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ON NUMBERS THAT CANNOT BE EXPRESSED AS A PLUS-MINUS WEIGHTED SUM OF A FIBONACCI NUMBER AND A PRIME

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Abstract

In 2012, Lenny Jones proved that there are infinitely many positive integers that cannot be expressed as $F_n - p$ or $F_n + p$, where F_n is the n^{th} term of the Fibonacci sequence and p denotes a prime. The smallest integer with this property he could find has 950 digits.

We prove that the numbers 135225 and 208641 have this property as well. We also show that there are infinitely many integers that cannot be written as $\pm F_n \pm p$, for any choice of the signs. In particular, 64369395 is such a number. We answer a question of Jones by showing that there exist infinitely many integers k such that neither k nor $k+1$ can be written as $F_n \pm p$. Finally, we prove that there exist infinitely many perfect squares and infinitely many perfect cubes that cannot be written as $F_n \pm p$.

1. Introduction

In 1849, de Polignac [3] conjectured that every odd integer k can be written as the sum of a prime number and a power of 2. It is well known that de Polignac's conjecture is false. Erdős [1] proved that there are infinitely many odd numbers that cannot be written as $2^n \pm p$ where n is a positive integer and p is a prime. More specifically, Erdős constructed infinitely many integers k such that $|k - 2^n|$ is composite for every positive integer n.

Consider the well known Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Very recently, Jones [2] considered the following variation of de Polignac's problem: are there any integers k that cannot be written in the form $F_n \pm p$ no matter how we choose the nonnegative integer n and the prime number p?

Jones solves the problem in the affirmative and constructs an infinite set of numbers with this property, the smallest of which is 950 digits long. Since both Erdős' and Jones' constructions have at their root the technique of covering systems, it is useful to present Erdős' approach first.

Definition 1. *[1] A finite covering system of the integers is a system of congruences $n \equiv r_i \pmod{m_i}$, with $1 \le i \le t$, such that every integer n satisfies at least one of the congruences. To avoid a trivial situation, we require $m_i > 1$ for all i.*

Erdős then considers triples $(r_i, m_i, p_i)_{i=1}^{i=t}$ with the following properties

$$\text{the set } \{(r_i, m_i)\}_{i=1}^{i=t} \text{ is a covering system of the integers.} \tag{1}$$

$$p_1, p_2, \ldots, p_t \text{ are distinct odd primes such that } ord_{p_i}(2) = m_i. \tag{2}$$

Here $ord_p(2)$ denotes the multiplicative order of 2 modulo the prime p, that is, the smallest positive integer m such that $2^m \equiv 1 \pmod{p}$. We are looking for a positive integer k with the property that $|k - 2^n|$ is composite for any choice of n.

Impose the system of congruences $k \equiv 2^{r_i} \pmod{p_i}$ for $i \in \{1, ..., t\}$. Since all primes p_i are odd, each of the above congruences has a solution for k in the form of an arithmetic sequence modulo p_i. Then, as $p_1, ..., p_t$ are distinct primes, by the Chinese Remainder Theorem it follows that all odd positive k that satisfy the system of congruences form an arithmetic progression modulo $2p_1 \cdots p_t$ (the fact that k must be odd translates into the additional congruence $k \equiv 1 \pmod{2}$). All such k have the property that $k - 2^n$ is always divisible by at least one p_i for $i \in \{1, ..., t\}$.

Indeed, let n be an arbitrary nonnegative integer. By (1), $n \equiv r_i \pmod{m_i}$ for some $1 \le i \le t$. Recall that from our choice (2) we have $2^{m_i} \equiv 1 \pmod{p_i}$ for every i. It follows that $k - 2^n \equiv k - 2^{r_i} \equiv 0 \pmod{p_i}$, where the last congruence follows from our choice of k. Thus, if $|k - 2^n| > \max\{p_i \mid i = 1, \ldots, t\}$, then $|k - 2^n|$ is composite for all n as desired.

