}{n!} \qquad (|t| < \pi), \tag{4.1}$$

$$\log \widetilde{\mathbf{G}}(t) = \frac{1}{2} \sum_{n \geq 1} (-1)^n \widetilde{\mathbf{G}}_n \frac{t^n}{n!} \quad (|t| < \pi). \tag{4.2}$$

Proof. Eq. (4.1) follows by its definition. Integrating the right-hand side differential equation of Lemma 4.1, we derive that

$$\log \widetilde{\mathbf{G}}(t) = \int \left(\frac{1}{2} \widetilde{\mathbf{G}}(t) - 1 \right) dt = \frac{1}{2} \sum_{n \geq 1} \widetilde{\mathbf{G}}_n \frac{t^n}{n!} - t + C$$

with a constant C. Since $\log \widetilde{\mathbf{G}}(0) = 0$, we obtain $C = 0$. Note that $(-1)^n \widetilde{\mathbf{G}}_n = \widetilde{\mathbf{G}}_n$ for $n \geq 2$ and $\widetilde{\mathbf{G}}_1 = 1$. With $t\, \widetilde{\mathbf{G}}_1/2 - t = -t\, \widetilde{\mathbf{G}}_1/2$ we finally get (4.2). $\square$

We also have a connection with hyperbolic functions, where we casually obtain the known coefficients of the following function by (4.2).

Lemma 4.3. *We have*

$$\log \widetilde{\mathbf{G}}(t) = -\frac{t}{2} - \log \cosh \left(\frac{t}{2} \right).$$

Proof. This follows by

$$e^{t/2} \cosh \left(\frac{t}{2} \right) = \frac{e^t + 1}{2} = \widetilde{\mathbf{G}}(t)^{-1}.$$

$\square$

Proposition 4.4. *Define* $\mathbf{F}_0(x) = \widehat{\mathbf{F}}_0(x) = 1$ *and* $\psi(x,t) = 1 - x(e^t - 1)$. *Then we have*

(a)

$$\mathcal{G}\mathbf{F}(x,t) = \frac{1}{\psi(x,t)}, \qquad \int \mathcal{G}\mathbf{F}(x,t) dx = -\widetilde{\mathbf{B}}(t) \log \psi(x,t),$$

(b)

$$\mathcal{G}\widehat{\mathbf{F}}(x,t) = -\frac{\log \psi(x,t)}{\psi(x,t)}, \qquad \int \mathcal{G}\widehat{\mathbf{F}}(x,t) dx = \frac{1}{2} \widetilde{\mathbf{B}}(t) \left(\log \psi(x,t) \right)^2.$$

Proof. Set $u = e^t - 1$. Note that $1/u = \widetilde{\mathbf{B}}(t)$ and $1 - xu = \psi(x,t)$. We need the following generating functions (cf. [10, p. 351]):

$$(e^t - 1)^k = \sum_{n \geq k} \left\langle {n \atop k} \right\rangle \frac{t^n}{n!}, \tag{4.3}$$

$$-\frac{\log(1-t)}{1-t} = \sum_{n \geq 1} \mathbf{H}_n t^n. \tag{4.4}$$

(a) Using (4.3) we obtain that

$$\frac{1}{1 - x(e^t - 1)} = \sum_{k \geq 0} (x(e^t - 1))^k = \sum_{k \geq 0} x^k \sum_{n \geq k} \left\langle {n \atop k} \right\rangle \frac{t^n}{n!} = \sum_{n \geq 0} \mathbf{F}_n(x) \frac{t^n}{n!} = \mathcal{G}\mathbf{F}(x,t). \tag{4.5}$$

We further deduce that

$$\int \mathcal{G}\mathbf{F}(x,t)dx = \int \frac{dx}{1-xu} = -\frac{\log(1-xu)}{u} = -\widetilde{\mathbf{B}}(t)\log\psi(x,t).$$

(b) First substitute t by $x(e^t-1)$ in (4.4). The result for $\mathcal{G}\widehat{\mathbf{F}}(x,t)$ is similarly derived as in (4.5) with an additional factor $\mathbf{H}_k$. The integral follows by

$$\int \frac{\log(1-xu)}{1-xu}dx = -\frac{\log(1-xu)^2}{2u}. \qquad \square$$

Proposition 4.5. *We have*

$$\mathbf{F}_n(-\tfrac{1}{2}) = \widetilde{\mathbf{G}}_{n+1} \quad (n \geq 1),$$

$$\widehat{\mathbf{F}}_n(-\tfrac{1}{2}) = \begin{cases} -\frac{1}{2}, & \textit{if } n = 1, \\ -\frac{n-1}{2}\widetilde{\mathbf{G}}_n, & \textit{if even } n \geq 2, \\ \frac{1}{2}\sum\limits_{\nu=1}^{n}\binom{n}{\nu}\widetilde{\mathbf{G}}_\nu\widetilde{\mathbf{G}}_{n+1-\nu}, & \textit{if odd } n \geq 3. \end{cases}$$

