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9−MODULARITY AND GCD PROPERTIES OF GENERALIZED FIBONACCI NUMBERS

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Abstract

We study 9-modularity properties of generalized Fibonacci numbers that give rise to well-known quasigroups. In this paper we also study GCD and divisibility properties of generalized Fibonacci numbers. This study includes necessary and sufficient conditions to determine whether the GCD of two generalized Fibonacci numbers is a regular Fibonacci number.

1. Introduction

The *generalized Fibonacci sequence* $\{G_n(a,b)\}_{n\in\mathbb{N}}$ is defined recursively by $G_n(a,b) = G_{n-1}(a,b)+G_{n-2}(a,b)$ for $n \geq 3$, with initial conditions $G_1(a,b) = a$ and $G_2(a,b) = b$, where $a, b \in \mathbb{Z}$. If $a = b = 1$, then we obtain the regular Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$. If $a = 1$ and $b = 3$, then we obtain the Lucas sequence $\{L_n\}_{n\in\mathbb{N}}$.

The basic properties of Fibonacci numbers have been extensively studied. However, their modular arithmetic properties still have many open questions. For instance, if $k \in \mathbb{N}$ and p is a prime number, then what are the necessary and sufficient

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conditions on the integer n for which $F_n \equiv 0 \bmod p^k$? The conditions on n are known when $k = 1$ and $p \in \{2, 3, 5\}$; see [7, Ch. 16].

A natural question to ask for generalized Fibonacci numbers is: What are the necessary and sufficient conditions on n, and the integers a and b, for which $G_n(a, b) \equiv 0 \bmod p^k$? In this paper we answer this question for $p = 3$ and $k = 2$. Moreover, we discover that the complete characterization of the integers n, a and b for which $G_n(a, b) \equiv 0 \bmod 9$ gives rise to eight isotopic quasigroups and the subtraction quasigroup of order 6. In particular, we construct Latin squares whose entries are either integers modulo 12 or integers modulo 4. We border these Latin squares and add a binary operation to obtain nine quasigroups. We glue these quasigroups to obtain a 9×9 table that completely characterizes the 9-modularity of generalized Fibonacci numbers.

Those familiar with Hosoya triangle [4, 5, 6, 7] may find the results in this paper strongly connected with the results in [5]. The Hosoya triangle is a good tool to study the connection between the algebra and the geometry of Fibonacci numbers. For instance, Flórez and Junes [4] and Flórez, Higuita and Junes [5] have proved and conjectured several GCD properties in the Hosoya triangle and the generalized Hosoya triangle.

In this paper we also prove some new properties and identities for generalized Fibonacci numbers. Some of them are generalizations of well-known properties of regular Fibonacci numbers. We also study divisibility properties, and provide necessary and sufficient conditions to determine when the GCD of two generalized Fibonacci numbers is a regular Fibonacci number.

2. Preliminaries

In this section we give some known results. All parts of Lemma 2.1 are proved in [7]. Proposition 2.2 and Lemma 2.3 are proved in [8] and [4], respectively.

Lemma 2.1 ([7]). *If m and n are natural numbers, then*

(1) $\gcd(F_m, F_n) = F_{\gcd(m,n)}$,

(2) $F_m \mid F_n$ *if* $m \mid n$,

(3) $F_n \equiv 0 \bmod 3$ *if and only if* $n \equiv 0 \bmod 4$,

(4) $L_n \equiv 0 \bmod 3$ *if and only if* $n \equiv 2 \bmod 4$.

Proposition 2.2 ([8]). *If n is a natural number, then*

(1) $F_n \equiv 0 \bmod 9$ *if and only if* $n \equiv 0 \bmod 12$,

(2) $L_n \equiv 0 \bmod 9$ *if and only if* $n \equiv 6 \bmod 12$.

Lemma 2.3 ([4]). *Let* m, n, s *and* t *be positive integers. If* $|m-n|$ *and* $|s-t|$ *are in* $\{1, 2\}$*, then*

(1) $\gcd(F_m, F_n) = 1$,

(2) $\gcd(F_m F_s, F_n F_t) = \gcd(F_m, F_t) \gcd(F_s F_n) = F_{\gcd(m,t)} F_{\gcd(n,s)}$.

3. Generalized Fibonacci Congruences

In this section we study the necessary and sufficient conditions to determine whether a generalized Fibonacci number $G_n(a, b)$ is congruent to zero modulo nine. Those conditions depend on a, b and n. We summarize those results in Section 4 by constructing some finite quasigroups. So, we obtain a finite arithmetic that allows us to classify the generalized Fibonacci numbers that are divisible by 9. In particular, Propositions 3.1, 3.2, and 3.3 provide the complete classification.

We denote by $\{G_n(a, b)\}_{n\in\mathbb{N}}$ the *generalized Fibonacci sequence* with integers a and b. That is, $G_1(a, b) = a$, $G_2(a, b) = b$ and $G_n(a, b) = G_{n-1}(a, b) + G_{n-2}(a, b)$ for all $n \in \mathbb{N} \setminus \{1, 2\}$. It is clear that $G_n(1, 1) = F_n$ and $G_n(1, 3) = L_n$. When there is no ambiguity, we denote the nth term of the generalized Fibonacci sequence by G_n. The identity $G_n(a, b) = aF_{n-2} + bF_{n-1}$ is proved in [7], and it will play an important role in this paper.