Erdős uses the following system of triples $\{(r_i, m_i, p_i)\}_{i=1}^{i=t}$

$$\{(0, 2, 3), (0, 3, 7), (1, 4, 5), (3, 8, 17), (7, 12, 13), (23, 24, 241)\}.$$

It is easy to check that the above system satisfies both properties (1) and (2). The system of congruences $k \equiv 1 \pmod{2}$, $k \equiv 2^{r_i} \pmod{p_i}$, $1 \le i \le 6$, has the general solution $k \equiv 7629217 \pmod{2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241}$. It is not difficult to prove that this arithmetic sequence contains infinitely many numbers not of the form $2^n \pm p$. The last condition to be satisfied is $|k - 2^n| > 241$ so that we make certain $|k - 2^n|$ does not equal one of the primes p_i. Since the length of the intervals determined by two consecutive powers of 2 increases exponentially as n goes to infinity, we see that there are infinitely many terms in the arithmetic sequence above for which $|k - 2^n| > 241$ holds true. In particular, as $2^{23} - 7629217 = 759391$ and $7629217 - 2^{22} = 3434913$, it follows that $k = 7629217$ has the desired property.

2. The Fibonacci Variation: Jones' Approach

What is crucial in Erdős' construction is the fact that given any odd prime p, the sequence $(2^n)_{n\geq 0}$ is periodic modulo p and the length of the period is exactly $ord_p(2)$, the multiplicative order of 2 modulo p. Moreover, within a period, all values of $2^n \pmod{p}$ are distinct. We give several illustrations below:

$$p = 3,\ ord_3(2) = 2, \qquad (2^n \pmod{3})_{n\geq 0} = 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$$
$$p = 5,\ ord_5(2) = 4, \qquad (2^n \pmod{5})_{n\geq 0} = 1, 2, 4, 3, 1, 2, 4, 3, 1, 2, 4, 3, 1, 2, 4, 3, \ldots$$
$$p = 7,\ ord_7(2) = 3, \qquad (2^n \pmod{7})_{n\geq 0} = 1, 2, 4, 1, 2, 4, 1, 2, 4, 1, 2, 4, 1, 2, 4, 1, \ldots$$
$$p = 13,\ ord_{13}(2) = 12, \quad (2^n \pmod{13})_{n\geq 0} = 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, \ldots.$$

As Jones [2] noticed, the situation is far from being as simple if one considers the Fibonacci sequence instead. Fortunately, it is still true that for any given positive integer M the Fibonacci sequence is periodic modulo M; for a proof one can consult [4]. We denote this period by $h(M)$; this quantity is usually known as the M^{th} *Pisano period*. Although no general formula for $h(M)$ is known, there are extensive tables containing values of $h(M)$ for all $M \leq 10000$ - see for instance sequence A001175 in the Online Encyclopedia of Integer Sequences and the included references.

Jones constructs a system $\mathcal{C} = \{(r_i, h(p_i), p_i)\}$ with the following properties

$$p_1, p_2, \ldots, p_t \text{ are distinct prime numbers.} \tag{3}$$
$$\text{for every integer } n, \text{there exists an } i \text{ such that } n \equiv r_i \pmod{h(p_i)}. \tag{4}$$

Note that condition (4) is simply saying that the pairs $(r_i, h(p_i))_{i=1}^{i=t}$ form a covering system of the integers. Jones then imposes the conditions $k \equiv F_{r_i} \pmod{p_i}$ and argues that for such a k and for every nonnegative integer n we have that $|k - F_n| \equiv 0 \pmod{p_i}$ for some i in $\{1, 2, \ldots, t\}$.

Indeed, since $n \equiv r_i \pmod{h(p_i)}$ we have that $k - F_n \equiv k - F_{r_i} \equiv 0 \pmod{p_i}$ by our choice for k. However, finding a covering with the properties (3) and (4) is quite difficult. In order to achieve the covering we would like the values of $h(p_i)$ to be "nice" in the sense that $L = \text{lcm}(h(p_1), h(p_2), \ldots, h(p_t))$ is not too large and still, has plenty of divisors. Adding to the challenge, not every integer is an $h(p)$. Most notably, small integers such as 2, 6 and 12, which are very desirable as moduli in an economical covering, are not Pisano periods for any prime p.

Quite miraculously, Jones succeeds in constructing such a covering: his system has $t = 133$ triples $(r_i, h(p_i), p_i)$ and $L = 453600$. For future reference we list below the first few triples in Jones' covering:

$$(0, 3, 2), (0, 8, 3), (3, 20, 5), (6, 16, 7), (1, 10, 11), (2, 28, 13), (29, 36, 17), \ldots. \tag{5}$$

As a result, Jones ends up with an arithmetic sequence $k \equiv k_0 \pmod{P}$, infinitely many of whose terms are not expressible in the form $F_n \pm p$. Here k_0 is a 950-digit number and $P = p_1 \cdot p_2 \cdots p_{133}$.