Proof. From Proposition 4.4 we conclude that

$$\psi(-\tfrac{1}{2},t) = \frac{e^t+1}{2} = \widetilde{\mathbf{G}}(t)^{-1}.$$

Hence,

$$\mathcal{G}\mathbf{F}(-\tfrac{1}{2},t) = \widetilde{\mathbf{G}}(t) \quad \text{and} \quad \mathcal{G}\widehat{\mathbf{F}}(-\tfrac{1}{2},t) = \widetilde{\mathbf{G}}(t)\log\widetilde{\mathbf{G}}(t).$$

By (4.1) we derive that

$$\mathbf{F}_n(-\tfrac{1}{2}) = \langle t^n \rangle\, \mathcal{G}\mathbf{F}(-\tfrac{1}{2},t) = \langle t^n \rangle\, \widetilde{\mathbf{G}}(t) = \widetilde{\mathbf{G}}_{n+1}.$$

Similarly, we obtain that

$$\widehat{\mathbf{F}}_n(-\tfrac{1}{2}) = \langle t^n \rangle\, \mathcal{G}\widehat{\mathbf{F}}(-\tfrac{1}{2},t) = \langle t^n \rangle\, \widetilde{\mathbf{G}}(t)\log\widetilde{\mathbf{G}}(t) = c_n,$$

where c_n arises from the convolution sum caused by the Cauchy product of $\widetilde{\mathbf{G}}(t)$ and $\log\widetilde{\mathbf{G}}(t)$. By means of (4.1) and (4.2) we achieve that

$$c_n = \frac{1}{2}\sum_{\nu=1}^{n}\binom{n}{\nu}(-1)^\nu\widetilde{\mathbf{G}}_\nu\widetilde{\mathbf{G}}_{n+1-\nu}.$$

Case even n: The indices ν and $n+1-\nu$ have different parity. Since $\widetilde{\mathbf{G}}_\nu = 0$ for odd $\nu > 1$ and $\widetilde{\mathbf{G}}_1 = 1$, the sum simplifies to

$$c_n = \frac{1}{2}\left(-\binom{n}{1}\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_n + \binom{n}{n}\widetilde{\mathbf{G}}_n\widetilde{\mathbf{G}}_1\right) = -\frac{n-1}{2}\widetilde{\mathbf{G}}_n.$$

Case odd n: For $n = 1$ we compute $c_1 = -\frac{1}{2}$. Let $n \geq 3$. Because of the same parity of the indices, we finally infer that

$$c_n = \frac{1}{2} \sum_{\nu=1}^{n} \binom{n}{\nu} \widetilde{\mathbf{G}}_\nu \widetilde{\mathbf{G}}_{n+1-\nu}$$

by omitting the factor $(-1)^\nu$. $\square$

Proposition 4.6. *We have*

$$\int_{-1}^{0} \mathbf{F}_n(x)dx = \mathbf{B}_n \quad \textit{and} \quad \int_{-1}^{0} \widehat{\mathbf{F}}_n(x)dx = -\frac{n}{2}\mathbf{B}_{n-1} \quad (n \geq 1).$$

Proof. Using Proposition 4.4, we obtain that $\psi(0,t) = 1$ and $\psi(-1,t) = e^t$. Therefore

$$\int_{-1}^{0} \mathcal{G}\mathbf{F}(x,t)dx = -\widetilde{\mathbf{B}}(t) \log \psi(x,t)\Big|_{x=-1}^{0} = \widetilde{\mathbf{B}}(t)t = \mathbf{B}(t)$$

and

$$\int_{-1}^{0} \mathcal{G}\widehat{\mathbf{F}}(x,t)dx = \frac{1}{2}\widetilde{\mathbf{B}}(t) \left(\log \psi(x,t)\right)^2 \Big|_{x=-1}^{0} = -\frac{1}{2}\widetilde{\mathbf{B}}(t)t^2 = -\frac{t}{2}\mathbf{B}(t).$$

Since the integrals above are independent of t, the operator $\langle t^n \rangle$ commutes with integration. Applying $\langle t^n \rangle$ to these equations easily yields the results. $\square$

Remark. The generating function $\mathcal{G}\mathbf{F}(x,t)$ can be found in [19, (9), p. 736] and [3, (3.14), p. 3853]. The value of $\mathbf{F}_n(-\frac{1}{2})$ was posed as an exercise in [10, 6.76, p. 559] and also given in [3, (3.29), p. 3855]; see [18, p. 288] for a short proof using the theory of Riordan arrays.