Proposition 3.1. *Let* $G_n(a, b)$ *be a generalized Fibonacci number and let* t *and* j *be nonnegative integers. If* $a \equiv 2^t \bmod 9$ *and* $b \equiv 2^{t+j} \bmod 9$*, then*

$$G_n(a, b) \equiv 0 \bmod 9 \textit{ if and only if } n \equiv \begin{cases} 4k \bmod 12 & \textit{if } j = 2k, \\ 4k-1 \bmod 12 & \textit{if } j = 2k+1 \end{cases}$$

for some integer k.

Proof. Since $G_n(a, b) \equiv 2^t F_{n-2} + 2^{t+j} F_{n-1} \bmod 9$, and 2^t is not divisible by 9, we have

$$G_n(a, b) \equiv 0 \bmod 9 \iff F_{n-2} + 2^j F_{n-1} \equiv 0 \bmod 9 \qquad (3.1)$$

where j and t are nonnegative integers.

The sequence $\{2^j\}_{j\in\mathbb{Z}_{\geq 0}}$ has a period of 6 modulo 9. Therefore, it is enough to prove the proposition for $j = 0, 1, \ldots, 5$.

Case $j = 0$. This follows from (3.1) and Proposition 2.2 part (1).

Case $j = 1$. From (3.1) and the recursive definition of Fibonacci numbers we know that $G_n(a, b) \equiv 0 \bmod 9$ if and only if $F_{n-2} + 2F_{n-1} \equiv 0 \bmod 9$ if and only if $F_n + F_{n-1} \equiv 0 \bmod 9$. This and Proposition 2.2 part (1) imply that

$$G_n(a, b) \equiv 0 \bmod 9 \iff n + 1 \equiv 0 \bmod 12.$$

Case $j = 2$. From (3.1) we obtain that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $F_{n-2} + 2^2 F_{n-1} \equiv 0 \bmod 9$ if and only if $2F_{n-2} + 2^3 F_{n-1} \equiv 0 \bmod 9$ if and only if $2F_{n-2} - F_{n-1} \equiv 0 \bmod 9$. Since $F_{n-1} = F_{n-2} + F_{n-3}$ and $F_{n-2} = F_{n-3} + F_{n-4}$, we conclude that $2F_{n-2} - F_{n-1} \equiv 0 \bmod 9$ is equivalent to

$$[F_{n-2} + (F_{n-3} + F_{n-4})] - (F_{n-2} + F_{n-3}) \equiv 0 \bmod 9 \iff F_{n-4} \equiv 0 \bmod 9.$$

This and Proposition 2.2 part (1) imply that $G_n(a,b) \equiv 0 \bmod 9 \iff n - 4 \equiv 0 \bmod 12$.

Case $j = 3$. From (3.1) we know that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $F_{n-2} + 2^3 F_{n-1} \equiv 0 \bmod 9$ if and only if $F_{n-2} - F_{n-1} \equiv 0 \bmod 9$. This, the identity $F_{n-2} - F_{n-1} = -F_{n-3}$, and Proposition 2.2 part (1) imply that

$$G_n(a,b) \equiv 0 \bmod 9 \iff n - 3 \equiv 0 \bmod 12.$$

Case $j = 4$. From (3.1) we obtain that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $F_{n-2} + 2^4 F_{n-1} \equiv 0 \bmod 9$ if and only if $F_{n-2} - 2F_{n-1} \equiv 0 \bmod 9$. This, Proposition 2.2 part (2) and the identities

$$F_{n-2} - 2F_{n-1} = -(F_{n-1} + F_{n-3}), \text{ and } F_{n-1} + F_{n-3} = L_{n-2}$$

imply that

$$G_n(a,b) \equiv 0 \bmod 9 \iff n - 2 \equiv 6 \bmod 12.$$

Case $j = 5$. The proof of this case is similar to case $j = 4$. Therefore, we omit it. This proves the proposition. $\square$

Proposition 3.2. *Let $G_n(a,b)$ be a generalized Fibonacci number. Suppose that $t \in \mathbb{Z}_{\geq 0}$ and that $j = 0, 1$.*

(1) If $a \equiv 2^t \bmod 9$ and $b \equiv (-1)^{t+j} 3 \bmod 9$, then

$$G_n(a,b) \equiv 0 \bmod 9 \quad \textit{if and only if} \quad n \equiv 4j + 6 \bmod 12.$$

(2) If $a \equiv (-1)^{t+j} 3 \bmod 9$ and $b \equiv 2^t \bmod 9$, then

$$G_n(a,b) \equiv 0 \bmod 9 \quad \textit{if and only if} \quad n \equiv 4j + 5 \bmod 12.$$

