TOTIENOMIAL COEFFICIENTS

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Abstract

We prove that the generalized binomial coefficients associated with the Jordan totient functions are all integers. In the process, we also demonstrate the integrality of generalized binomials coming from other number-theoretic sequences including the Dedekind psi function. We finish by initiating the search for combinatorial interpretations of these coefficients.

1. Introduction

The binomial coefficients are sometimes defined as the ratio of factorials

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

instead of as the number of n-subsets of an m-set. A problem with this is that a novice may not believe these numbers are integral; however, the formula intrigues upon the realization that the ratio, though undefined for $n > m$, is in fact an integer when $n \leq m$. Using this definition of binomial coefficients, it makes sense to construct "generalized binomial coefficients" for any sequence of nonzero integers in the following way (see [7]). If $C = (c_1, c_2, c_3, \ldots)$ is a sequence of nonzero integers then we define the C-factorial by

$$(m!)_C = \begin{cases} c_m c_{m-1} \cdots c_1 & \text{when } m \neq 0 \\ 1 & \text{when } m = 0. \end{cases}$$

Next, we use the C-factorial to define the generalized binomial coefficients associated with C, called the C-nomial coefficients:

$$\binom{m}{n}_C = \begin{cases} \frac{(m!)_C}{(n!)_C((m-n)!)_C} & \text{when } 0 \leq n \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Using the sequence of nonzero natural numbers leads to the traditional binomial coefficients. As a warm-up exercise, we define, for each $k \geq 1$, the sequence of k-th powers $N^k = (1^k, 2^k, 3^k, 4^k, \ldots)$. The following computation shows that the associated generalized binomial coefficients are integers:

$$\binom{m}{n}_{N^k} = \frac{m^k \cdot (m-1)^k \cdots 2^k \cdot 1^k}{n^k \cdot (n-1)^k \cdots 1^k \cdot (m-n)^k \cdot (m-n-1)^k \cdots 1^k} = \binom{m}{n}^k. \qquad (1)$$

The construction of generalized binomial coefficients can be performed for any nonzero integer sequence; however, it is unlikely the process will result in integers. Nonetheless, the following lemma, proven in [7], provides a sufficient condition for integrality.

Lemma 1 (Knuth and Wilf [7]). *For any sequence, C, of nonzero integers the C-nomial coefficients are all integers if* $\gcd(C_i, C_j) = C_{\gcd(i,j)}$ *for all* $i, j > 0$.

Many families of sequences have been shown to have integral generalized binomial coefficients. For instance, using the Fibonacci sequence leads to the Fibotorials and Fibonomial coefficients (see [4], [3], or [5]). Using q-analogs of the natural numbers leads to the q-factorial and the Gaussian binomial coefficients (see [6] or [7]). Additionally, in [2], we describe a simple family of sequences for which the binomial coefficients are integral and are related to arithmetic in different bases.

The purpose of this note is to investigate the generalized factorial and binomial coefficients for the family of Jordan totient functions, J_k, where $J_k(n)$ counts the number of k-tuples of positive integers less than or equal to n that form a coprime $(k+1)$-tuple with n. These functions generalize $J_1 = \varphi$, Euler's totient function, which counts the number of positive integers less than or equal to and coprime to n. For example, the sequences for $\varphi = J_1$ and J_2 through J_{10} are A000010, A007434, A059376, A059377, A059378, and A069091-A06905 in [1]. Unfortunately, the totient functions do not satisfy the conditions of Lemma 1; for example, $\varphi(\gcd(7, 13)) = 1$ while $\gcd(\varphi(7), \varphi(13)) = 6$.

Given a totient function, J_k, and a natural number m we write $m!_{J_k}$ for the J_k-factorial; we call this the k-totientorial (see A001088 for the 1-totientorial in [1]). Likewise, we let $\binom{m}{n}_{J_k}$ represent the associated binomial coefficients and call these the k-totienomial coefficients. Even though the totient sequences do not satisfy Lemma 1, we will demonstrate that the totienomial coefficients are all integers (see Figure 1). On our path to the proof, we will encounter a few other sequences that have integral generalized binomial coefficients as well.