5. Convolutions

We use the notations

$$\begin{aligned} (\alpha_r + \beta_s)^n &= \sum_{\nu=0}^{n} \binom{n}{\nu} \alpha_{r+\nu}\beta_{s+n-\nu}, \\ \{\alpha_r + \beta_s\}^n &= \sum_{\nu=0}^{n} \alpha_{r+\nu}\beta_{s+n-\nu} \end{aligned} \qquad (n, r, s \geq 0)$$

for symmetric binomial, resp. usual convolutions of two sequences $(\alpha_\nu)_{\nu \geq 0}$ and $(\beta_\nu)_{\nu \geq 0}$. Unless otherwise noted, we generally assume $n \geq 1$ for convolutions in this section. We first need a simple lemma (cf. [9, p. 82]).

Lemma 5.1. *If $n \geq 1$ and $r, s \geq 0$, then*

$$(\alpha_r + \beta_s)^n = (\alpha_r + \beta_{s+1})^{n-1} + (\alpha_{r+1} + \beta_s)^{n-1},$$
$$(\widetilde{\alpha}_0 + \widetilde{\beta}_0)^n = \frac{1}{n}(\widetilde{\alpha}_0 + \beta_0)^n + \frac{1}{n}(\alpha_0 + \widetilde{\beta}_0)^n,$$

where for the second part $\alpha_0 = \widetilde{\alpha}_0 = \beta_0 = \widetilde{\beta}_0 = 0$ and $\widetilde{\alpha}_\nu = \alpha_\nu/\nu$, $\widetilde{\beta}_\nu = \beta_\nu/\nu$ for $\nu \geq 1$.

Proof. The first part follows by $\binom{n}{\nu} = \binom{n-1}{\nu} + \binom{n-1}{\nu-1}$, the second part by the identity $1/(\nu(n-\nu)) = 1/(n\nu) + 1/(n(n-\nu))$. □

The Euler polynomials $\mathbf{E}_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n \geq 0} \mathbf{E}_n(x) \frac{t^n}{n!} \quad (|t| < \pi). \tag{5.1}$$

Compared to (1.4) and (4.1), formulas with $\mathbf{E}_n(x)$ are naturally transferred to Genocchi numbers by the relation $\mathbf{E}_n(0) = \widetilde{\mathbf{G}}_{n+1}$. The well-known identity ([15, (17), p. 135], [11, (51.6.2), p. 346])

$$(\mathbf{E}_0(x) + \mathbf{E}_0(y))^n = 2(1 - x - y)\mathbf{E}_n(x + y) + 2\mathbf{E}_{n+1}(x + y)$$

leads to basic convolutions of the Genocchi numbers

$$(\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_1)^n = 2\widetilde{\mathbf{G}}_{n+2} + 2\widetilde{\mathbf{G}}_{n+1}, \tag{5.2}$$
$$(\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_2)^n = \widetilde{\mathbf{G}}_{n+3} + \widetilde{\mathbf{G}}_{n+2}, \tag{5.3}$$

where the latter equation is derived by Lemma 5.1. Note that (5.2) also follows immediately by Lemma 4.1 and (4.1). More general convolution identities can be found in [5] for Bernoulli and Euler polynomials, that cover some known convolutions as special cases.

As a result of Proposition 4.5 in the last section, the convolution

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n = \frac{1}{2}(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^{n+1} \tag{5.4}$$

has appeared, which resists a simple evaluation for odd $n > 1$; the right-hand side follows by Lemma 5.1. We shall give some arguments in the following that this remains an open problem. Convolutions with different indices in the shape of $(\mathbf{G}_j + \mathbf{G}_k)^n$ for $j + k \geq 0$ and $(\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_k)^n$ for $k \geq 1$ are discussed in [2] and [1], respectively. A connection with the last-mentioned convolutions is established by the following lemma.

Lemma 5.2. *If $n \geq 1$, then*

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n = \sum_{k=1}^{n} (\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_k)^{n-k}.$$

Proof. By Lemma 5.1 we obtain the recurrences

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_k)^{n-k+1} = (\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_{k+1})^{n-k} + (\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_k)^{n-k}$$

for $k = 1, \ldots, n$. Since $(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_{n+1})^0 = 0$, this gives the claimed sum. □

However, this will not simplify (5.4), see below. As before, we set $\widetilde{\mathbf{B}}_0(x) = 0$ and $\widetilde{\mathbf{B}}_n(x) = \mathbf{B}_n(x)/n$ ($n \geq 1$) for the Bernoulli polynomials. We can translate (5.4) as follows.

