A DIVISIBILITY OBSTRUCTION FOR CERTAIN WALKS ON GAUSSIAN INTEGERS

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Abstract

A number of authors have studied several interesting questions concerning walks in the ring of Gaussian integers $\mathbb{Z}[i]$ and other rings. In this article, we investigate a phenomenon related to walks of unit step jumps in the ring of Gaussian integers that exhibit a divisibility obstruction. Our study is motivated in part by the following elementary question: Consider a finite or infinite sequence of Gaussian integers z_j such that $|z_{j+1} - z_j| = 1$ for all j and a pair of indices j_1 and j_2 for which $z_{j_2} - z_{j_1} = n$, where n is a positive integer. Can one find a pair of indices j_3 and j_4 such that $z_{j_4} - z_{j_3} = k$ for any $1 \leq k \leq n$? We discover that the answer is no. Furthermore, we show that our result on chains of Gaussian integers possessing unit step jumps generalizes to chains of elements in a vector space.

1. Introduction and Statement of Results

Several interesting questions concerned with walks in the ring of Gaussian integers $\mathbb{Z}[i]$ and other rings have been considered by a number of authors. We shall use a geometric representation of the complex number $z = a + ib$ as the point with Cartesian coordinates (a, b), referred to orthogonal axes. This representation is known as the Argand diagram. The number a is called the real part of the complex number z, the number b is called the imaginary part, the x-axis in the Argand diagram is called the real axis, and the y-axis is called the imaginary axis. Using this geometric representation of complex numbers, we find that the Gaussian primes $\pm 1 \pm i, \pm 1 \pm 2i, \pm 2 \pm i, \pm 3 \pm 3i, \pm 2 \pm 3i, \pm 3 \pm 2i, \pm 4 \pm i, \pm 1 \pm 4i, \pm 5 \pm 2i, \pm 2 \pm 5i, \ldots$ make an interesting pattern.

Motzkin and Gordon posed the question of whether one can start in the vicinity of the origin of the complex plane and walk to infinity using the Gaussian primes as stepping stones and only taking steps of bounded length. (See Problem A16 in Guy's well-known problem book on number theory [1].) Jordan and Rabung [2] showed that steps of length at least 4 would be required to make the journey. They proved the existence of a region of composites of width $\sqrt{10}$ that completely surrounds the origin.

Gethner and Stark [3] showed that, starting anywhere in the complex plane and taking steps of length at most 2, one cannot reach infinity. Gethner, Wagon, and Wick [4] provided an explicit construction of such a region of composites of sizes 4 and $\sqrt{18}$, described a computational proof that a region of composites of width $\sqrt{26}$ exists, and proved that there is no line in the complex plane along which one can walk to infinity.

Vardi [6] examined the question of whether there exists an unbounded walk of bounded step size along Gaussian primes. He constructed a random model of Gaussian primes and showed that an unbounded walk of step size $k\sqrt{\log|z|}$ at z exists with probability 1 if $k > \sqrt{2\pi\lambda_c}$, and does not exist with probability 1 if $k < \sqrt{2\pi\lambda_c}$, where $\lambda_c \approx 0.35$ is a constant in continuum percolation. Finally, the question of whether there exists an infinite sequence of pairwise different Gaussian primes with bounded difference was also studied by Loh [5].

In the present article, we investigate a phenomenon related to walks of unit step jumps in the ring of Gaussian integers $\mathbb{Z}[i]$ that exhibit a divisibility obstruction, which can then be generalized to chains of elements in a vector space. Suppose we have a finite or infinite sequence of Gaussian integers z_j such that $|z_{j+1} - z_j| = 1$ for all j. Suppose further that we have a pair of indices j_1 and j_2 for which $z_{j_2} - z_{j_1}$ belongs to $\mathbb{Z}$; say $z_{j_2} - z_{j_1} = n$ with $n \geq 1$. One may ask the question whether this implies that there exists a pair of indices j_3 and j_4 such that $z_{j_4} - z_{j_3} = k$ for any $1 \leq k \leq n$. The answer turns out to be that this is not the case.

Let us consider the following finite sequence (see Figure 1):

$$\begin{aligned}
&z_0 = 0, z_1 = i, z_2 = 2i, z_3 = 3i, z_4 = 4i, z_5 = 1+4i, z_6 = 2+4i,\\
&z_7 = 3+4i, z_8 = 4+4i, z_9 = 4+3i, z_{10} = 4+2i, z_{11} = 4+i, z_{12} = 4,\\
&z_{13} = 4-i, z_{14} = 4-2i, z_{15} = 5-2i, z_{16} = 6-2i, z_{17} = 6-i, z_{18} = 6,\\
&z_{19} = 6+i, z_{20} = 6+2i, z_{21} = 7+2i, z_{22} = 8+2i, z_{23} = 8+i, z_{24} = 8,\\
&z_{25} = 8-i, z_{26} = 8-2i, z_{27} = 8-3i, z_{28} = 8-4i, z_{29} = 9-4i,\\
&z_{30} = 10-4i, z_{31} = 11-4i, z_{32} = 12-4i, z_{33} = 12-3i, z_{34} = 12-2i,\\
&z_{35} = 12-i, z_{36} = 12.
\end{aligned}$$