(3) If $a \equiv 2^t j \bmod 9$ and $b \equiv 2^t(1 - j) \bmod 9$, then

$$G_n(a,b) \equiv 0 \bmod 9 \quad \textit{if and only if} \quad n \equiv j + 1 \bmod 12.$$

Proof. We prove part (1). We need some preliminaries. An easy argument by induction on n shows that $(-1)^n 3 \equiv 2^n 3 \bmod 9$. Therefore, $2^t F_{n-2} + (-1)^{t+j} 3F_{n-1} \equiv 0 \bmod 9$ if and only if $2^t F_{n-2} + 2^{t+j} 3F_{n-1} \equiv 0 \bmod 9$, where $j = 0, 1$ and t is a nonnegative integer. Since $G_n(a,b) \equiv 2^t F_{n-2} + (-1)^{t+j} 3F_{n-1} \bmod 9$ and 2^t is not divisible by 9, we have

$$G_n(a,b) \equiv 0 \bmod 9 \iff F_{n-2} + 2^j\, 3F_{n-1} \equiv 0 \bmod 9. \tag{3.2}$$

We now prove part (1) for all cases of j.

Case $j = 0$. From (3.2) and the identity $F_{n-2} + 3F_{n-1} = F_{n+1} + F_{n-1}$ we obtain that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $F_{n+1} + F_{n-1} \equiv 0 \bmod 9$. This, the identity $F_{n+1} + F_{n-1} = L_n$, and Proposition 2.2 part (2) imply that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $n \equiv 6 \bmod 12$.

Case $j = 1$. From (3.2) we obtain that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $F_n + 5F_{n-1} \equiv 0 \bmod 9$. So,

$$\begin{aligned} G_n(a,b) \equiv 0 \bmod 9 \quad &\iff F_{n+1} + 4F_{n-1} \equiv 0 \bmod 9 \\ &\iff 4(F_{n-1} - 2F_{n+1}) \equiv 0 \bmod 9 \\ &\iff -F_n - F_{n+1} = -F_{n+2} \equiv 0 \bmod 9. \end{aligned}$$

This and Proposition 2.2 part (1) imply that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $n + 2 \equiv 0 \bmod 12$. This proves part (1).

The proof of part (2) follows from Proposition 2.2 part (2) and the fact that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $2^j\, 3F_{n-2} + F_{n-1} \equiv 0 \bmod 9$, where $j = 0, 1$. Therefore, we omit the details.

We prove part (3) case $j = 0$, the proof of part (3) case $j = 1$ is similar and we omit it. Since $a \equiv 0 \bmod 9$, we conclude that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $2^t F_{n-1} \equiv 0 \bmod 9$. Thus, $G_n(a,b) \equiv 0 \bmod 9$ if and only if $F_{n-1} \equiv 0 \bmod 9$. This and Proposition 2.2 part (1) imply that $G_n(a,b) \equiv 0 \bmod 9$ if and only if $n - 1 \equiv 0 \bmod 12$. This proves part (3). □

Proposition 3.3. *Let $G_n(a,b)$ be a generalized Fibonacci number and let $j \in \{0,1\}$.*

(1) If $a \equiv b \equiv \pm 3 \bmod 9$, then

$$G_n(a,b) \equiv 0 \bmod 9 \quad \textit{if and only if} \quad n \equiv 0 \bmod 4.$$

(2) If $a \equiv \pm 3 \bmod 9$ and $b \equiv \mp 3 \bmod 9$, then

$$G_n(a,b) \equiv 0 \bmod 9 \quad \textit{if and only if} \quad n \equiv 3 \bmod 4.$$

(3) If $a \equiv \pm 3 \cdot j \bmod 9$ and $b \equiv \pm 3 \cdot (1-j) \bmod 9$, then

$$G_n(a,b) \equiv 0 \bmod 9 \quad \textit{if and only if} \quad n \equiv j + 1 \bmod 4.$$

Proof. We first prove part (1). Since $a \equiv b \equiv \pm 3 \bmod 9$, we conclude that

$$G_n(a,b) \equiv 0 \bmod 9 \text{ if and only if } \pm 3(F_{n-2} + F_{n-1}) \equiv 0 \bmod 9.$$

Therefore, $F_{n-2} + F_{n-1} = F_n \equiv 0 \bmod 3$. This and Lemma 2.1 part (3) imply that $G_n(a, b) \equiv 0 \bmod 9$ if and only if $n \equiv 0 \bmod 4$.

We now prove part (2) case $a \equiv 3 \bmod 9$ and $b \equiv -3 \bmod 9$, the proof of case $a \equiv -3 \bmod 9$ and $b \equiv 3 \bmod 9$ is similar and we omit it. It is easy to see that $G_n(a, b) \equiv 0 \bmod 9$ is equivalent to $3(F_{n-2} + 2F_{n-1}) \equiv 0 \bmod 9$. Thus,

$$\begin{aligned} G_n(a,b) \equiv 0 \bmod 9 \quad &\iff F_{n-2} + 2F_{n-1} \equiv 0 \bmod 3 \\ &\iff F_n + F_{n-1} = F_{n+1} \equiv 0 \bmod 3. \end{aligned}$$

This and Lemma 2.1 part (3) imply that $G_n(a, b) \equiv 0 \bmod 9$ if and only if $n \equiv 3 \bmod 4$.