Lemma 5.3. *We have*

$$\frac{1}{4}(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n = (\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{B}}_0)^n - 2^n(\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{B}}_0(\tfrac{1}{2}))^n \quad (n \geq 1).$$

Proof. By (1.5) we have $\widetilde{\mathbf{G}}_\nu \widetilde{\mathbf{G}}_{n-\nu}/4 = (1 - 2^\nu - 2^{n-\nu} + 2^n)\widetilde{\mathbf{B}}_\nu \widetilde{\mathbf{B}}_{n-\nu}$. It is well known that $\mathbf{B}_n(\frac{1}{2}) = (2^{1-n} - 1)\mathbf{B}_n$. Observing the symmetry, the result follows from $(2^{n-\nu} - 2^{n-1})\widetilde{\mathbf{B}}_\nu = 2^{n-1}(2^{1-\nu} - 1)\widetilde{\mathbf{B}}_\nu = 2^{n-1}\widetilde{\mathbf{B}}_\nu(\frac{1}{2})$. □

Proposition 5.4 (Gessel [9, (12), p. 81]). *If $n \geq 1$, then*

$$\frac{n}{2}\{\widetilde{\mathbf{B}}_0(x) + \widetilde{\mathbf{B}}_0(x)\}^n - (\widetilde{\mathbf{B}}_0 + \mathbf{B}_0(x))^n = \mathbf{H}_{n-1}\mathbf{B}_n(x) + \frac{n}{2}\mathbf{B}_{n-1}(x).$$

For $x = 0$ this reduces to Miki's identity [14, Theorem, p. 297]:

$$\{\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{B}}_0\}^n - (\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{B}}_0)^n = 2\mathbf{H}_n\widetilde{\mathbf{B}}_n + \mathbf{B}_{n-1}.$$

For $x = \frac{1}{2}$ this gives the Faber-Pandharipande-Zagier identity [8, Lemma 4, p. 22]:

$$\frac{n}{2}\{\widetilde{\mathbf{B}}_0(\tfrac{1}{2}) + \widetilde{\mathbf{B}}_0(\tfrac{1}{2})\}^n - (\widetilde{\mathbf{B}}_0 + \mathbf{B}_0(\tfrac{1}{2}))^n = \mathbf{H}_{n-1}\mathbf{B}_n(\tfrac{1}{2}) + \frac{n}{2}\mathbf{B}_{n-1}(\tfrac{1}{2}).$$

Note that $(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n$ is mainly transferred to $(\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{B}}_0)^n$, $(\widetilde{\mathbf{B}}_0 + \mathbf{B}_0(\frac{1}{2}))^n$, and $(\mathbf{B}_0 + \widetilde{\mathbf{B}}_0(\frac{1}{2}))^n$ by Lemmas 5.3 and 5.1. The Miki type convolutions above show that one can replace binomial convolutions by usual convolutions, but this does not lead to a simplification compared to (5.2).

Agoh showed the formula below for $k \geq 3$ in a slightly different form. In view of (5.2) and (5.3) this is also valid for $k = 1, 2$.

Proposition 5.5 (Agoh [1, Theorem, p. 61]). *If $k, n \geq 1$, then*

$$(\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_k)^n = 2\left(\widetilde{\mathbf{G}}_{n+k} - \widetilde{\mathbf{B}}_k\widetilde{\mathbf{G}}_{n+1} + \frac{1}{k}(\mathbf{B}_0 + \widetilde{\mathbf{G}}_{n+1})^k\right).$$

As an application we derive a different Miki type convolution as follows.

Proposition 5.6. *For $n \geq 1$ we have*

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n = 2\left(\mathbf{H}_{n-1}\widetilde{\mathbf{G}}_{n+1} - \frac{1}{2}\widetilde{\mathbf{G}}_n + \mathbf{G}_n - \{\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{G}}_2\}^{n-1} + (\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{G}}_2)^{n-1}\right).$$

Proof. The case $n = 1$ is trivial, let $n \geq 2$. We combine Lemma 5.2 and Proposition 5.5, where we split the summations. Note that $\widetilde{\mathbf{B}}_0 = 0$ and $\mathbf{B}_0 = 1$. We then obtain that