Here, we note that $z_{36} - z_0 = 12$. One can check that the set of differences

$$\mathcal{A} = \{z_{j'} - z_j \colon 0 \leq j, j' \leq 36\}$$

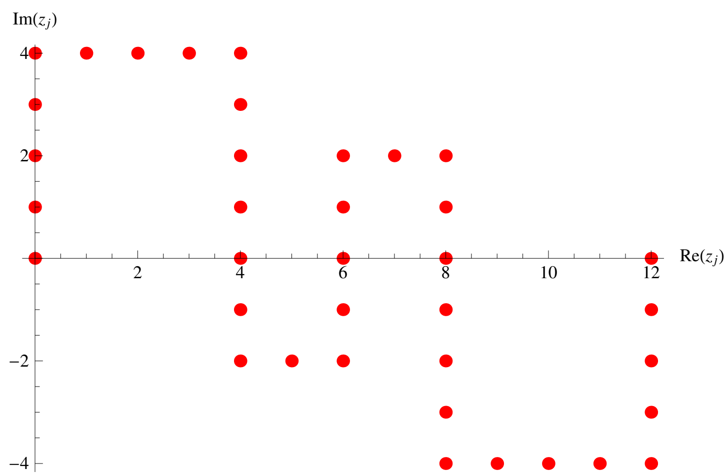

Figure 1: A finite sequence of 37 Gaussian integers z_j.

contains the positive integers $1, 2, 3, 4$, and 6. However, the set $\mathcal{A}$ does not contain the positive integer 5. This leads us to speculate that divisibility may play an important role here. The main goal of the present article is to prove that one does have such a divisibility obstruction in this problem. Namely, if the set of differences $\mathcal{A}$ contains a nonzero $n \in \mathbb{Z}$, then it contains all the divisors of n in $\mathbb{Z}$.

Theorem 1. *Let $(z_j)_{j\in\mathcal{J}}$ be a finite or infinite sequence of Gaussian integers such that $|z_{j+1} - z_j| = 1$ for all $j \in \mathcal{J}$ with $j + 1 \in \mathcal{J}$. Consider the set of differences*

$$\mathcal{A} = \{z_{j'} - z_j : j, j' \in \mathcal{J}\}.$$

Then for any nonzero $n \in \mathbb{Z}$ which belongs to $\mathcal{A}$, every divisor d of n in $\mathbb{Z}$ also belongs to $\mathcal{A}$.

As a consequence of Theorem 1, if $(z_j)_{j\in\mathcal{J}}$ is a finite or infinite sequence of Gaussian integers as in the statement of the theorem, and if j_1 and j_2 are such that $z_{j_2} - z_{j_1}$ is an element of $\mathbb{Z}$ not equal to -1, 0, and 1, then there exist j_3 and j_4 such that the ratio $(z_{j_2} - z_{j_1})/(z_{j_4} - z_{j_3})$ is a prime in $\mathbb{Z}$.

Finally, let us point out that Theorem 1 generalizes to chains of elements in a vector space.

Theorem 2. *Let V be a vector space over a field K of characteristic zero. Fix two vectors $u, v \in V$ which are linearly independent over K. Let n be a positive integer and let w_j with $0 \le j \le m$ be elements of V such that $w_m - w_0 = nu$ and $w_j - w_{j-1} \in \{u, -u, v, -v\}$ for $1 \le j \le m$. Then for any divisor d of n there are indices j and j' with $0 \le j, j' \le m$ for which $w_{j'} - w_j = du$.*

2. Proofs of Theorems 1 and 2

Theorem 2 follows immediately from Theorem 1. Indeed, if w_j with $0 \le j \le m$ are as in the statement of Theorem 2, then each w_j can be uniquely written in the form $w_j = a_j u + b_j v$ with $a_j, b_j \in K$. Moreover, the coefficients a_j and b_j lie in $\mathbb{Z}$, and we may associate to the sequence of elements w_j a finite sequence of Gaussian integers $z_j = a_j + ib_j$ with $0 \le j \le m$ satisfying the properties from the statement of Theorem 1. By Theorem 1, for any divisor d of n there exist indices j and j' with $0 \le j, j' \le m$ for which $z_{j'} - z_j = d$. This further gives $w_{j'} - w_j = du$, which proves Theorem 2.

We now proceed to prove Theorem 1. Suppose toward a contradiction that the theorem fails. Without any loss of generality, we may assume that there exist integers $m, n \ge 1$ and integral points (a_j, b_j) with $0 \le j \le m$ whose Cartesian coordinates satisfy $a_0 = b_0 = b_m = 0$, $a_m = n$, and $|a_j - a_{j-1}| + |b_j - b_{j+1}| = 1$ for all $1 \le j \le m$, and there exists a positive divisor d of n such that no pairs of indices $0 \le j, j' \le m$ satisfy $a_{j'} = a_j + d$ and $b_{j'} = b_j$.

However, if such numbers do exist, then let us fix two numbers n and d for which such counterexamples exist. With n and d fixed, let us consider the set of values of m for which such counterexamples exist. In what follows, we denote by k the smallest element of this set. Hence, for any $m < k$ and any sequence $z_0, \ldots, z_m$ as in the statement of the theorem, the conclusion of the theorem holds for the above fixed values of n and d, while for $m = k$ there are counterexamples for these values of n and d.

To proceed, let us denote by $\mathcal{Z}_{n,k}$ the set of finite sequences of Gaussian integers of the form $z_0, \ldots, z_k$ for which $z_0 = 0$, $z_k = n$, and $|z_{j+1} - z_j| = 1$ for $j = 0, \ldots, k-1$. Let $\mathcal{E}_{n,d,k}$ denote the subset of $\mathcal{Z}_{n,k}$ consisting of those elements of $\mathcal{Z}_{n,k}$ which provide counterexamples for our fixed values of n and d (recall that k is uniquely determined by n and d, and so k is also fixed). In other words, an element of $\mathcal{Z}_{n,k}$ belongs to $\mathcal{E}_{n,d,k}$ if and only if, as a finite sequence $z_0, \ldots, z_k$, it has the property that there is no pair of indices j with $j' \in \{0, \ldots, k\}$ for which $z_{j'} - z_j = d$.