We prove part (3) case $a \equiv 0 \bmod 9$ and $b \equiv 3 \bmod 9$, the proof of other cases are similar and we omit them.

Since $a \equiv 0 \bmod 9$ and $b \equiv 3 \bmod 9$, we conclude that $G_n(a, b) \equiv 0 \bmod 9$ if and only if $0F_{n-2} + 3F_{n-1} \equiv 0 \bmod 9$ if and only if $F_{n-1} \equiv 0 \bmod 3$. This and Lemma 2.1 part (3) imply that $G_n(a, b) \equiv 0 \bmod 9$ if and only if $n \equiv 1 \bmod 4$. □

We would like to mention that the proofs of Propositions 3.1, 3.2, 3.3 can be also derived from the fact that the restricted period of the Fibonacci sequence modulo 9 has a maximal length 12. (See for example [2, 3].)

4. Latin squares

In this section we study Latin squares and quasigroups to construct a table (see Table 3) that summarizes Propositions 3.1, 3.2 and 3.3. In [1] there is a complete study of Latin squares and quasigroups. Van Lint and Wilson [9, Chs. 17 and 22] have two chapters dedicated to the study of Latin squares.

We start this section with the following fact taken from [1, Ch. 1]: If $n \in \mathbb{N}$ and $Q = \{0, 1, \ldots, n-1\}$, then the binary operation "$*$" defined on Q as

$$a * b = ha + kb + l,$$

where the addition is modulo n and h, k and l are fixed integers with h and k prime to n, defines a quasigroup on the set Q.

Our goal is to construct quasigroups from Propositions 3.1, 3.2 and 3.3. The following discussion leads to this.

The entries of Table 1 part (a) form a Latin Square of size 6×6. These entries correspond to numbers calculated modulo 12 and have the form $4k$ or $4k - 1$. The numbers located in the leftmost column and uppermost row of this table are the b's and a's, respectively, of the generalized Fibonacci numbers such that

$$G_n(a, b) \equiv 0 \bmod 9 \iff n \equiv x_{a,b} \bmod 12,$$

where $x_{a,b}$ corresponds to the entry of Table 1 part (a) that lies in the intersection of the column labeled by a and row labeled by b. In addition, the a's and b's of Table 1 part (a) are powers of 2 modulo 9. That is,

$$\begin{array}{lll} 2^0 \equiv 1 \bmod 9; & 2 \equiv 2 \bmod 9; & 2^2 \equiv 4 \bmod 9; \\ 2^3 \equiv 8 \bmod 9; & 2^4 \equiv 7 \bmod 9; & 2^5 \equiv 5 \bmod 9. \end{array}$$

Notice that $\{1, 2, 4, 5, 7, 8\}$ is a complete system of residues on base 2 modulo 9. That is, for every $j \in \mathbb{Z}_{\geq 0}$, $2^j \equiv r \bmod 9$ for some $r \in \{1, 2, 4, 5, 7, 8\}$. Therefore, Table 1 part (a) summarizes Proposition 3.1. We use these observations to prove the following proposition.

Proposition 4.1. *Proposition 3.1 gives rise to a quasigroup isotopic to the quasigroup $Q = \{0, 1, \ldots, 5\}$ with binary operation $a * b = a - b$.*

Proof. Since none of the elements $\{0, 7, 8, 3, 4, 11\}$ of Table 1 part (a) occur twice within any row or column, it is clear that the entries of Table 1 part (a) form a Latin Square. To avoid confusion, we rename the elements $0, 7, 8, 3, 4, 11$ as q_1, q_2, q_3, q_4, q_5 and q_6, respectively, to form the entries of Table 1 part (b). If we bordered this Latin Square with q_1, q_2, q_3, q_4, q_5 and q_6 as shown in Table 1 part (b), then we obtain the multiplication table of a quasigroup. This proves that Proposition 3.1 give rise to a quasigroup.

We now prove that the quasigroup defined by Table 1 part (b) is isotopic to the quasigroup $Q = \{0, 1, \ldots, 5\}$ with binary operation $a * b = a - b$, where the subtraction is taken modulo 6. Dénes and Keedwell [1] prove that Q is a quasigroup. It is easy to see that the multiplication table of Q is given by

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

We now define three injections τ, η and ζ from $\{q_1, q_2, q_3, q_4, q_5, q_6\}$ into Q as follows:

$$\begin{array}{llllll} \tau(q_1) = 0, & \tau(q_2) = 1, & \tau(q_3) = 2, & \tau(q_4) = 3, & \tau(q_5) = 4, & \tau(q_6) = 5; \\ \eta(q_1) = 0, & \eta(q_2) = 1, & \eta(q_3) = 2, & \eta(q_4) = 3, & \eta(q_5) = 4, & \eta(q_6) = 5; \\ \zeta(q_1) = 0, & \zeta(q_2) = 5, & \zeta(q_3) = 4, & \zeta(q_4) = 3, & \zeta(q_5) = 2, & \zeta(q_6) = 1. \end{array}$$