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n = \widetilde{\mathbf{G}}_n + \sum_{k=1}^{n-1}(\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_{n-k})^k = \widetilde{\mathbf{G}}_n + 2(S_1 + S_2 + S_3),$$

where

$$S_1 = (n-1)\widetilde{\mathbf{G}}_n, \quad S_2 = -\sum_{k=1}^{n-1}\widetilde{\mathbf{B}}_k\widetilde{\mathbf{G}}_{n+1-k} = -\{\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{G}}_2\}^{n-1},$$

$$\begin{aligned} S_3 &= \sum_{k=1}^{n-1}\frac{1}{k}\sum_{\nu=0}^{k}\binom{k}{\nu}\mathbf{B}_\nu\widetilde{\mathbf{G}}_{n+1-\nu} = \mathbf{H}_{n-1}\widetilde{\mathbf{G}}_{n+1} + \sum_{k=1}^{n-1}\sum_{\nu=1}^{k}\binom{k-1}{\nu-1}\widetilde{\mathbf{B}}_\nu\widetilde{\mathbf{G}}_{n+1-\nu} \\ &= \mathbf{H}_{n-1}\widetilde{\mathbf{G}}_{n+1} + \sum_{\nu=0}^{n-1}\binom{n-1}{\nu}\widetilde{\mathbf{B}}_\nu\widetilde{\mathbf{G}}_{n+1-\nu} = \mathbf{H}_{n-1}\widetilde{\mathbf{G}}_{n+1} + (\widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{G}}_2)^{n-1}. \end{aligned}$$

Summing up the terms establishes the result. □

As a result of Section 3, we have yet another formula.

Proposition 5.7. *For odd $n > 1$ we have*

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n = 2\left(\widetilde{\mathbf{G}}_{n+1} + \sum_{\substack{\nu=1 \\ \text{odd } \nu}}^{n-2}\lambda_{n,\nu}(-\tfrac{1}{2})\widetilde{\mathbf{G}}_{\nu+1}\right),$$

where the polynomials $\lambda_{n,\nu} \in \mathfrak{S}_{-1/2}$ are defined as in Proposition 3.4.

Proof. By Propositions 3.4 and 3.5 we have

$$\widehat{\mathbf{F}}_n(x) = \mathbf{F}_n(x) + (n-1)x\mathbf{F}_{n-1}(x) + \sum_{\nu=1}^{n-2}\lambda_{n,\nu}(x)\mathbf{F}_\nu(x),$$

where $\lambda_{n,\nu} \in \mathfrak{S}_{-1/2}$. Along with Proposition 4.5 we derive the result by setting $x = -\frac{1}{2}$ and omitting the terms where $\widetilde{\mathbf{G}}_{\nu+1} = 0$. □

The last two propositions show that one can resolve the convolution in (5.4) by

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n = \gamma_{n+1}\widetilde{\mathbf{G}}_{n+1} + \gamma_{n-1}\widetilde{\mathbf{G}}_{n-1} + \cdots + \gamma_2\widetilde{\mathbf{G}}_2 \quad (\text{odd } n > 1)$$

with some coefficients γ_ν, but this is again unimproved compared to the convolution itself. Either the coefficients γ_ν are connected with Bernoulli numbers or with polynomials that have to be recursively computed.

6. p-Adic Analysis

Let $\mathbb{Z}_p$ be the ring of p-adic integers and $\mathbb{Q}_p$ be the field of p-adic numbers. Define $\operatorname{ord}_p x$ as the p-adic valuation of x. Define $[x]$ as the integer part of $x \in \mathbb{R}$.

Lemma 6.1 ([17, p. 37]). *If $n \geq 1$, then*

$$\operatorname{ord}_p \sum_{\nu=0}^{n} x_\nu \geq \min_{0 \leq \nu \leq n} \operatorname{ord}_p x_\nu \quad (x_\nu \in \mathbb{Q}_p),$$

where equality holds, if there exists an index m such that $\operatorname{ord}_p x_m < \operatorname{ord}_p x_\nu$ for all $\nu \neq m$.

Lemma 6.2 ([17, p. 241]). *If $n \geq 1$, then*

$$\operatorname{ord}_p n! = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ is the sum of the digits of the p-adic expansion of n.

For even $n > 0$ the numbers $\widetilde{\mathbf{G}}_n$ are p-adically interesting for $p = 2$, whereas the numbers $\mathbf{G}_n$ are odd integers.