2. Radinomials

Given any natural number n, we define the radical of n, denoted $\mathrm{rad}(n)$, as the product of primes dividing n, i.e. $\mathrm{rad}(n) = \prod_{p|n} p$. The radical sequence is A007947

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0	0
3	1	2	2	1	0	0	0	0	0
4	1	2	4	2	1	0	0	0	0
5	1	4	8	8	4	1	0	0	0
6	1	2	8	8	8	2	1	0	0
7	1	6	12	24	24	12	6	1	0
8	1	4	24	24	48	24	24	4	1

Figure 1: The first 8 rows of the 1-totienomial coefficients.

and the "raditorial" is A048803 in [1]. In a similar fashion, we define the related sequence $\mathcal{R}$ by $\mathcal{R}(n) = \prod_{p|n}(p-1)$ where again the product is only over primes (see A173557 in [1]). For each of these sequences, we construct the associated binomial coefficients, and call them the radinomial and $\mathcal{R}$-nomial coefficients, denoted $\binom{m}{n}_{\text{rad}}$ and $\binom{m}{n}_{\mathcal{R}}$ respectively.

Proposition 2. *For any two natural numbers n and m with $n \leq m$, the radinomial coefficient $\binom{m}{n}_{\text{rad}}$ is an integer.*

Proof. Notice that a prime p divides both n and m if and only if p divides $\gcd(n, m)$. Thus, $\gcd(\text{rad}(n), \text{rad}(m)) = \text{rad}(\gcd(n, m))$; the result follows by Lemma 1. □

This seems to be previously known since the sequence of radinomial coefficients can be found at A048804 in [1].

Corollary 3. *For any two natural numbers n and m with $n \leq m$, the $\mathcal{R}$-nomial coefficient $\binom{m}{n}_{\mathcal{R}}$ is an integer.*

Proof. For natural numbers n and m with $n \leq m$, Proposition 2 implies that $\binom{m}{n}_{\text{rad}}$ is an integer. Suppose that $\{p_1, ..., p_s\}$ is the set of primes less than or equal to m. Then, let $(m!)_{\text{rad}} = \prod_{i=1}^{s} p_i^{r_i}$ where each $r_i > 0$, and $(n!)_{\text{rad}} \cdot ((m-n)!)_{\text{rad}} = \prod_{i=1}^{s} p_i^{t_i}$ where each $t_i \geq 0$. It follows that

$$\binom{m}{n}_{\text{rad}} = \frac{\prod_{i=1}^{s} p_i^{r_i}}{\prod_{i=1}^{s} p_i^{t_i}} = \prod_{i=1}^{s} p_i^{r_i - t_i}.$$

The integrality of $\binom{m}{n}_{\text{rad}}$ implies that $r_i \geq t_i$ for all i. Furthermore,

$$\binom{m}{n}_{\mathcal{R}} = \frac{\prod_{i=1}^{s}(p_i - 1)^{r_i}}{\prod_{i=1}^{s}(p_i - 1)^{t_i}} = \prod_{i=1}^{s}(p_i - 1)^{r_i - t_i}.$$

Since $r_i \geq t_i$ for all i, we see that $\binom{m}{n}_{\mathcal{R}}$ is an integer. □

These sequences will allow us to prove that each totienomial coefficient $\binom{m}{n}_{J_1}$ is an integer. To be more general, we need extended versions of the radical and $\mathcal{R}$. For any natural number $k \geq 1$, we define the sequence rad_k by $\mathrm{rad}_k(n) = \mathrm{rad}(n)^k = \prod_{p|n} p^k$ and the sequence $\mathcal{R}_k$ by $\mathcal{R}_k(n) = \prod_{p|n}(p^k - 1)$ (once again the products are both over all primes dividing n). For instance, see A078615 in [1] for rad_2. We call these the k-radinomials and $\mathcal{R}_k$-nomials respectively.