3. The Fibonacci Variation: The New Idea

Given a prime number p, let us take a closer look at

$$T := (F_0 \pmod p, F_1 \pmod p, \ldots, F_{h(p)-1} \pmod p), \tag{6}$$

the entries of $F_n \pmod p$ within the first cycle. A few particular examples are going to be useful.

$$p = 2,\ h(p) = 3,\ T = (0, 1, 1). \tag{7}$$

$$p = 3,\ h(p) = 8,\ T = (0, 1, 1, 2, 0, 2, 2, 1). \tag{8}$$

$$p = 5,\ h(p) = 20,\ T = (0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1). \tag{9}$$

$$p = 7,\ h(p) = 16,\ T = (0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1). \tag{10}$$

$$p = 11,\ h(p) = 10,\ T = (0, 1, 1, 2, 3, 5, 8, 2, 10, 1). \tag{11}$$

Notice that Jones used $(r_1, h(p_1), p_1) = (0, 3, 2)$ as the first triple in (5): with this, all values of n for which $F_n \equiv 0 \pmod 2$ are covered. In other words, if one chooses $k \equiv 0 \pmod 2$ then $k - F_n \equiv 0 \pmod 2$ for every $n \equiv 0 \pmod 3$; that is, one third of the integers are covered. Notice that the corresponding value of S in (7) is $S = (0, 1, 1)$. This means that $F_n \equiv 1 \pmod 2$ **twice as often as** $F_n \equiv 0 \pmod 2$.

Accordingly, if one chooses $k \equiv 1 \pmod 2$ instead, we have that $k - F_n \equiv 0 \pmod 2$ whenever $n \equiv 1, 2 \pmod 3$, so we already have two thirds of the integers covered.

Let us examine now the second triple in (5): $(0, 8, 3)$. By choosing $k \equiv 0 \pmod 3$, we have that $k - F_n \equiv 0 \pmod 3$ as soon as $n \equiv 0 \pmod 4$, which means one fourth of the integers n are covered. However, we can do better. Notice that in (8) we have that $S = (0, 1, 1, 2, 0, 2, 2, 1)$ for $p = 3$. This means that $F_n \equiv 1 \pmod 3$ three-eights of the time and the same is true for $F_n \equiv 2 \pmod 3$. Hence, by choosing the congruence carefully, we can cover 3/8 of the integers instead of 1/4.

Similar discussions can be carried on for the other primes. Jones needs so many triples in his covering because for each prime p_i, the corresponding triple $(r_i, h(p_i), p_i)$ covers $1/h(p_i)$ of the integers. Using our approach however, we cover a multiple of that fraction. The difference is going to be seen shortly. The previous discussion prompts us to introduce the following definition.

Definition 2. *For every prime p and every integer r let*

$$S(r, p) = \left\{ n \in \mathbb{Z}^+ \mid F_n \equiv r \pmod p \right\}. \tag{12}$$

For instance,

$$S(8,3) = S(2,3) = \{3,\,5,\,6\} + 8\mathbb{Z}^+ = \{3,\,5,\,6,\,11,\,13,\,14,\,19,\,21,\,22,\ldots\}.$$
$$S(-1,7) = S(6,7) = \{7,\,9,\,10,\,14\} + 16\mathbb{Z}^+ = \{7,\,9,\,10,\,14,\,23,\,25,\,26,\,30,\ldots\}.$$

Suppose we can find sets $S(r_1,\,p_1)$, $S(r_2,\,p_2),\ldots,\,S(r_t,\,p_t)$, with $p_1,\,p_2,\,\ldots p_t$ distinct primes such that

$$S(r_1,\,p_1) \cup S(r_2,\,p_2) \cup \cdots \cup S(r_t,\,p_t) = \mathbb{Z}^+. \tag{13}$$

Then, choose k such that $k \equiv r_i \pmod{p_i}$ for every i in $\{1,\,2,\,\ldots,\,t\}$. The existence of such a number is guaranteed by the Chinese Remainder Theorem. We claim that with this choice of k, for every integer n, we have that $F_n - k \equiv 0 \pmod{p_i}$ for some $i \in \{1,\,2,\,\ldots,\,t\}$. Indeed, let n be an arbitrary integer. By (13), there exists an i such that $n \in S(r_i,\,p_i)$. Then $F_n \equiv r_i \equiv k \pmod{p_i}$, where the first congruence follows from the definition of $S(r_i, p_i)$ and the second congruence is a consequence of our choice for k.

We are now in position to prove our first result.