Proposition 6.3. *For $n \in 2\mathbb{N}$ the numbers $\widetilde{\mathbf{G}}_n \in \mathbb{Q}_2 \backslash \mathbb{Z}_2$, while $\widetilde{\mathbf{G}}_n \in \mathbb{Z}_p$ for $p > 2$. More precisely, $\operatorname{ord}_2 \widetilde{\mathbf{G}}_n = -\operatorname{ord}_2 n$ and $\operatorname{ord}_2 \mathbf{G}_n = 0$.*

Proof. Let $n \in 2\mathbb{N}$. It is well known that the tangent numbers (cf. [10, p. 287])

$$2^n(1-2^n)\widetilde{\mathbf{B}}_n = 2^{n-1}\widetilde{\mathbf{G}}_n$$

are integers, here defined with different sign. The right-hand side follows by (1.5). Hence the numbers $\widetilde{\mathbf{G}}_n$ are p-integers for $p > 2$. For $p = 2$ we derive that

$$\operatorname{ord}_2 \widetilde{\mathbf{G}}_n = \operatorname{ord}_2(2\widetilde{\mathbf{B}}_n) = -\operatorname{ord}_2 n,$$

where we have used the fact that $\operatorname{ord}_2(2\mathbf{B}_n) = 0$, which follows by the von Staudt-Clausen theorem, see [12, Theorem 3, p. 233]. Since n is even, we infer that $-\operatorname{ord}_2 n < 0$ and consequently that $\widetilde{\mathbf{G}}_n \in \mathbb{Q}_2 \backslash \mathbb{Z}_2$. By the same arguments, it follows from (1.5) that $\operatorname{ord}_2 \mathbf{G}_n = 0$. $\square$

It remains of interest to evaluate the convolution $(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n$ for even n. We will see that the 2-adic valuation of $(\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_1)^n$ has a simple form, while the 2-adic valuation of $(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n$ is more complicated.

Proposition 6.4. *Let $n \geq 1$ and $m = [n/2]$. Then*

$$\operatorname{ord}_2 (\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_1)^n = -\operatorname{ord}_2(m+1).$$

Proof. By (5.2) it follows that

$$\operatorname{ord}_2 (\widetilde{\mathbf{G}}_1 + \widetilde{\mathbf{G}}_1)^n = 1 + \operatorname{ord}_2(\widetilde{\mathbf{G}}_{n+2} + \widetilde{\mathbf{G}}_{n+1}) =: 1 + g,$$

where either $\widetilde{\mathbf{G}}_{n+2}$ or $\widetilde{\mathbf{G}}_{n+1}$ vanishes. Note that $n+2 = 2(m+1)$ for even n and $n+1 = 2(m+1)$ for odd n. We conclude that $g = -\operatorname{ord}_2(2(m+1))$ using Proposition 6.3, which gives the result. $\square$

Proposition 6.5. *We have*

$$\operatorname{ord}_2 (\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n = \begin{cases} \infty, & \text{if } n = 1, \\ 1 - \operatorname{ord}_2(n-1), & \text{if odd } n \geq 3, \\ 1 - \operatorname{ord}_2 n - [\log_2 n] + 2\omega_2(n), & \text{if even } n \geq 2, \end{cases}$$

where $\omega_2(n) = 1$, if n is a power of 2, otherwise $\omega_2(n) = 0$.

Proof. The cases $n = 1, 2, 4$ are handled separately with $\widetilde{\mathbf{G}}_1 = 1$ and $\widetilde{\mathbf{G}}_2 = -\frac{1}{2}$. For odd $n \geq 3$ we have $(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n = 2n\widetilde{\mathbf{G}}_{n-1}$ by symmetry and different parity of indices. The result follows by Proposition 6.3. Now, let n even and $n \geq 6$. We first obtain by Lemma 5.1 that

$$(\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^n = \frac{2}{n}(\widetilde{\mathbf{G}}_0 + \mathbf{G}_0)^n. \tag{6.1}$$

Set $\mathcal{I} = \{2, 4, \ldots, n-2\}$ and $L = [\log_2 n] - \omega_2(n)$. Further define $\ell_2(x)$ as the number of digits of $x \in \mathbb{N}$ in base 2. Note that $\operatorname{ord}_2 \binom{n}{\nu} = -s_2(n) + s_2(\nu) + s_2(n-\nu)$ by Lemma 6.2. With the help of Proposition 6.3 and Lemma 6.1, we can evaluate $\operatorname{ord}_2 (\widetilde{\mathbf{G}}_0 + \mathbf{G}_0)^n$ and obtain that

$$\operatorname{ord}_2 \sum_{\nu \in \mathcal{I}} \binom{n}{\nu} \widetilde{\mathbf{G}}_\nu \mathbf{G}_{n-\nu} \geq -s_2(n) + \min_{\nu \in \mathcal{I}} \big(s_2(\nu) + s_2(n-\nu) - \operatorname{ord}_2 \nu \big). \tag{6.2}$$

We will show that there is only one minimum on the right-hand side to get equality. To be more precise, this takes place for $\nu_m = 2^L$, the greatest power of 2 in $\mathcal{I}$, where we have

$$s_2(\nu_m) + s_2(n - \nu_m) - \operatorname{ord}_2 \nu_m = 1 + s_2(n - 2^L) - L. \tag{6.3}$$

We have now to distinguish between two cases, whether n is a power of 2 or not.