In the following, we shall assume that the above set $\mathcal{E}_{n,d,k}$ is nonempty and then obtain a contradiction. In order to achieve this goal, in a sequence of lemmas below, we show that if the elements of $\mathcal{E}_{n,d,k}$ do exist, they must have rather peculiar shapes. As a matter of notation, from now on we shall denote the elements of $\mathcal{Z}_{n,k}$ by $\mathbf{z}$, where $\mathbf{z}$ is a $(k+1)$-tuple $(z_0, \ldots, z_k)$. Also, the real and imaginary parts of each z_j will be denoted in what follows by a_j and b_j, respectively.

Lemma 1. *Let $\mathbf{z} \in \mathcal{Z}_{n,k}$. Assume that there exist $0 \le s < t \le k$ with $t - s \ge 3$ such that $a_s = a_t, a_{s+1} = a_{s+2} = \cdots = a_{t-1} = a_s - 1$ and $b_j \ne 0$ for $s + 2 \le j \le t - 2$. Then $\mathbf{z} \notin \mathcal{E}_{n,d,k}$.*

Proof. Let $\mathbf{z}$ be as in the statement of the lemma. We need to show that $\mathbf{z} \notin \mathcal{E}_{n,d,k}$.

Let us suppose on the contrary that $\mathbf{z} \in \mathcal{E}_{n,d,k}$. Let $\mathcal{M}$ denote the collection of integral points $P_j = (a_j, b_j)$ with $0 \le j \le k$. If there is a configuration of integral points $P_s, P_{s+1}, P_{s+2}, \ldots, P_{t-1}, P_t$ in $\mathcal{M}$ of the form as in the statement of the lemma, then we may select such a configuration whose vertical length $t-s$ is as short as possible.

We consider translations of this configuration to the right and to the left by d, thereby obtaining points $P'_s, \ldots, P'_t$ and $P''_s, \ldots, P''_t$, respectively. (See Figure 2.) There are two possibilities, here. If any of the points $P'_s, \ldots, P'_t$ or $P''_s, \ldots, P''_t$ belong to $\mathcal{M}$, then there is nothing to prove, because then this point and its translate to the right or to the left by d belong to $\mathcal{M}$. Otherwise, we must find two points in $\mathcal{M}$, say P_{i^*} and P_{j^*}, that lie on the same horizontal line segment such that $a_{j^*} = a_{i^*} + d$ and $b_{j^*} = b_{i^*}$.

We consider the points $V_1, V_2, \ldots, V_{t-s-3}$ on the vertical line segment $[P_s, P_t]$ through the points P_s and P_t. By the minimality property of k, it follows that the sequence of points $P_0, P_1, \ldots, P_s, V_1, V_2, \ldots, V_{t-s-3}, P_t, P_{t+1}, \ldots, P_k$ does not produce a counterexample to the statement of the theorem for the given values of n and d. We deduce that there are points P_{i^*} and P_{j^*} in the set $\{P_0, P_1, \ldots, P_s, V_1, V_2, \ldots, V_{t-s-3}, P_t, P_{t+1}, \ldots, P_k\}$ such that $a_{j^*} = a_{i^*} + d$ and $b_{j^*} = b_{i^*}$. If these two points are in the subset $\{P_0, P_1, \ldots, P_s, P_t, P_{t+1}, \ldots, P_k\}$, then we are done.

Note that the points cannot both be in the subset $\{V_1, V_2, \ldots, V_{t-s-3}\}$, since this set consists of points that are on a vertical line segment. We now consider the remaining case when one of the points is in the set $\{P_0, P_1, \ldots, P_s, P_t, P_{t+1}, \ldots, P_k\}$ and the other is in the set $\{V_1, V_2, \ldots, V_{t-s-3}\}$. Say that $i^* \in \{0, 1, \ldots, s, t, t+1, \ldots, k\}$ and $l \in \{1, 2, \ldots, t-s-3\}$ are such that the points P_{i^*} and V_l lie on the same horizontal line and at distance d. Hence, P_{i^*} is either the right shift or the left shift of V_l by d. Without any loss of generality, we may suppose the former. Also, $P_{i^*} \in \mathcal{M}$. Since none of the points $P'_s, P'_{s+1}, \ldots, P'_{t-1}, P'_t$ is in $\mathcal{M}$, the neighbors of P_{i^*} that are in $\mathcal{M}$ must lie either above, below, or to the right of P_{i^*}.

Let $v \ge i^*$ denote the largest index for which all the points $P_{i^*+1}, P_{i^*+2}, \ldots, P_v$ lie on the same vertical line segment as P_{i^*}. Likewise, let $u \le i^*$ denote the smallest index for which all the points $P_u, P_{u+1}, \ldots, P_{i^*-1}$ lie on the same vertical line segment as P_{i^*}. Since, by the minimality of k, the points $P_0, P_1, \ldots, P_k$ are clearly distinct, the y coordinates of the points $P_u, P_{u+1}, \ldots, P_{i^*-1}, P_{i^*}, P_{i^*+1}, P_{i^*+2}, \ldots, P_v$ form a set of consecutive integers. Since these points $P_u, \ldots, P_v$ all lie on the vertical line segment $[P'_s, P'_t]$, and since they are distinct from P'_s and P'_t, the vertical distance between P_u and P_v is strictly less than the vertical distance between P'_s and P'_t.

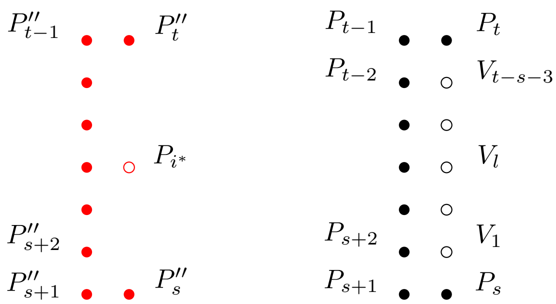

Figure 2: Left translation by d in red and right translation by d in blue.

We now distinguish two cases.