Theorem 1. *There are infinitely many integers of the form $k = 208641 + 312018\,Z$ which cannot be written as $F_n \pm p$ for any nonnegative integer n and any prime p. In particular, $k_0 = 208641$ is such a number.*

Proof. Consider the sets

$$S(1,2), S(0,3), S(6,7), S(0,17), S(2,19), S(8,23),$$

as defined in (12). We claim that

$$S(1,2) \cup S(0,3) \cup S(6,7) \cup S(0,17) \cup S(2,19) \cup S(8,23) = \mathbb{Z}^+. \tag{14}$$

Since $\operatorname{lcm}\{h(2),\,h(3),\,h(7),\,h(17),\,h(19),\,h(23)\} = \operatorname{lcm}\{3,\,8,\,16,\,36,\,18,\,48\} = 144$, it suffices to prove that

$$A := \{0,\,1,\ldots,\,143\} \subseteq S(1,2)\cup S(0,3)\cup S(6,7)\cup S(0,17)\cup S(2,19)\cup S(8,23). \tag{15}$$

Straightforward computations show that

$$\begin{aligned}
S(1,2) \cap A &= \{1,\,2,\,4,\,5,\,7,\,8,\,10,\,11,\ldots,\,136,\,137,\,139,\,140,\,142,\,143\},\\
S(0,3) \cap A &= \{0,\,4,\,8,\,12,\,16,\,20,\,24,\,28,\,\ldots,\,120,\,124,\,128,\,132,\,136,\,140\},\\
S(6,7) \cap A &= \{7,\,9,\,10,\,14,\,23,\,25,\,26,\,30,\,\ldots,\,122,\,126,\,135,\,137,\,138,\,142\},\\
S(0,17) \cap A &= \{0,\,9,\,18,\,27,\,36,\,45,\,54,\,63,\,72,\,81,\,90,\,99,\,108,\,117,\,126,\,135\},\\
S(2,19) \cap A &= \{3,\,8,\,15,\,21,\,26,\,33,\,39,\,44,\,51,\,\ldots,\,116,\,123,\,129,\,134,\,141\},\\
S(8,23) \cap A &= \{6,\,18,\,54,\,66,\,102,\,114\},
\end{aligned}$$

which means that (15), and therefore (14), is satisfied.

One may write down the explicit congruences in this covering as follows:

$$\mathcal{C} = \{(1,2;3),(0,4;8),(7,9,10,14;16),(0,9,18,27;36),(3,8,15;18),(6,18;48)\},$$

where, for example, $(1,2;3)$ means that the covering contains the two congruences $1 \pmod 3$ and $2 \pmod 3$.

Choose k as the solution of the system of congruences:

$$\begin{array}{lll} k \equiv 1 \pmod{2}, & k \equiv 0 \pmod{3}, & k \equiv 6 \pmod{7}, \\ k \equiv 0 \pmod{17}, & k \equiv 2 \pmod{19}, & k \equiv 8 \pmod{23}. \end{array}$$

It follows that $k = 208641 + 312018Z$, where Z is an arbitrary integer. For these particular k, we have that $\gcd(|F_n - k|, 2 \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 23) > 1$ for all nonnegative integers n. It follows that $|F_n - k|$ is composite unless $|F_n - k| \in \{2, 3, 7, 17, 19, 23\}$.

Note that $F_{31} = 1346269 > 3 \cdot 312018$, which means that the interval $[F_{32}, F_{33}]$ contains at least three numbers of the form $208641 + 312018\,Z$. Denote by ω_{32} one of these numbers, other than the smallest or the largest. Then, $F_{33} - \omega_{32} \geq 312018$ and $\omega_{32} - F_{32} \geq 312018$; this implies that $|F_n - \omega_{32}| \geq 312018$ for every nonnegative integer n, and therefore, as per our previous argument, $|F_n - \omega_{32}|$ is always composite.

In general, for every $s \geq 32$, the interval $[F_s, F_{s+1}]$ contains at least one number ω_s of the form $208641 + 312018\,Z$, for which $F_{s+1} - \omega_s \geq 312018$ and $\omega_s - F_s \geq 312018$. It follows that $|F_n - \omega_s| \geq 312018$ and therefore, it is composite for all $n \geq 0$. We thus have infinitely many numbers of the form $208641 + 312018Z$ that cannot be written as $F_n \pm p$.