Case $n = 2^{L+1}$: The right-hand side of (6.3) reduces to $2 - L$. We derive for $\nu \in \mathcal{I} - \{\nu_m\}$ that

$$2 - L < s_2(\nu) + s_2(n-\nu) - \operatorname{ord}_2 \nu,$$

since $2 \leq s_2(\nu) + s_2(n-\nu)$ and $-L < -\operatorname{ord}_2 \nu$ by construction.

Case $n \neq 2^{L+1}$: One observes that $\ell_2(n) = \ell_2(\nu_m)$. Regarding (6.3) we then conclude that $1 + s_2(n - 2^L) - L = s_2(n) - L$. As above, for $\nu \in \mathcal{I} - \{\nu_m\}$ we have

$$\operatorname{ord}_2 \nu - L < 0 \leq \operatorname{ord}_2 \binom{n}{\nu},$$

which is equivalent to

$$s_2(n) - L < s_2(\nu) + s_2(n-\nu) - \operatorname{ord}_2 \nu.$$

Both cases show that we have exactly one minimum. Thus (6.2) becomes

$$\operatorname{ord}_2 \sum_{\nu \in \mathcal{I}} \binom{n}{\nu} \widetilde{\mathbf{G}}_\nu \mathbf{G}_{n-\nu} = -s_2(n) + s_2(\nu_m) + s_2(n-\nu_m) - \operatorname{ord}_2 \nu_m =: M,$$

where we compute in case $n = 2^{L+1}$ that

$$M = -s_2(n) + 2 - L = 1 - L = -[\log_2 n] + 2\omega_2(n),$$

otherwise $\omega_2(n) = 0$ and

$$M = -s_2(n) + s_2(n) - L = -L = -[\log_2 n] + 2\omega_2(n).$$

Together with (6.1) this gives the result. □

7. Proofs of Theorems

Proof of Theorem 1.2. (a) We have two proofs either by Proposition 2.1 or by Proposition 4.6. (b) This is shown by Proposition 4.5. (c) This is given by Corollary 3.3. □

Proof of Theorem 1.3. (a) We have two proofs either by Proposition 2.2 or by Proposition 4.6. (b) This is given by Proposition 4.5. (c) We have two different methods. The first proof is derived by Proposition 4.5. Second proof: For even n we then obtain by Corollary 3.6 and Lemma 3.1 that

$$\widehat{\mathbf{F}}_n(x)/x - (n-1)\mathbf{F}_{n-1}(x) = 0$$

for $x = -\frac{1}{2}$. (d) For even n we get by Propositions 4.5 and 6.3 that

$$\operatorname{ord}_2 \widehat{\mathbf{F}}_n(-\tfrac{1}{2}) = -1 + \operatorname{ord}_2 \widetilde{\mathbf{G}}_n = -1 - \operatorname{ord}_2 n.$$

For odd n we derive by Propositions 4.5, 6.5, and Eq. (5.4) that

$$\begin{aligned}\operatorname{ord}_2 \widehat{\mathbf{F}}_n(-\tfrac{1}{2}) &= \operatorname{ord}_2 \left(\frac{1}{2} (\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_1)^n \right) = \operatorname{ord}_2 \left(\frac{1}{4} (\widetilde{\mathbf{G}}_0 + \widetilde{\mathbf{G}}_0)^{n+1} \right) \\ &= -1 - \operatorname{ord}_2(n+1) - [\log_2(n+1)] + 2\omega_2(n+1).\end{aligned}$$

If $n+1 = 2^r$ with $r \geq 1$, then the latter expression simplifies to $-1 - 2(r-1)$, since $\operatorname{ord}_2(n+1) = [\log_2(n+1)] = r$ and $\omega_2(n+1) = 1$ in that case; otherwise $\omega_2(n+1)$ vanishes. (e) This is a consequence of Propositions 3.4 and 3.5. (f) This is Corollary 3.6. □

To prove Theorem 1.5 we have to introduce some transformations. The Hadamard product ([6, pp. 85–86]) of two formal series

$$f(x) = \sum_{\nu \geq 0} a_\nu x^\nu, \quad g(x) = \sum_{\nu \geq 0} b_\nu x^\nu \tag{7.1}$$

is defined to be

$$(f \odot g)(x) = \sum_{\nu \geq 0} a_\nu b_\nu x^\nu.$$

For a sequence $(s_n)_{n \geq 0}$ its binomial transform $(s_n^*)_{n \geq 0}$ is defined by

$$s_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k s_k.$$

Since the inverse transform is also given as above, we have $(s_n^{**})_{n \geq 0} = (s_n)_{n \geq 0}$, see [10, p. 192]. The following transformation is due to Euler ([4, Ex. 3, p. 169]), where we prove a finite case.