That is, ζ operates on the elements of the Latin Square of Table 1 part (b), while τ and η operate on the borders. A straightforward computation shows that

$$\tau(q_i) - \eta(q_j) = \zeta(q_i * q_j) \text{ for all } i, j \in \{1, \ldots, 6\}.$$

Thus, the quasigroup defined by Table 1 part (b) is isotopic to Q. This completes the proof. $\square$

$b \backslash a$	1	2	4	8	7	5
1	0	7	8	3	4	11
2	11	0	7	8	3	4
4	4	11	0	7	8	3
8	3	4	11	0	7	8
7	8	3	4	11	0	7
5	7	8	3	4	11	0

(a)

$*$	q_1	q_2	q_3	q_4	q_5	q_6
q_1	q_1	q_2	q_3	q_4	q_5	q_6
q_2	q_6	q_1	q_2	q_3	q_4	q_5
q_3	q_5	q_6	q_1	q_2	q_3	q_4
q_4	q_4	q_5	q_6	q_1	q_2	q_3
q_5	q_3	q_4	q_5	q_6	q_1	q_2
q_6	q_2	q_3	q_4	q_5	q_6	q_1

(b)

Table 1: Subtraction quasigroup.

We now analyze Proposition 3.2 parts (1) and (2) and Proposition 3.3 parts (1) and (2). These propositions, with their corresponding parts, are summarized in Table 2. Indeed, the entries of all Latin squares in Table 2 (a)–(g) correspond to numbers calculated modulo 12, the entries of Table 2 (h) are calculated modulo 4. The numbers located in the leftmost column and uppermost row of the first seven tables are the b's and a's, respectively, of the generalized Fibonacci numbers such that

$$G_n(a,b) \equiv 0 \bmod 9 \text{ if and only if } n \equiv x_{a,b} \bmod 12,$$

where $x_{a,b}$ corresponds to the entry of Table 2 (a)–(g) that lies in the intersection of the column labeled by a and row labeled by b. This is given by Proposition 3.2 parts (1) and (2).

The numbers located in the leftmost column and uppermost row of the last table in Table 2 part (h) are the b's and a's, respectively, of the generalized Fibonacci numbers such that

$$G_n(a,b) \equiv 0 \bmod 9 \text{ if and only if } n \equiv x_{a,b} \bmod 4,$$

where $x_{a,b}$ corresponds to the entry of Table 2 (h) that lies in the intersection of the column labeled by a and row labeled by b. This is given by Proposition 3.3 parts (1) and (2).

Using a similar argument as in Proposition 4.1, we can see that all tables in Table 2 give rise to eight quasigroups. Since there is only one quasigroup of order 2 up to isotopy, the proof of the following proposition is straightforward.

Proposition 4.2. *Proposition 3.2 parts (1) and (2) and Proposition 3.3 parts (1) and (2) give rise to eight isotopic quasigroups.*

(a)

	1	2
3	6	10
6	10	6

(b)

	4	8
3	10	6
6	6	10

(c)

	7	5
3	6	10
6	10	6

(d)

	1	2
3	6	10
6	10	6

(e)

	3	6
1	5	9
2	9	5

(f)

	3	6
4	5	9
8	9	5

(g)

	3	6
7	5	9
5	9	5

(h)

	3	6
3	0	3
6	3	0

Table 2: Isotopic quasigroups of order 2.

We "glue" Tables 1 and 2 to construct a new table. (See Table 3.) Propositions 3.1, 3.2 and 3.3 are summarized by Table 3. However, reading this new table requires some precaution. That is, the upper row and the leftmost column in Table 3 correspond to a's and b's in Propositions 3.2 and 3.3. The first five 2's in the lowermost row and the first five 1's on the rightmost column in Table 3, correspond to $x_{a,b}$ modulo 12. The last three numbers in the bottommost row and the last three numbers in the rightmost column correspond to $x_{a,b}$ modulo 4. Therefore, we obtain that the entries in the bottommost square on the right of the Table 3 correspond to numbers calculated modulo 4. The other entries of the Table 3 correspond to numbers calculated modulo 12.

For instance, if $a \equiv 5 \bmod 9$ and $b \equiv 4 \bmod 9$, then from Table 3 we obtain

$$G_n(a, b) \equiv 0 \bmod 9 \iff n \equiv 3 \bmod 12.$$

Similarly, if $a \equiv 6 \bmod 9$ and $b \equiv 3 \bmod 9$, then from Table 3 we obtain

$$G_n(a, b) \equiv 0 \bmod 9 \iff n \equiv 3 \bmod 4.$$

Notice that the entry corresponding to $a = 9$ and $b = 9$ in Table 3 has been left blank. This is done on purpose to indicate that, if $a \equiv 0 \bmod 9$ and $b \equiv 0 \bmod 9$, then $G_n(a, b) \equiv 0 \bmod 9$ for every $n \in \mathbb{N}$.