In particular, $k_0 = 208641$ has the property that $|F_n - k_0| \geq 12223$, and therefore, $|F_n - k_0|$ is composite for all $n \geq 0$. This implies that k_0 cannot be written as $F_n \pm p$ where p is a prime. □

Note added in proof. One of the referees noticed that there are infinitely many integers of the form $k = 135225 + 1647030\,Z$ which cannot be written as $F_n \pm p$ for any nonnegative integer n and any prime p. In particular, $k_0 = 135225$ is such a number. To see this, consider the following collection of S-sets:

$$\{S(1,2), S(0,3), S(0,5), S(6,7), S(2,11), S(8,23), S(3,31)\}.$$

Reasoning as above, it is straightforward to show that this collection corresponds to the following covering:

$$\{(1,2;3),(0,4;8),(0,5,10,15;20),(7,9,10,14;16),(3,7;10),(6,18;48),(4,9,21;30)\},$$

which produces $k = 135225 + 1647030\,Z$. It would be interesting to find the smallest integer k^*, that cannot be written in the form $F_n \pm p$. As per our previous discussion, we have that $k^* \leq 135225$.

4. Adding One More Restriction: $k \neq \pm F_n \pm p$

In this section we prove that there are infinitely many integers k that cannot be written as $\pm F_n \pm p$ for any choice of the signs $\pm$ and any integer n, and prime p. We achieve this by extending the approach presented in the previous section.

Recall the definition

Definition 3. *For every prime p and every integer r let*

$$S(r,p) = \left\{ n \in \mathbb{Z}^+ \mid F_n \equiv r \pmod{p} \right\}. \tag{16}$$

Suppose we can find sets $S(r_1, p_1)$, $S(r_2, p_2), \ldots, S(r_t, p_t)$, with $p_1, p_2, \ldots p_t$ distinct primes such that

$$S(r_1, p_1) \cup S(r_2, p_2) \cup \cdots \cup S(r_t, p_t) = \mathbb{Z}^+ \text{ and} \tag{17}$$

$$S(-r_1, p_1) \cup S(-r_2, p_2) \cup \cdots \cup S(-r_t, p_t) = \mathbb{Z}^+. \tag{18}$$

Then, choose k such that $k \equiv r_i \pmod{p_i}$ for every i in $\{1, 2, \ldots, t\}$. As before, the existence of such a number is guaranteed by the Chinese Remainder Theorem.

We claim that with this choice of k, for every integer n, we have that $F_n - k \equiv 0 \pmod{p_i}$ for some $i \in \{1, 2, \ldots, t\}$ and $F_n + k \equiv 0 \pmod{p_j}$ for some $j \in \{1, 2, \ldots, t\}$. Indeed, let n be an arbitrary integer. Then by (17), there exists an i, $1 \le i \le t$, such that $n \in S(r_i, p_i)$. This implies that

$$F_n \equiv r_i \equiv k \pmod{p_i},$$

where the first congruence follows from the definition of $S(r,p)$ and the second one results from our choice of k. We obtain $F_n - k \equiv 0 \pmod{p_i}$ as claimed.

Similarly, by (18), there exists a j, $1 \le j \le t$ such that $n \in S(-r_j, p_j)$. This gives that

$$F_n \equiv -r_j \equiv -k \pmod{p_j},$$

where the first congruence follows from the definition of $S(r,p)$ and the second one results from our choice of k. In this case, we have $F_n + k \equiv 0 \pmod{p_j}$ as desired.

Theorem 2. *There are infinitely many integers of the form $k = 64369395 + 531990690\,Z$ that cannot be written as $\pm F_n \pm p$ for any integer n and any prime p. In particular $k_0 = 64369395$ is such a number.*

Proof. Consider the sets $S(r_i, p_i)$, $1 \le i \le 9$ defined below

$$S(1,2), S(0,3), S(6,7), S(0,17), S(17,19), S(8,23), S(0,5), S(2,11), S(3,31).$$

Notice that the corresponding sets $S(-r_i, p_i)$ can be written as

$$S(1,2), S(0,3), S(1,7), S(0,17), S(2,19), S(15,23), S(0,5), S(9,11), S(28,31).$$

We claim that

$$\bigcup_{i=1}^{9} S(r_i, p_i) = \mathbb{Z}^+ \qquad \text{and} \qquad \bigcup_{i=1}^{9} S(-r_i, p_i) = \mathbb{Z}^+.$$