Proposition 7.1. *If f, g are polynomials as defined in* (7.1)*, then the Hadamard product is given by*

$$(f \odot g)(x) = \sum_{\nu \geq 0} (-1)^\nu a_\nu^* \frac{g^{(\nu)}(x)}{\nu!} x^\nu. \tag{7.2}$$

Proof. We may assume that $f \cdot g \neq 0$. Let $N = \deg g$. Define $g_n(x) = \sum_{\nu=0}^{n} b_\nu x^\nu$ where $g_N(x) = g(x)$. We use induction on n up to N. For $n = 0$ we have

$$(f \odot g_0)(x) = a_0^* \, g_0(x) = a_0 b_0.$$

Now assume the result holds for $n \geq 0$. Since $g_{n+1}(x) = b_{n+1} x^{n+1} + g_n(x)$, we consider the difference of (7.2) for $n+1$ and n. Thus

$$\begin{aligned}(f \odot g_{n+1})(x) - (f \odot g_n)(x) &= \sum_{\nu=0}^{n+1} (-1)^\nu a_\nu^* \frac{b_{n+1}(n+1)_\nu \, x^{n+1-\nu}}{\nu!} x^\nu \\ &= b_{n+1} x^{n+1} \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} (-1)^\nu a_\nu^* \\ &= a_{n+1} b_{n+1} x^{n+1}\end{aligned}$$

showing the claim for $n+1$. □

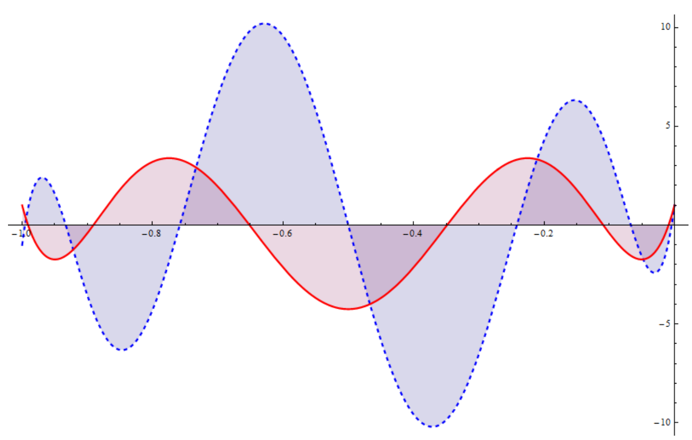
Proof of Theorem 1.5. The binomial transform

$$-\frac{1}{n} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \mathbf{H}_k \tag{7.3}$$

is well known, cf. [10, pp. 281–282]. Using Proposition 7.1 with $f(x) = \sum_{\nu=1}^{n} \mathbf{H}_\nu x^\nu$ and $g = \mathbf{F}_n$, where $f \odot g = \widehat{\mathbf{F}}_n$, gives the result by means of (7.3). □

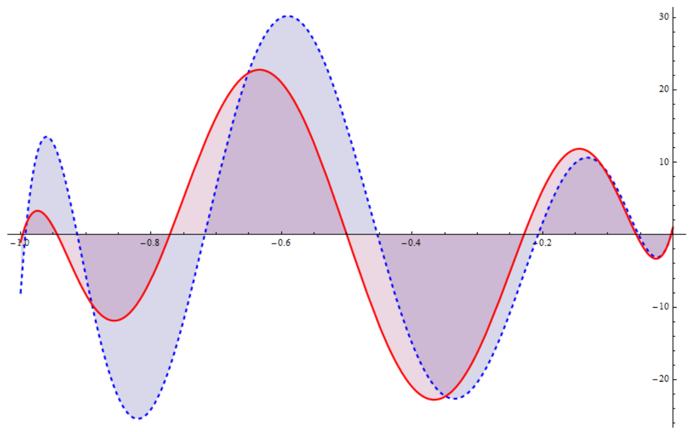
A. Figures

Figure A.1: Functions $\mathbf{F}_n$



$\mathbf{F}_n(x)/x$ with $n = 8$ (dashed blue line), $\mathbf{F}_n(x)/x$ with $n = 7$ (red line).

Figure A.2: Functions $\widehat{\mathbf{F}}_n$



$\widehat{\mathbf{F}}_n(x)/x$ with $n = 8$ (dashed blue line),
$\widehat{\mathbf{F}}_n(x)/x - (n-1)\mathbf{F}_{n-1}(x)$ with $n = 8$ (red line).

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