`Case 1:` If the point P_u is not the initial point P_0 and if the point P_v is not the endpoint P_k, then the point P_{u-1} lies on the right of P_u on the same horizontal line, at distance 1 from P_u and, likewise, the point P_{v+1} lies on the same horizontal line as P_v, at distance 1 and on the right of P_v. This is because the point that lies on the same horizontal line as P_v and to its left is in the set $\{P'_{s+2}, \ldots, P'_{t-2}\}$.

Hence, the points $P_{u-1}, P_u, \ldots, P_v, P_{v+1}$ form a configuration as in the statement of the lemma. Its vertical length is strictly less than the vertical length of the configuration formed by the points $P_s, P_{s+1}, \ldots, P_t$, and this contradicts the minimality condition in our original assumption.

`Case 2:` If either the point P_u coincides with the initial point P_0 or the point P_v coincides with the endpoint P_k, then the horizontal axis passes through either the point P_u or the point P_v. Therefore, it passes through one of the points $P_{s+2}, \ldots, P_{t-2}$, and this contradicts our assumption that $b_j \neq 0$ for $s+2 \leq j \leq t-2$.

This completes the proof of Lemma 1. $\square$

Lemma 1 shows that if an element $\mathbf{z} \in \mathcal{Z}_{n,k}$ does belong to $\mathcal{E}_{n,d,k}$, then geometrically it cannot contain configurations $P_s, \ldots, P_t$, as shown in Figure 2. The next lemma plays a symmetric role.

Lemma 2. *Let $\mathbf{z} \in \mathcal{Z}_{n,k}$. Assume that there exist $0 \leq s < t \leq k$ with $t - s \geq 3$ such that $a_s = a_t, a_{s+1} = a_{s+2} = \cdots = a_{t-1} = a_s + 1$ and $b_j \neq 0$ for $s + 2 \leq j \leq t - 2$. Then $\mathbf{z} \notin \mathcal{E}_{n,d,k}$.*

Proof. The proof is similar to that of Lemma 1. Here, the configuration opens to the left. $\square$

In the next two lemmas, we consider configurations which are similar to those in the previous two lemmas, except that the new configurations open downward or upward rather than right or left. Here, we are less successful and cannot show that for any element $\mathbf{z} \in \mathcal{Z}_{n,k}$ which exhibits such a configuration one must have $\mathbf{z} \notin \mathcal{E}_{n,d,k}$. For a hypothetical $\mathbf{z} \in \mathcal{E}_{n,d,k}$ having such a configuration, we do obtain, however, an obstruction which will play a role later in the process of completing the proof of Theorem 1.

Lemma 3. *Let $\mathbf{z} \in \mathcal{E}_{n,d,k}$. Assume that there exist $0 \le s < t \le k$ with $t - s \ge 3$ such that $b_s = b_t, b_{s+1} = b_{s+2} = \cdots = b_{t-1} = b_s + 1$ and $a_j \ne 0$ for $s+2 \le j \le t-2$. Then $b_s \ge 0$ and there exists an index $j \in \{s+2, \ldots, t-2\}$ such that d is a divisor of a_j.*

Proof. Let $\mathcal{M} = \{P_j = (a_j, b_j)\colon 0 \le j \le k\}$. Consider the configuration of integral points $P_s, P_{s+1}, P_{s+2}, \ldots, P_{t-1}, P_t$ in $\mathcal{M}$ as in the statement of the lemma. We translate it to the right and to the left by d. We obtain the points $P'_s, \ldots, P'_t$ and $P''_s, \ldots, P''_t$, respectively. By our assumption that $\mathbf{z} \in \mathcal{E}_{n,d,k}$, none of these points belongs to $\mathcal{M}$.

Consider next the points $V_1, V_2, \ldots, V_{t-s-3}$ on the horizontal line segment $[P_s, P_t]$, in the direction from P_s to P_t. By the minimality of k, it follows that the points $P_0, P_1, \ldots, P_s, V_1, V_2, \ldots, V_{t-s-3}, P_t, P_{t+1}, \ldots, P_k$ furnish a sequence that satisfies the statement of the theorem for the given values of n and d. Hence, there exist P_{i^*} and P_{j^*} in the set $\{P_0, P_1, \ldots, P_s, V_1, V_2, \ldots, V_{t-s-3}, P_t, P_{t+1}, \ldots, P_k\}$ such that $a_{j^*} = a_{i^*} + d$ and $b_{j^*} = b_{i^*}$.

Note that, by our assumption that $\mathbf{z} \in \mathcal{E}_{n,d,k}$, one cannot have both points in the subset $\{P_0, P_1, \ldots, P_s, P_t, P_{t+1}, \ldots, P_k\}$. Also, they cannot both be in the subset $\{V_1, V_2, \ldots, V_{t-s-3}\}$, since this set consists of points that are on the horizontal line segment $[P_s, P_t]$, and if two of them are at distance d, then the two points in Figure 3 that lie exactly above them will be in $\mathcal{M}$ and at distance d, contradicting our assumption that $\mathbf{z} \in \mathcal{E}_{n,d,k}$.

It remains to consider the case when there exist indices $i^* \in \{0, 1, \ldots, s, t, t+1, \ldots, k\}$ and $l \in \{1, 2, \ldots, t-s-3\}$ such that P_{i^*} and V_l lie on the same horizontal line at distance d. It follows that P_{i^*} is either the right or the left shift of V_l by d. Suppose that P_{i^*} is the right shift of V_l. Also, $P_{i^*} \in \mathcal{M}$. Since none of the points $P'_s, P'_{s+1}, \ldots, P'_{t-1}, P'_t$ is in $\mathcal{M}$, it follows that the neighbors of P_{i^*} that are in $\mathcal{M}$ must lie either to the right, or to the left or below P_{i^*}.