Since $\operatorname{lcm}\{h(2), h(3), h(7), h(17), h(19), h(23), h(5), h(11), h(31)\} = 720$, it would suffice to prove that

$$A \subseteq \bigcup_{i=1}^{9} S(r_i, p_i) \qquad \text{and} \qquad A \subseteq \bigcup_{i=1}^{9} S(-r_i, p_i),$$

where $A := \{0, 1, \ldots, 719\}$. Straightforward calculations show that

$$\begin{aligned}
S(1,2) \cap A &= \{1, 2, 4, 5, 7, 8, 10, 11, \ldots, 709, 710, 712, 713, 715, 716, 718, 719\}\\
S(0,3) \cap A &= \{0, 4, 8, 12, 16, 20, 24, \ldots, 692, 696, 700, 704, 708, 712, 716\}\\
S(6,7) \cap A &= \{7, 9, 10, 14, 23, 25, 26, 30, \ldots, 697, 698, 702, 711, 713, 714, 718\}\\
S(1,7) \cap A &= \{1, 2, 6, 15, 17, 18, 22, 31, \ldots, 690, 694, 703, 705, 706, 710, 719\}\\
S(0,17) \cap A &= \{0, 9, 18, 27, 36, 45, 54, 63, \ldots, 657, 666, 675, 684, 693, 702, 711\}\\
S(17,19) \cap A &= \{10, 28, 46, 64, 82, 100, 118, \ldots, 622, 640, 658, 676, 694, 712\}\\
S(2,19) \cap A &= \{3, 8, 15, 21, 26, 33, 39, 44, \ldots, 681, 687, 692, 699, 705, 710, 717\}\\
S(8,23) \cap A &= \{6, 18, 54, 66, 102, 114, 150, \ldots, 582, 594, 630, 642, 678, 690\}\\
S(15,23) \cap A &= \{30, 42, 78, 90, 126, 138, 174, \ldots, 606, 618, 654, 666, 702, 714\}\\
S(0,5) \cap A &= \{0, 5, 10, 15, 20, 25, 30, 35, \ldots, 685, 690, 695, 700, 705, 710, 715\}\\
S(2,11) \cap A &= \{3, 7, 13, 17, 23, 27, 33, 37, \ldots, 687, 693, 697, 703, 707, 713, 717\}\\
S(9,11) \cap A &= \{3, 21, 27, 45, 51, 69, 75, 93, \ldots, 651, 669, 675, 693, 699, 717\}\\
S(3,31) \cap A &= \{4, 9, 21, 34, 39, 51, 64, 69, \ldots, 651, 664, 669, 681, 694, 699, 711\}\\
S(28,31) \cap A &= \{26, 56, 86, 116, 146, 176, 206, \ldots, 566, 596, 626, 656, 686, 716\}.
\end{aligned}$$

A simple computer program can be now used to verify that the desired conditions are indeed satisfied. Choose k as the solution of the system of congruences:

$$\begin{array}{lll}
k \equiv 1 \pmod{2}, & k \equiv 0 \pmod{3}, & k \equiv 6 \pmod{7},\\
k \equiv 0 \pmod{17}, & k \equiv 17 \pmod{19}, & k \equiv 8 \pmod{23},\\
k \equiv 0 \pmod{5}, & k \equiv 2 \pmod{11}, & k \equiv 3 \pmod{31}.
\end{array}$$

It follows that $k = 64369395 + 531990690Z$, where Z is an arbitrary integer. For these particular k, we have that $\gcd(|F_n \pm k|, 2 \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 23 \cdot 5 \cdot 11 \cdot 31) > 1$ for all nonnegative integers n. It follows that $|F_n \pm k|$ is composite unless $|F_n \pm k| \in \{2, 3, 5, 7, 11, 17, 19, 23, 31\}$.

Note that $F_{42} = 267914296 > 3 \cdot 531990690$, which means that the interval $[F_{43}, F_{44}]$ contains at least three numbers of the form $k = 64369395 + 531990690Z$. Denote by ω_{43} one of these numbers, other than the smallest or the largest.

Then, $F_{44} - \omega_{43} \geq 531990690$ and $\omega_{43} - F_{43} \geq 531990690$; this implies that $|F_n - \omega_{43}| \geq 531990690$ for every nonnegative integer n, and therefore, as per our previous argument, $|F_n \pm \omega_{43}|$ is always composite. The same argument can be used for every $n \geq 43$. We thus have infinitely many numbers of the form $64369395 + 531990690Z$ that cannot be written as $\pm F_n \pm p$.