Proposition 4. *For all $k \geq 1$, the k-radinomials and $\mathcal{R}_k$-nomials are all integers.*

Proof. The fact that $\binom{m}{n}_{\mathrm{rad}_k}$ is integral follows from an argument analogous to the warm-up exercise given in Equation (1). In particular,

$$\binom{m}{n}_{\mathrm{rad}_k} = \left(\binom{m}{n}_{\mathrm{rad}}\right)^k,$$

and the latter is an integer by Proposition 2. In turn, a proof similar to that of Corollary 3 demonstrates that the integrality of $\binom{m}{n}_{\mathcal{R}_k}$ follows directly from the integrality of $\binom{m}{n}_{\mathrm{rad}_k}$. $\square$

We finish this section with one final family of sequences. For $k \in \mathbb{N}$, we let $\text{d-rad}_k(n) = \frac{n^k}{\mathrm{rad}_k(n)}$ (see for instance A003557 in [1]). Once again, the associated binomial coefficients are integers.

Proposition 5. *Let $k \geq 1$. Then*

1. *For all $m, n \in \mathbb{N}$ with $n \leq m$, $\binom{m}{n}_{\text{d-rad}_k}$ is an integer.*
2. *For all $m, n \in \mathbb{N}$ with $n \leq m$, $\binom{m}{n}_{\text{d-rad}_k} = \left(\binom{m}{n}_{\text{d-rad}_1}\right)^k$.*
3. *For all $m, n \in \mathbb{N}$ with $n \leq m$*

$$\binom{m}{n}_{\text{d-rad}_k} = \frac{\binom{m}{n}_{N^k}}{\binom{m}{n}_{\mathrm{rad}_k}}$$

and so $\binom{m}{n}_{\mathrm{rad}_k}$ evenly divides $\binom{m}{n}_{N^k}$.

Proof. In a similar fashion to the radical, we see that d-rad_1 satisfies the conditions of Lemma 1, giving one case for part (1). Next, a computation analogous to Equation (1) shows that $\binom{m}{n}_{\text{d-rad}_k} = \binom{m}{n}^k_{\text{d-rad}_1}$, which gives part (2) and completes part (1). Finally, part (3) follows directly from the definitions of the generalized binomial coefficients and factorials. $\square$

3. Totienomials

Armed with the technical results from the previous section, the fact that the totienomials are integral now follows from a basic fact found in [8, pg. 219].

Theorem 6. *Let $k \in \mathbb{N}$ and let J_k be the k-th Jordan totient function. Then for $n \in \mathbb{N}$,*

$$J_k(n) = n^k \cdot \prod_{p|n} \left(1 - \frac{1}{p^k}\right) = \frac{n^k \cdot \prod_{p|n}(p^k - 1)}{\prod_{p|n} p^k} = \frac{N^k(n) \cdot \mathcal{R}_k(n)}{\mathrm{rad}_k(n)}.$$

This theorem leads us to the following.

Theorem 7. *Let $k \in \mathbb{N}$ and let J_k be the k-th Jordan totient function. The following all hold.*

1. *For all $m, n \in \mathbb{N}$ with $n \leq m$,*
$$\binom{m}{n}_{J_k} = \frac{\binom{m}{n}_{N^k} \cdot \binom{m}{n}_{\mathcal{R}_k}}{\binom{m}{n}_{\mathrm{rad}_k}}.$$
2. *For all $m, n \in \mathbb{N}$ with $n \leq m$, $\binom{m}{n}_{J_k} = \binom{m}{n}_{\text{d-rad}_k} \cdot \binom{m}{n}_{\mathcal{R}_k}$.*
3. *Each k-totienomial coefficient is an integer.*

Proof. The first result is a consequence of the definitions of generalized factorials and generalized binomial coefficients coupled with Theorem 6. Part (2) follows from part (1) along with Proposition 5. Finally, the third result follows from part (2) along with Propositions 4 and 5, which show that $\binom{m}{n}_{\text{d-rad}_k}$ and $\binom{m}{n}_{\mathcal{R}_k}$ are integral. $\square$

This appears to be a new result as we could not find it in the literature or the triangular array of any of the totienomials in the OEIS. Consequently, we have added a few examples to the database: see A238453 for 1-totienomials, A238688 for 2-totienomials, A238743 for 3-totienomials, A238754 for 4-totienomials and A239633 for 5-totienomials.