5. GCD Properties of Generalized Fibonacci Numbers

In this section we give necessary and sufficient conditions to determine whether the GCD of two generalized Fibonacci numbers is a Fibonacci number. These results have some important applications to the geometry of the star of David in the generalized Hosoya triangle (see [5]).

We start this section with three preliminary results. Lemma 5.1 is a technical result that will be used in Theorem 5.4, the main theorem of this section.

$b \backslash a$	1	2	4	8	7	5	3	6	9
1	0	7	8	3	4	11	5	9	1
2	11	0	7	8	3	4	9	5	1
4	4	11	0	7	8	3	5	9	1
8	3	4	11	0	7	8	9	5	1
7	8	3	4	11	0	7	5	9	1
5	7	8	3	4	11	0	9	5	1
3	6	10	6	10	6	10	0	3	1
6	10	6	10	6	10	6	3	0	1
9	2	2	2	2	2	2	2	2	

Table 3:

Lemma 5.1. *If k, n, and w are natural numbers, then*

(1) $$G_n(a,b) = \begin{cases} (a-b)F_{n-2} + bF_{wk}, & \textit{if } n = wk, \\ (a+b)F_{n-1} - aF_{wk}, & \textit{if } n-3 = wk, \\ aF_{n+1} + (b-2a)F_{n-1}, & \textit{for all } n. \end{cases}$$

(2) $$G_{n+w}(a,b) = \begin{cases} (a-b)F_{n+w-2} + bF_{w(k+1)}, & \textit{if } n = wk, \\ aF_{n+w-2} + bF_{w(k+1)}, & \textit{if } n-1 = wk, \\ aF_{w(k+1)} + bF_{n-1+w}, & \textit{if } n-2 = wk, \\ (a+b)F_{n-1+w} - aF_{w(k+1)}, & \textit{if } n-3 = wk, \\ aF_{n+1+w} + (b-2a)F_{n-1+w}, & \textit{for all } n. \end{cases}$$

Proof. We know that

$$G_n(a,b) = aF_{n-2} + bF_{n-1}. \tag{5.1}$$

From (5.1) it is easy to see that

$$G_n(a,b) = aF_{n-2} + b(F_n - F_{n-2}) = (a-b)F_{n-2} + bF_n. \tag{5.2}$$

We prove part (1). The case $n = wk$ follows directly from 5.2. If $n-3 = wk$, then using (5.1) we can write

$$\begin{aligned} G_n(a,b) &= a(F_{n-1} - F_{n-3}) + bF_{n-1} = (a+b)F_{n-1} - aF_{n-3} \\ &= (a+b)F_{n-1} - aF_{wk}. \end{aligned} \tag{5.3}$$

If $n \in \mathbb{N}$, then using (5.1) we can write

$$\begin{aligned} G_n(a,b) &= a(F_n - F_{n-1}) + bF_{n-1} = a((F_{n+1} - F_{n-1}) - F_{n-1}) + bF_{n-1} \\ &= aF_{n+1} + (b-2a)F_{n-1}. \end{aligned}$$

This proves part (1).

We now prove part (2). If we replace n by $n+w$ in (5.2), and use that $n = wk$, then we obtain part (2) case $n = wk$. If $n-1 = wk$, then from (5.1) we get that

$$G_{n+w}(a,b) = aF_{n+w-2} + bF_{n-1+w} = aF_{n+w-2} + bF_{wk+w} = aF_{n+w-2} + bF_{w(k+1)}.$$

If $n-2 = wk$, then using again (5.1) it is easy to see that

$$G_{n+w}(a,b) = aF_{n-2+w} + bF_{n-1+w} = aF_{wk+w} + bF_{n-1+w} = aF_{w(k+1)} + bF_{n-1+w}.$$

If we replace n by $n+w$ in (5.3), and use that $n-3 = wk$, then we can easily obtain part (2) case $n-3 = wk$. If $n \in \mathbb{N}$, then replacing n by $n+w$ in the last case of part (1) we obtain the last case of part (2). This proves part (2). $\square$

Lemma 5.2 ([5, Lemma 3]). *If* $\gcd(a,b) = d$*,* $a' = a/d$ *and* $b' = b/d$*, then* $G_n(a,b) = dG_n(a',b')$ *for* $n \in \mathbb{N}$*.*

Proof. We know that $G_n(a,b) = aF_{n-2} + bF_{n-1}$ for all $n \in \mathbb{N}$. Thus,

$$G_n(a,b) = da'F_{n-2} + db'F_{n-1} = d(a'F_{n-2} + b'F_{n-1}) = dG_n(a',b').$$

This proves the Lemma. $\square$

Theorem 5.3 ([5, Theorem 5]). *Let* $d = \gcd(a,b)$*. If* n *and* w *are natural numbers, then* $\gcd(G_n, G_{n+w}) \mid d\,F_w$*. Moreover, if* $w = 1, 2$*, then* $\gcd(G_n, G_{n+w}) = d$*.*