In particular, $k_0 = 64369395$ has the property that $|F_n \pm k_0| \geq 1123409$, and therefore, $|F_n \pm k_0|$ is composite for all $n \geq 0$. This implies that k_0 cannot be written as $\pm F_n \pm p$ where p is a prime. □

5. Consecutive Numbers Not of the Form $F_n \pm p$

Jones [2] asked whether there do exist integers k, such that neither k nor $k+1$ can be written in the form $F_n + p$ or $F_n - p$. We prove that there are infinitely many such integers. The approach is similar to the one in the previous section, only more challenging.

Suppose we can find sets $S(r_1, p_1)$, $S(r_2, p_2), \ldots, S(r_t, p_t)$, with $p_1, p_2, \ldots, p_t$ distinct primes such that

$$S(r_1, p_1) \cup S(r_2, p_2) \cup \cdots \cup S(r_t, p_t) = \mathbb{Z}^+ \text{ and} \tag{19}$$

$$S(1 + r_1, p_1) \cup S(1 + r_2, p_2) \cup \cdots \cup S(1 + r_t, p_t) = \mathbb{Z}^+. \tag{20}$$

Then, choose k such that $k \equiv r_i \pmod{p_i}$ for every i in $\{1, 2, \ldots, t\}$. As before, the existence of such a number is guaranteed by the Chinese Remainder Theorem.

We claim that with this choice of k, for every integer n, we have that $F_n - k \equiv 0 \pmod{p_i}$ for some $i \in \{1, 2, \ldots, t\}$ and $F_n - (k+1) \equiv 0 \pmod{p_j}$ for some $j \in \{1, 2, \ldots, t\}$. Indeed, let n be an arbitrary integer. Then by (19), there exists an i, $1 \leq i \leq t$, such that $n \in S(r_i, p_i)$. This implies that

$$F_n \equiv r_i \equiv k \pmod{p_i},$$

where the first congruence follows from the definition of $S(r, p)$ and the second one results from our choice of k. We obtain $F_n - k \equiv 0 \pmod{p_i}$ as claimed.

Similarly, by (20), there exists a j, $1 \leq j \leq t$, such that $n \in S(1 + r_j, p_j)$. This gives that

$$F_n \equiv 1 + r_j \equiv k + 1 \pmod{p_j},$$

where the first congruence follows from the definition of $S(r, p)$ and the second one results from our choice of k. In this case, we have $F_n - k - 1 \equiv 0 \pmod{p_j}$ as desired.

The main difficulty in finding sets $S(r_i, p_i)$ with the properties (19) and (20) is that, more often than not, if $S(r, p)$ is "good" then $S(1+r, p)$ is "bad" and viceversa. Here "good" and "bad" are referring to the fraction of integers covered by the set $S(r, p)$.

Despite this, we were able to find such a covering system with $t = 19$ sets.

Theorem 3. *There are infinitely many integers of the form*

$$k = 13889533546318195209442052906782 7 + \prod_{i=1}^{19} p_i\, Z,$$

such that neither k nor $k+1$ can be written as $F_n \pm p$ for any integer n integer and any p prime. Here $\prod_{i=1}^{19} p_i = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 47 \cdot 61 \cdot 107 \cdot 181 \cdot 241 \cdot 541 \cdot 1103 \cdot 2161 \cdot 2521$.

Proof. Consider the table below.

r_i	1	2	2	0	0	0	2	21	4	1	8
p_i	2	3	5	7	11	17	19	23	31	41	61
$h(p_i)$	3	8	20	16	10	36	18	48	30	40	60

r_i	3	54	563	63	54	959	179	205
p_i	47	107	2161	2521	241	1103	181	541
$h(p_i)$	32	72	80	120	240	96	90	90

We claim that for the choices above we have that

$$\bigcup_{i=1}^{19} S(r_i, p_i) = \bigcup_{i=1}^{19} S(1 + r_i, p_i) = \mathbb{Z}^+,$$

and therefore conditions (19) and (20) are satisfied.

Notice that $\text{lcm}(h(p_1), \ldots, h(p_{19})) = 1440$, and therefore it would suffice to check that

$$\{0, 1, \ldots, 1439\} \subseteq \bigcup_{i=1}^{19} S(r_i, p_i) \quad \text{and} \quad \{0, 1, \ldots, 1439\} \subseteq \bigcup_{i=1}^{19} S(1 + r_i, p_i).$$

A simple computer verification is needed. For the reader curious to test the result, it would suffice to consider the differences $F_n - k$ and $F_n - k - 1$ for $0 \leq n \leq 1439$ and check that each such number is divisible by some p_i. The difficulty was finding the sets $S(r_i, p_i)$ that satisfy (19) and (20). Once they were found, it is quite straightforward to check that they have the desired properties. □

6. Perfect Squares and Perfect Cubes Not of the Form $F_n \pm p$

Jones [2] asked whether there exist infinitely many powers not of the form $F_n \pm p$. We prove that this is indeed true in the cases of perfect squares and perfect cubes.