Let $v \ge i^*$ denote the largest index for which all the points $P_{i^*+1}, P_{i^*+2}, \ldots, P_v$ lie on the same horizontal line segment as P_{i^*}. Let $u \le i^*$ denote the smallest index for which all the points $P_u, P_{u+1}, \ldots, P_{i^*-1}$ lie on the same horizontal line segment as P_{i^*}. Since the points $P_0, P_1, \ldots, P_k$ are distinct, it follows that the x coordinates of the points $P_u, P_{u+1}, \ldots, P_{i^*-1}, P_{i^*}, P_{i^*+1}, P_{i^*+2}, \ldots, P_v$ form a set of consecutive integers. Since these points $P_u, \ldots, P_v$ lie inside the horizontal line segment $[P'_s, P'_t]$, and since they are distinct from P'_s and P'_t, the horizontal distance between P_u and P_v is strictly less than the horizontal distance between P'_s and P'_t.

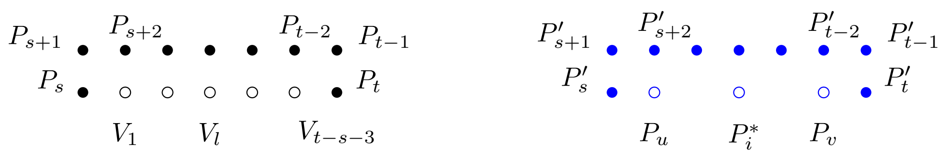

Figure 3: Right translation by d in blue.

We distinguish three possibilities.

`Case 1:` If the point P_u is the initial point P_0, then the vertical line through the origin passes through P_u. Then there exists an index $j \in \{s+2, \dots, t-2\}$ for which the vertical line $x = -d$ passes through the point P_j. Hence, $a_j = -d$. Also, $b_s = b_u = 0$, and so the statement of the lemma holds true in this case.

`Case 2:` If the point P_v is the endpoint P_k, then the vertical line $x = n$ passes through P_v. Then there exists an index $j \in \{s+2, \dots, t-2\}$ for which $a_j = n - d$, which is also a multiple of d. Here $b_s = b_v = 0$ again, and the statement of the lemma holds true in this case as well.

`Case 3:` If the point P_u is not the initial point P_0 and if the point P_v is not the endpoint P_k, then the points P_{u-1} and P_{v+1} exist and there is a smaller configuration formed by the points $P_{u-1}, P_u, \dots, P_v, P_{v+1}$. Here, the point P_{u-1} lies on the same vertical line as the point P_u, at distance 1 below P_u, and the point P_{v+1} lies on the same vertical line as the point P_v, at distance 1 below P_v.

The procedure may be repeated, using the smaller configuration in Case 3 in place of the configuration formed by the points $P_s, P_{s+1}, P_{s+2}, \dots, P_{t-1}, P_t$, until it terminates in either Case 1 or Case 2. Taking into account the translation to the left by d, we see that in the last configuration a_j equals either $\pm d$ or $n \pm d$, and the configuration prior to this one is shifted either to the right or to the left by d, so that the vertical lines passing through it are shifted accordingly by $\pm d$, and thus they necessarily intersect the previous configuration.

Also, in this process, each configuration lies one level below the previous one. In other words, the y-coordinates decrease by 1. Since the procedure terminates in either Case 1 or Case 2, where the configurations touch the x-axis, it follows that in the above process all the configurations, including the initial one, lie in the upper half plane. Hence, we must have $b_s \geq 0$, as claimed.

This completes the proof of Lemma 3. $\square$

Lemma 4. *Let* $\mathbf{z} \in \mathcal{E}_{n,d,k}$. *Assume that there exist* $0 \leq s < t \leq k$ *with* $t - s \geq 3$ *such that* $b_s = b_t, b_{s+1} = b_{s+2} = \cdots = b_{t-1} = b_s - 1$ *and* $a_j \neq 0$ *for* $s+2 \leq j \leq t-2$. *Then* $b_s \leq 0$ *and there exists an index* $j \in \{s+2, \dots, t-2\}$ *such that* d *is a divisor of* a_j.

Proof. The proof is similar to that of Lemma 3. Here, the configuration opens upward. □

Next, let us fix an arbitrary element $\mathbf{z} \in \mathcal{E}_{n,d,k}$. Given such a $\mathbf{z}$, we look at the first nonzero y-coordinate b_i. To be precise, let i_0 be such that $0 = b_0 = \cdots = b_{i_0} \neq b_{i_0+1}$. Note that if no such index i_0 exists, then every b_i will vanish and every point will be spaced by 1 on the horizontal line segment passing through the origin and through the point $(n, 0)$ with no two points being identical. Hence, each point will correspond to an integer number between 0 and n. In particular, one has the pair of points $(0, 0)$ and $(d, 0)$, which contradicts our assumption that $\mathbf{z} \in \mathcal{E}_{n,d,k}$. So there is an i_0 as above.

Now let j_0 denote the smallest index larger than $i_0 + 1$ for which $b_{j_0} = 0$. We know that such an index exists, for recall that $b_k = 0$. The next lemma severely restricts the possible shape of any element $\mathbf{z} \in \mathcal{E}_{n,d,k}$ by showing that for any such $\mathbf{z}$, and i_0 and j_0 defined in terms of $\mathbf{z}$ as above, the a_j's with j between i_0 and j_0 form a monotonic sequence.

In this connection, since $b_{j_0} = 0 = b_{i_0}$ and by the minimality of the length k, we cannot have $a_{j_0} = a_{i_0}$ also. Hence, in the next lemma we distinguish two cases, according as to whether a_{j_0} is strictly larger or strictly smaller than a_{i_0}.