Theorem 5.4. *Let* $n \in \mathbb{N}$*,* $a' = a/d$*,* $b' = b/d$ *where* $d = \gcd(a,b)$*.*

(1) If $w \in \{3,4,5\}$ *and* $\gcd\left(G_n(a,b), G_{n+w}(a,b)\right) = dF_w$*, then at least one of the following statements is true:*

(i) $a' \equiv b' \bmod F_w$ *and* $n \equiv 0 \bmod w$*,*

(ii) $a' \equiv 0 \bmod F_w$ *and* $n \equiv 1 \bmod w$*,*

(iii) $b' \equiv 0 \bmod F_w$ *and* $n \equiv 2 \bmod w$*,*

(iv) $b' \equiv -a' \bmod F_w$ *and* $n \equiv 3 \bmod w$*,*

(v) $w = 5$*,* $b' \equiv 2a' \bmod F_w$ *and* $n \equiv 4 \bmod w$*.*

(2) If one of the statements (i)–(v) from part (1) holds for any $w \in \mathbb{N}$*, then*

$$\gcd\left(G_n(a,b), G_{n+w}(a,b)\right) = dF_w.$$

Proof. We prove part (1); suppose that $\gcd(G_n(a,b), G_{n+w}(a,b)) = dF_w$ for $w \in \{3,4,5\}$. Lemma 5.2 shows that $\gcd(G_n(a,b), G_{n+w}(a,b)) = dF_w$ is equivalent to

$$\gcd(G_n(a',b'), G_{n+w}(a',b')) = F_w.$$

Therefore, we can assume $\gcd(G_n(a',b'), G_{n+w}(a',b')) = F_w$ and prove that at least one of the statements (i)–(v) holds.

It is known that if n is a natural number, then there is $0 \le r < w$ such that $n \equiv r \bmod w$. Therefore, there is $k \in \mathbb{N}$ such that $wk = n - r$. This and Lemma 2.1 part (2) imply that

$$F_w \mid F_{n-r}. \tag{5.4}$$

We analyze all occurrences of r where $w \in \{3,4,5\}$. That is, $r \in \{0,\dots,4\}$.

`Case 1.` $r = 0$. Thus, $wk = n$. Therefore, Lemma 5.1 part (1) shows that $G_n(a',b') = (a'-b')F_{n-2} + b'F_{wk}$. Since $F_w \mid G_n(a',b')$ and $F_w \mid b'F_{wk}$,

$$F_w \mid (a'-b')F_{n-2}. \tag{5.5}$$

Lemma 2.3 part (1) shows that $\gcd(F_n, F_{n-2}) = 1$. This and (5.4) imply that $\gcd(F_w, F_{n-2}) = 1$. Thus, (5.5) implies that $F_w \mid (a'-b')$. Therefore, $a' \equiv b' \bmod F_w$. This shows part (i).

`Case 2.` $r = 1$. The hypothesis and (5.4) imply that $F_w \mid G_n(a',b') = (a'F_{n-2} + b'F_{n-1})$ and $F_w \mid F_{n-1}$. Therefore,

$$F_w \mid a'F_{n-2}. \tag{5.6}$$

Lemma 2.3 part (1) shows that $\gcd(F_{n-1}, F_{n-2}) = 1$. This and (5.4) imply that $\gcd(F_w, F_{n-2}) = 1$. Thus, (5.6) implies that $F_w \mid a'$. So, $a' \equiv 0 \bmod F_w$. This shows part (ii).

`Case 3.` $r = 2$. Thus, $wk = n-2$. Since $G_n(a',b') = a'F_{n-2} + b'F_{n-1}$ and $F_w \mid G_n(a',b')$, we have $F_w \mid (b'F_{n-1} + a'F_{wk})$. It is clear that $F_w \mid a'F_{wk}$. Thus, $F_w \mid b'F_{n-1}$. Using again Lemma 2.3 part (1) and (5.4), we can conclude that $F_w \mid b'$. So, $b' \equiv 0 \bmod F_w$. This shows part (iii).

`Case 4.` $r = 3$. Thus, $wk = n-3$. From Lemma 5.1 part (1) we have

$$G_n(a',b') = (a'+b')F_{n-1} - a'F_{wk}.$$

Since $F_w \mid G_n(a',b')$ and $F_w \mid a'F_{wk}$, we have $F_w \mid (a'+b')F_{n-1}$. Using Lemma 2.3 part (1) and (5.4) we obtain that $F_w \mid (a'+b')$. This proves part (iv).

`Case 5.` $r = 4$. Therefore, $n \equiv 4 \bmod w$. If $w = 3$ or $w = 4$, then statements (ii) and (i) are true by the proof of cases $r = 1$ and $r = 0$, respectively. We now suppose that $w = 5$, and we prove $b' \equiv 2a' \bmod F_w$. It is easy to see that $n+1 \equiv 0 \bmod w$. Therefore, there is a $k \in \mathbb{N}$ such that $n+1 = wk$. This and Lemma 2.1 part (2) imply that $F_w \mid F_{n+1}$. From Lemma 2.3 we have $\gcd(F_{n-1}, F_{n+1}) = 1$. Therefore, it is easy to see that $\gcd(F_{n-1}, F_w) = 1$. Thus,

$$F_w \mid F_{n+1} \text{ and } \gcd(F_{n-1}, F_w) = 1. \tag{5.7}$$

From $F_w \mid G_n(a', b')$ and Lemma 5.1 part (1) we conclude that

$$F_w \mid (a'F_{n+1} + (b' - 2a')F_{n-1}).$$

This and (5.7) imply that $F_w \mid (b' - 2a')$; thus, $b' \equiv 2a' \mod F_w$. This shows part (v).