Theorem 4. *There are infinitely many integers* $k^2 = (6945512265 + 465395333370\,Z)^2$ *that cannot be written as* $F_n \pm p$ *for any integer* n *and any prime* p.

Proof. It is straightforward to check that the union

$$S(1,2) \cup S(0,3) \cup S(0,5) \cup S(1,11) \cup S(0,17) \cup S(2,31) \cup S(8,41) \cup S(22,61) \cup S(99,107)$$

covers the integers. Notice that for every set $S(r_i, p_i)$, $1 \le i \le 9$ in the above relation, the equation $k^2 \equiv r_i \pmod{p_i}$ has solutions, that is, r_i is a quadratic residue modulo p_i. If one chooses k subject to the following congruences:

$$\begin{array}{lll} k \equiv 1 \pmod{2}, & k \equiv 0 \pmod{3}, & k \equiv 0 \pmod{5}, \\ k \equiv 10 \pmod{11}, & k \equiv 0 \pmod{17}, & k \equiv 23 \pmod{31}, \\ k \equiv 7 \pmod{41}, & k \equiv 49 \pmod{61}, & k \equiv 62 \pmod{107}, \end{array}$$

then it is immediate that $k^2 \equiv r_i \pmod{p_i}$ for every $1 \le i \le 9$, and therefore for every $n \ge 0$, $\gcd(k^2 - F_n, 2 \cdot 3 \cdot 5 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 61 \cdot 107) > 1$.

This means that $|k^2 - F_n|$ cannot be a prime unless it equals one of the p_i. Since F_n grows exponentially while the sequence $k^2 = (6945512265 + 465395333370\,Z)^2$ has quadratic growth, it follows that for n large enough the interval $[F_n, F_{n+1}]$ contains at least three such k^2. This implies that for infinitely many values of k, the difference $|k^2 - F_n|$ is always large. The proof is complete. □

A similar result is valid for perfect cubes.

Theorem 5. *There are infinitely many integers* $k^3 = (56145 + 1647030\,Z)^3$ *that cannot be written as* $F_n \pm p$ *for any integer* n *and any prime* p.

Proof. It is straightforward to check that

$$S(1,2) \cup S(0,3) \cup S(0,5) \cup S(6,7) \cup S(1,11) \cup S(8,23) \cup S(2,31) = \mathbb{Z}^+.$$

Notice that for every set $S(r_i, p_i)$, $1 \le i \le 7$ in the above relation, the equation $k^3 \equiv r_i \pmod{p_i}$ has solutions, that is, r_i is a cubic residue modulo p_i. If one chooses k subject to the following congruences:

$$\begin{array}{llll} k \equiv 1 \pmod{2}, & k \equiv 0 \pmod{3}, & k \equiv 0 \pmod{5}, & k \equiv 5 \pmod{7} \\ k \equiv 1 \pmod{11}, & k \equiv 2 \pmod{23}, & k \equiv 4 \pmod{31}, & \end{array}$$

then it is immediate that $k^3 \equiv r_i \pmod{p_i}$ for every $1 \le i \le 7$, and therefore for every $n \ge 0$, $\gcd(k^3 - F_n, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31) > 1$.

This means that $|k^3 - F_n|$ cannot be a prime unless it equals one of the primes p_i. Reasoning similarly as above, it can be easily shown that there exist infinitely many values of k for which this does not happen. This concludes the proof. □

7. An Open Question

Erdős proved that there exist infinitely many integers that cannot be written as $2^n \pm p$. Jones showed a similar result if 2^n is replaced by F_n, the general term of the Fibonacci sequence. What these two sequences have in common is that they grow exponentially as n increases. The following conjecture seems reasonable.

Conjecture 1. *Let $(x_n)_{n\geq 0}$ be an integer sequence defined by a second order recurrence relation $x_{n+2} = ax_{n+1} + bx_n$, where a and b are integers. Further assume that $\lim_{n\to\infty} |x_n| = +\infty$. Then, there exist integers k which cannot be written in the form $\pm x_n \pm p$ for any n and any prime p.*

In other words, we expect a sequence that behaves similarly to a geometric sequence not to be dense enough, in combination with all the primes, to cover all the integers.

References

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