Remark. Unlike the usual binomial coefficients, the rows in the triangular array of the totienomial coefficients may not be unimodal, and the middle term(s) may not be the largest number in the row. For instance, if we consider $\varphi = J_1$, then the following table shows $\binom{10}{n}_\varphi$ for different values of n (i.e. row 10 in the triangular array).

n	0	1	2	3	4	5	6	7	8	9	10
$\binom{10}{n}_\varphi$	1	4	24	48	144	72	144	48	24	4	1

In the final section, we will briefly discuss the 1-totienomial coefficients. Before we do, we use the previous results to demonstrate the integrality of generalized binomial coefficients for one more sequence: the Dedekind psi function (see A001615 in [1]). For a nonzero natural number n, let $\psi(n) = n \cdot \prod_{p|n}\left(1+\frac{1}{p}\right)$ where again the product is over all primes dividing n.

Corollary 8. *Let m and n be natural numbers with $n \leq m$. The following hold.*

1. *The ψ-nomial coefficient $\binom{m}{n}_\psi$ is an integer.*
2. *The ψ-nomial coefficient is given by*

$$\binom{m}{n}_\psi = \frac{\binom{m}{n}_{J_2}}{\binom{m}{n}_{J_1}}.$$

Proof. To see this, we note that $\prod_{p|n}\left(1+\frac{1}{p}\right) = \prod_{p|n}\left(\frac{p+1}{p}\right)$, and the sequence given by $\prod_{p|n}(p+1)$ has integral binomial coefficients by a modification of the argument from Corollary 3. Part (1) is then analogous to Theorem 7, and part (2) follows from the well known fact that $\psi(n) = \frac{J_2(n)}{J_1(n)}$. $\square$

We have added the ψ-nomial coefficients to [1] as A238498.

4. Combinatorial Interpretation

In this final section, we limit our focus to the 1-totientorial and the 1-totienomial coefficients, i.e. the coefficients associated with Euler's totient function $J_1 = \varphi$. Nonetheless, the questions described here are interesting for all sequences with integral binomial coefficients (see [5] or [3]).

Since $\binom{m}{n}_\varphi$ is a positive integer, we would like to find a class of objects counted by $\binom{m}{n}_\varphi$. For $m \in \mathbb{N}$, the 1-totientorial $m!_\varphi$ is the determinant of the $m \times m$ matrix, M, given by $M_{i,j} = \gcd(i,j)$ (see [9]). There is possibly some interpretation of $\binom{m}{n}_\varphi$ in terms of this matrix.

Moreover, $m!_\varphi$ gives the order of the abelian group $G = U_1 \times U_2 \times \cdots \times U_{m-1} \times U_m$, where U_i is multiplicative group of units in $\mathbb{Z}_i$. It would be interesting to find a set that G acts on in order to interpret $\binom{m}{n}_\varphi$ as the size of an orbit (using the orbit-stabilizer theorem). In relation to this, the integrality of $\binom{m}{n}_\varphi$ guarantees the existence of a $\binom{m}{n}_\varphi$-to-one function

$$f : U_{n+1} \times U_{n+2} \times \cdots \times U_m \to U_1 \times U_2 \times \cdots \times U_{m-n}.$$

It would be interesting to describe and understand functions of this type (to determine, for example, if such an f could be a homomorphism).

For our last point, we define the φ-Catalan numbers as the sequence given by

$$C_n^{\varphi} = \frac{1}{\varphi(n+1)} \binom{2n}{n}_{\varphi}.$$

Upon inspection of the first 5000 terms of this sequence, this sequence appears to be an integer sequence. A proof of integrality, especially via a combinatorial interpretation, is desirable.

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