Lemma 5. *Let $\mathbf{z} \in \mathcal{E}_{n,d,k}$ and let i_0 and j_0 be defined as above.*

(i) *If $a_{j_0} > a_{i_0}$, then $a_{s+1} \geq a_s$ for all $i_0 \leq s \leq j_0 - 1$.*

(ii) *If $a_{j_0} < a_{i_0}$, then $a_{s+1} \leq a_s$ for all $i_0 \leq s \leq j_0 - 1$.*

Proof. We only prove part (i), the proof of part (ii) being similar. Suppose part (i) does not hold true. Then there exists an index s_0 in the given range such that $a_{s_0+1} < a_{s_0}$. Taking the smallest index s_0 possible, we see that the point (a_{s_0+1}, b_{s_0+1}) must lie to the left of the point (a_{s_0}, b_{s_0}), and the previous point (a_{s_0-1}, b_{s_0-1}) must either lie above or below the point (a_{s_0}, b_{s_0}).

`Case 1:` $a_{s_0} > a_{i_0}$. Let us look at the set $\mathcal{J} = \{i_0 \leq j \leq s_0 - 1 \colon a_j < a_{s_0}\}$. This set is nonempty, as it contains the index i_0. Let now $u = \max \mathcal{J}$. Then one sees that $a_u = a_{s_0} - 1$ and $a_{u+1} = a_{u+2} = \cdots = a_{s_0-1} = a_{s_0}$. Hence, the sequence of points $(a_u, b_u), \ldots, (a_{s_0+1}, b_{s_0+1})$, create a configuration as in the statement of Lemma 2.

Note also that, as a consequence of the definition of i_0 and j_0, this configuration lies entirely above the x-axis, or entirely below the x-axis. Thus, each b_j as in the statement of Lemma 2 is nonzero, as required. Hence, Lemma 2 is applicable and implies that $\mathbf{z} \notin \mathcal{E}_{n,d,k}$, which contradicts our assumption on $\mathbf{z}$.

`Case 2:` $a_{s_0} \leq a_{i_0}$. Since $a_{j_0} > a_{i_0} \geq a_{s_0}$, there exists an index v between s_0 and j_0 such that $a_{v+1} > a_v$. We take the smallest such index v. Then $a_v \leq a_{v-1} \leq a_{v-2} \leq \cdots \leq a_{s_0+1} < a_{s_0}$. Let us look at the set $\mathcal{J}' = \{s_0 \leq l \leq v - 1 \colon a_l > a_v\}$. This set is nonempty, since it contains the index s_0. Putting $l_0 = \max \mathcal{J}'$, we obtain $a_{l_0} > a_v$, so that $a_v = a_{v-1} = \cdots = a_{l_0+1} < a_{l_0} = a_v + 1$. Examining the sequence of points $(a_{i_0}, b_{i_0}), \ldots, (a_{s_0}, b_{s_0}), \ldots, (a_{l_0}, b_{l_0}), \ldots, (a_v, b_v), \ldots, (a_{j_0}, b_{j_0})$, we find that the points with indices ranging from l_0 to $v+1$ form a configuration as in the statement of Lemma 1 which, as in the previous case, leads to a contradiction.

This completes the proof of Lemma 5. □

Before going any further, let us remark that the proof above works in more generality. More precisely, if we replace i_0 by any index $i_1 < k$ which has the property that $0 = b_{i_1} \neq b_{i_1+1}$, and if we further let $j_1 \leq k$ denote the smallest index larger than i_1 for which $b_{j_1} = 0$, then the lemma still holds with i_0 and j_0 replaced by i_1 and j_1, respectively.

Hence, a hypothetical element $\mathbf{z} \in \mathcal{E}_{n,d,k}$ is restricted to have the following shape. It consists of a sequence of points, some of which lie on the x-axis including the first at $(0,0)$ and the last at $(n,0)$, and any time the sequence leaves the x-axis, and before it returns to the x-axis, the x-coordinates a_j form a monotonic sequence.

Next, we look at the y-coordinates b_j between any two such intersections with the x-axis. For simplicity, we look at the indices between i_0 and j_0. As before, the same results hold in more generality, where i_0 and j_0 are replaced by an i_1 and a corresponding j_1 as above.

Lemma 6. *Let* $\mathbf{z}$, i_0, *and* j_0 *be as in the statement of Lemma 5 and assume that* b_{i_0+1} *is positive. Then there exist two indices* u_0 *and* v_0 *such that the following statements hold true:*

(i) $i_0 + 1 \leq u_0 < v_0 \leq j_0 - 1$;

(ii) $b_{u_0} = b_{u_0+1} = \cdots = b_{v_0}$;

(iii) $b_{u_0} > b_{u_0-1}, b_{v_0} > b_{v_0+1}$;

(iv) $b_{v_0+1} \geq b_{v_0+2} \geq \cdots \geq b_{j_0-1} = 1$;

(v) $1 = b_{i_0+1} \leq b_{i_0+2} \leq \cdots \leq b_{u_0-1}$.

Proof. Let $\mathcal{N} = \{i_0 < j \leq j_0 - 1 \colon b_j > b_{j-1}\}$. This set is nonempty, as it contains the index $i_0 + 1$. Denote $u_0 = \max \mathcal{N}$. Next, let v_0 be the largest index for which $b_{u_0} = b_{u_0+1} = \cdots = b_{v_0}$. It is easy to see that for this choice of u_0 and v_0 parts (i), (ii), and (iii) in the statement of the lemma hold true.

Note that, in principle, there may exist more than one pair of indices u_0 and v_0 for which parts (i), (ii), and (iii) in the statement of the lemma simultaneously hold true. For the above particular choice of u_0 and v_0 part (iv) also holds, by the maximality property in the definition of u_0. As for part (v), it does not follow from the above considerations.

Alternatively, we could have worked with the minimum rather than the maximum element of $\mathcal{N}$, say $v_0' = \min \mathcal{N}$, and then let u_0' be the smallest index for which $b_{u_0'} = b_{u_0'+1} = \cdots = b_{v_0'}$. Then one sees that if one uses u_0' and v_0' in place of u_0 and v_0, respectively, parts (i), (ii), (iii), and (v) will hold true, but part (iv) will not necessarily follow.