We now prove part (2). Using Lemma 5.2 we only have to prove that

$$F_w = \gcd(G_n(a', b'), G_{n+w}(a', b')).$$

From Theorem 5.3 we know that $\gcd(G_n(a', b'), G_{n+w}(a', b'))$ divides F_w. Therefore, to complete the proof of part (2) it is enough to prove that

$$F_w \mid G_n(a', b') \text{ and } F_w \mid G_{n+w}(a', b').$$

We suppose that (1) part (i) holds. That is, $n \equiv 0 \mod w$ and $a' \equiv b' \mod F_w$. Therefore, there is a $k \in \mathbb{N}$ such that $wk = n$ and $F_w \mid (a' - b')$. From Lemma 2.1 part (2) we obtain that F_w divides both F_{wk} and $F_{w(k+1)}$. Thus,

$$F_w \mid (a' - b')F_{n-2} + bF_{wk} \quad \text{ and } \quad F_w \mid (a' - b')F_{n+w-2} + b'F_{w(k+1)}.$$

These and Lemma 5.1 parts (1) and (2) imply that

$$F_w \mid G_n(a', b') \quad \text{ and } \quad F_w \mid G_{n+w}(a', b').$$

We suppose that (1) part (ii) holds. That is, $n \equiv 1 \mod w$ and $a' \equiv 0 \mod F_w$. Therefore, there is a $k \in \mathbb{N}$ such that $n-1 = wk$ and $F_w \mid a'$. These and Lemma 2.1 part (2) imply that F_w divides both $a'F_{n-2}$ and $b'F_{n-1}$. Therefore, $F_w \mid a'F_{n-2} + b'F_{n-1} = G_n(a', b')$.

Since $F_w \mid a'$ and $F_w \mid F_{w(k+1)}$, we obtain $F_w \mid a'F_{n+w-2} + b'F_{w(k+1)}$. Therefore, by Lemma 5.1 part (2) we get that $F_w \mid G_{n+w}(a', b')$. This proves that

$$F_w \mid G_n(a', b') \quad \text{ and } \quad F_w \mid G_{n+w}(a', b').$$

We suppose that (1) part (iii) holds. This proof is similar to the previous case and we omit it.

We suppose that (1) part (iv) holds. That is, $n \equiv 3 \mod w$ and $b' \equiv -a' \mod F_w$. Therefore, there is a $k \in \mathbb{N}$ such that $n - 3 = wk$ and that $F_w \mid (a' + b')$. These and Lemma 2.1 part (2) imply that F_w divides both $(a' + b')F_{n-1}$ and $a'F_{wk}$. Thus, $F_w \mid (a' + b')F_{n-1} - a'F_{wk}$. This and Lemma 5.1 part (1) imply that F_w divides $G_n(a', b')$.

Since $F_w \mid (a'+b')$ and $F_w \mid F_{w(k+1)}$, we obtain $F_w \mid (a'+b')F_{n-1+w} - a'F_{w(k+1)}$. Therefore, by Lemma 5.1 part (2) we get that $F_w \mid G_{n+w}(a',b')$. This proves that

$$F_w \mid G_n(a',b') \quad \text{and} \quad F_w \mid G_{n+w}(a',b').$$

We suppose that (1) part (v) holds for $w = 5$. That is, $n \equiv 4 \bmod 5$ and $b' \equiv 2a' \bmod F_5$. Therefore, there is a $k \in \mathbb{N}$ such that $n+1 = 5k$ and $F_5 \mid (b - 2a')$. From Lemma 2.1 part (2), we know that $F_5 \mid F_{5k}$. That is, $F_5 \mid F_{n+1}$. Since $F_5 \mid (b' - 2a')$ and $F_5 \mid F_{n+1}$, we have $F_5 \mid (a'F_{n+1} + (b' - 2a')F_{n-1})$. This and Lemma 5.1 part (1) imply that $F_5 \mid G_n(a',b')$.

We now prove that $F_5 \mid G_{n+w}(a',b')$. Since $n+1 = 5k$ and $F_5 \mid F_{5k+5}$, we have $F_5 \mid F_{n+1+5}$. Therefore, $F_5 \mid (a'F_{n+1+5} + (b' - 2a')F_{n-1+5})$. This and Lemma 5.1 part (2) imply that $F_5 \mid G_{n+5}(a',b')$. Therefore,

$$F_w \mid G_n(a',b') \quad \text{and} \quad F_w \mid G_{n+w}(a',b').$$

This proves part (2). □

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