Now, the key observation is the following. The indices u_0' and v_0' must, in fact, coincide with u_0 and v_0, respectively. Indeed, if not, then one sees that by considering the points in our sequence with indices between v_0' and u_0, somewhere inside this sequence of points there will be a configuration as in the statement of Lemma 4. Moreover, this configuration will lie entirely on the upper half plane, contradicting the conclusion of Lemma 4. Hence, u_0' and v_0' must coincide with u_0 and v_0, respectively. Then all parts (i), (ii), (iii), (iv), and (v) will hold true.

This finishes the proof of Lemma 6. $\square$

Let us remark that, in the statement of the above lemma, the only reason we assumed that b_{i_0+1} is positive was in order to make a choice. In the case that b_{i_0+1} is negative, a similar statement with obvious modifications holds true.

Lemma 7. *Let* $\mathbf{z}$, i_0, *and* j_0 *be as in the statement of Lemma 5 and consider the set* $\mathcal{L} = \{r\colon$ *there exist two indices* i *and* j *such that* $0 \le i \le u_0, v_0 \le j \le j_0$ *for which* $a_j = a_i + r$ *and* $b_j = b_i\}$. *Then* $\mathcal{L} \supseteq \{v_0 - u_0, v_0 - u_0 + 1, \ldots, a_{j_0}\}$.

Proof. Clearly, $a_{j_0} \in \mathcal{L}$. Suppose $r \in \mathcal{L}$ and $r > v_0 - u_0$. We want to show that $r - 1 \in \mathcal{L}$. Since $r \in \mathcal{L}$, there exist points (a_i, b_i) and (a_j, b_j), with $0 \le i \le u_0$ and $v_0 \le j \le j_0$, such that $a_j = a_i + r$ and $b_j = b_i$. Since $r > v_0 - u_0$, these points cannot be on the line segment $[(a_{u_0}, b_{u_0}), (a_{v_0} b_{v_0})]$. Let us look at their neighbors.

We have $a_{i+1} \ge a_i$ and $b_{i+1} \ge b_i$. If the point (a_{i+1}, b_{i+1}) lies to the right of (a_i, b_i), then we have $r-1 \in \mathcal{L}$, since in this case we can select the points (a_{i+1}, b_{i+1}) and (a_j, b_j) which are on the same horizontal line, at distance $r - 1$. Similarly, if the point (a_{j-1}, b_{j-1}) lies to the left of (a_j, b_j), then again one has $r - 1 \in \mathcal{L}$.

By Lemmas 5 and 6, the only remaining case is when the point (a_{i+1}, b_{i+1}) lies directly above the point (a_i, b_i), and the point (a_{j-1}, b_{j-1}) lies directly above (a_j, b_j). In this way, we obtain a new pair of points at distance r, on a horizontal line situated above the horizontal line containing the previous pair of points at distance r, and we can repeat the same argument for the new pair of points.

This procedure will have to stop after finitely many steps, before it reaches the horizontal line passing through the points (a_{u_0}, b_{u_0}) and (a_{v_0}, b_{v_0}). Hence, one will eventually obtain two points on a horizontal line segment, at distance $r - 1$.

Hence, Lemma 7 is now proved. $\square$

We are now ready to complete the proof of Theorem 1. In order to prove the theorem, we need to show that the set $\mathcal{E}_{n,d,k}$ is empty. Let us assume on the contrary that $\mathcal{E}_{n,d,k}$ is nonempty, and let us fix an element $\mathbf{z} \in \mathcal{E}_{n,d,k}$.

Next, for this particular $\mathbf{z}$, define i_0, j_0, u_0, and v_0 as above. Let us remark that the points $(a_{u_0-1}, b_{u_0-1}), \ldots, (a_{v_0+1}, b_{v_0+1})$ form a configuration as in the statement of Lemma 3. Hence, there exists an index $u_0+1 \leq t \leq v_0-1$ for which d is a divisor of a_t. Here, $0 \leq a_{i_0} \leq a_{u_0} < a_t < a_{v_0} \leq a_{j_0}$. Since a_t is a multiple of d and since $a_t > 0$, we have that $a_t \geq d$. But $a_t < a_{j_0}$, and so $1 \leq d < a_{j_0}$.

If now d is smaller than $v_0 - u_0$, then we can select two integer points on the line segment $[(a_{u_0}, b_{u_0}), (a_{v_0} b_{v_0})]$ at distance d, and this contradicts our assumption that $\mathbf{z} \in \mathcal{E}_{n,d,k}$. On the other hand, if d is larger than $v_0 - u_0$, then d is in $\mathcal{L}$ by Lemma 7, and we arrive again at a contradiction.

The above arguments cover the case when $0 \leq a_{i_0} < a_{j_0}$. The same arguments also work in the case when $0 \geq a_{i_0} > a_{j_0}$. Let us remark that there are two cases where the above proof does not work. These cases are when $a_{i_0} < 0 < a_{j_0}$ and, respectively, when $a_{i_0} > 0 > a_{j_0}$. In both of these cases, the above proof breaks down because, in principle, a_t which we know is a multiple of d could be zero. Hence, in these cases, we cannot derive any useful inequality involving d. For all we know, d could be much larger than any of $|a_{i_0}|, \ldots, |a_{j_0}|$.

The key remark here, which completes the proof of the theorem, is that in these cases Lemmas 1 and 2 apply directly and, in fact, prevent these two cases from happening. To be precise, if $a_{i_0} < 0 < a_{j_0}$, then by examining our sequence of points restricted to indices between $i_0 - 1$ and j_0, we see that it must contain a configuration as in Lemma 1. Then, by this lemma, it follows that $\mathbf{z} \notin \mathcal{E}_{n,d,k}$, thus contradicting our hypothesis. Similarly, if $a_{i_0} > 0 > a_{j_0}$, then $\mathbf{z}$ will exhibit a configuration as in Lemma 2, and applying that lemma we again contradict our assumption that $\mathbf{z} \in \mathcal{E}_{n,d,k}$.

This finishes the proof of Theorem 1.

3. Further Examples and Open Problems

A natural question that arises from Theorem 1 is the following.

Question 1. Given a positive integer n, which positive integers belong to the intersection of the sets $\mathcal{A}$ where the intersection is taken over all the sequences of Gaussian integers as in the statement of Theorem 1 for which $n \in \mathcal{A}$?

From Theorem 1, we know that this intersection contains all the positive divisors of n, and one may ask for which values of n the intersection contains no other positive integers.

Question 2. For which positive integers n, the positive integers which belong to the intersection of the sets $\mathcal{A}$ are exactly the positive divisors of n?

An example of such an n is $n = 12$. Indeed, let us consider the following finite sequence of Gaussian integers, which starts at zero and ends at 12: $0, i, 1+i, 2+i, 3+i, 4+i, 5+i, 6+i, 6, 6-i, 7-i, 8-i, 9-i, 10-i, 11-i, 12-i, 12$.

In this case, the positive integers which belong to $\mathcal{A}$ are $1, 2, 3, 4, 5, 6$, and 12. Hence, if we intersect this set with the corresponding set $\mathcal{A}$ coming from the sequence in Figure 1 which, as noted earlier, does not contain the number 5, we obtain exactly the set of positive divisors of 12.

Let us also remark that, although the intersection appearing in Questions 1 and 2 is over infinitely many sequences, in the particular case $n = 12$, two such sequences are enough in order to obtain the intersection. How many such sequences are needed for a general n?

Question 3. Given a positive integer n, how many sequences as in the statement of Theorem 1 for which $n \in \mathcal{A}$ are needed in order for the intersection of their sets $\mathcal{A}$ to coincide with the intersection over all sequences?

Let us now consider the following infinite family of examples, where the shape of the corresponding figure is similar to the one in Figure 1. Choose any positive integers a, b, c, and d, with $a > c$ and $b > d$. Consider the finite sequence of Gaussian integers which goes up from zero to ai, then moves right to $b + ai$, then moves down to $b-ci$, right to $b+d-ci$, up to $b+d+ci$, right to $b+2d+ci$, down to $b+2d-ai$, right to $2b+2d-ai$, and lastly up to $2b+2d$. Hence, here $n = 2b+2d$. The example in Figure 1 corresponds to the particular case when $a = 4$, $b = 4$, $c = 2$, and $d = 2$.

Returning to the general case above, we look for examples which are close to providing counterexamples to Theorem 1. Let us observe that for any a, b, c, and d as above, $n/2$ equals the difference of two elements of the sequence, and the only positive integers strictly smaller than $n/2$ which are differences of two elements of the sequence are exactly the positive integers less than or equal to $\max\{b, 2d\}$.

Thus, if in one such example n has a divisor in the open interval $(\max\{b, 2d\}, n/2)$, one would have a counterexample to Theorem 1. Since the largest possible divisor of n strictly less than $n/2$ is $n/3$, let us restrict the above class of examples to those where n is a multiple of 3, so that $n/3$ is a divisor of n.

Next, for any such fixed n, let us arrange the parameters in such a way that $\max\{b, 2d\}$ is as small as possible, in order for $n/3$ to have a chance to lie inside the above open interval. It turns out that this minimum is exactly $n/3$, which shows that this family of examples is as close as possible to provide counterexamples, without actually providing counterexamples to Theorem 1.

In connection with the above family of examples, as noticed above, the divisors $n/3$ and $n/2$ have immediate neighbors (namely $n/3+1$ and $n/2-1$, respectively) which are not representable as differences of two elements of the sequence. One may ask if, more generally, one can construct examples where this property holds simultaneously for all divisors of n in the sense that, for each divisor d of n, at least one of its two immediate neighbors $d-1$ or $d+1$ is not representable as the difference of two elements of the sequence.

This is, of course, not the case since there are pairs of numbers n and d for which each of the numbers $d-1$, d, and $d+1$ is a divisor of n. Hence, in any example, all three numbers $d-1$, d, and $d+1$ are represented as differences of two elements of the corresponding sequence, by Theorem 1. Taking this into account, we raise the following modified question.

Question 4. For which numbers n there exists a sequence of Gaussian integers as in the statement of Theorem 1 such that $n \in \mathcal{A}$ and such that for each divisor d of n, with $1 < d < n$, either both $d-1$ and $d+1$ are divisors of n, or at least one of $d-1$ or $d+1$ is not in $\mathcal{A}$?

For example, one such number n is $n = 12$, and in this case a sequence satisfying the required properties is the one provided in Figure 1. Here, the divisors 4 and 6 of 12 have the common immediate neighbor 5 which does not belong to $\mathcal{A}$, while the divisors 2 and 3 are exempted since their immediate neighbors are also divisors of 12.

More generally, instead of immediate neighbors one may fix a positive integer K and consider neighbors at distance less than or equal to K. Then one may generalize the above problem in various ways. One variant is the following question.

Question 5. For which numbers n there exists a sequence of Gaussian integers as in the statement of Theorem 1 such that $n \in \mathcal{A}$ and such that for each divisor d of n with $K < d < n-K$, either all the numbers $d-K, d-K+1, \ldots, d+K$ are divisors of n or at least one of $d-K, d-K+1, \ldots, d+K$ is not in $\mathcal{A}$?

One may also consider weaker conditions, as follows.

Question 6. For which numbers n there exists a sequence of Gaussian integers as in the statement of Theorem 1 such that $n \in \mathcal{A}$ and such that for each divisor d of n at least one of the numbers $d-K, d-K+1, \ldots, d-1, d+1, \ldots, d+K$ is either a divisor of n or is not in $\mathcal{A}$?

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