SOME CONGRUENCES FOR BALANCING AND LUCAS-BALANCING NUMBERS AND THEIR APPLICATIONS

Prasanta Kumar Ray
International Institute of Information Technology Bhubaneswar, Odisha, India
prasanta@iiit-bh.ac.in

Received: 5/17/12, Revised: 12/4/13, Accepted: 1/4/14, Published: 1/30/14

Abstract

Balancing numbers n and balancers r are solutions of the Diophantine equation $1+2+\ldots+(n-1)=(n+1)+(n+2)+\ldots+(n+r)$. It is well-known that if n is a balancing number, then $8n^2+1$ is a perfect square and its positive square root is called a Lucas-balancing number. In this paper, some new identities involving balancing and Lucas-balancing numbers are obtained. Some divisibility properties of these numbers are also studied.

1. Introduction

The concept of balancing numbers was originally introduced by A. Behera and G.K. Panda [1] in connection with the Diophantine equation

$$1+2+\ldots+(n-1)=(n+1)+(n+2)+\ldots+(n+r),$$

where n is a balancing number, and r is a balancer corresponding to n. The numbers 6, 35 and 204 are balancing numbers with balancers 2, 14 and 84 respectively. The n^{th} balancing number is denoted by B_n and the balancing numbers satisfy the recurrence relation

$$B_{n+1}=6B_n-B_{n-1}; \quad n\geq 2, \tag{1}$$

with $B_1=1$ and $B_2=6$ [1]. The recurrence relation for Lucas-balancing numbers is also similar to balancing numbers and is given by

$$C_{n+1}=6C_n-C_{n-1}; \quad n\geq 2, \tag{2}$$

where $C_n=\sqrt{8B_n^2+1}$ denotes the n^{th} Lucas-balancing number with $C_1=3$ and $C_2=17$ [12]. In [6], K. Liptai searched for those balancing numbers which are also Fibonacci numbers and found that the only balancing number in the sequence of Fibonacci numbers is 1. In [7], he also proved that there is no Lucas number in the sequence of balancing numbers. L. Szalay, in [16], also obtained the same result. In a subsequent paper, K. Liptai et al. [8] added another interesting result to the theory of balancing numbers by generalizing these numbers. A. Berczes et al. [2] and P. Olajos [9] studied many interesting properties of generalized balancing numbers. G.K. Panda [12] established many useful identities involving balancing and Lucas-balancing numbers. Certain congruence properties of balancing numbers were also studied in [15]. In [13, 14], the author established some new product formulas for balancing and Lucas-balancing numbers. Recently, R. Keskin and O. Karaath [4] obtained some new properties for balancing numbers and square triangular numbers.

There are many well-known relationships between balancing and Lucas-balancing numbers. Most of the relationships were established from the Binet's formulas

$$B_n = \frac{\lambda^n - \lambda^{-n}}{2\sqrt{8}}, \quad C_n = \frac{\lambda^n + \lambda^{-n}}{2}, \tag{3}$$

where $\lambda = 3 + \sqrt{8}$ and $\lambda^{-1} = 3 - \sqrt{8}$. An interesting fact is that, for all integers n, $\lambda^n = \lambda B_n - B_{n-1}$ and $\lambda^{-n} = \lambda^{-1} B_n - B_{n-1}$.

In this paper, we establish some interesting sum formulas involving balancing and Lucas-balancing numbers and then obtain some congruences concerning these numbers. These congruences allow us to prove many known and new properties of balancing and Lucas-balancing numbers. With these congruences, certain results concerning divisibility properties are also discussed.

2. Sums and Congruences Concerning Balancing and Lucas-Balancing Numbers

The following lemma is useful for proving the subsequent important results.

Lemma 2.1. *If X is a square matrix of order 2 with $X^2 = 6X - I$ where I is the identity matrix of the same order as X, then $X^n = B_n X - B_{n-1} I$ for all integers n.*

Proof. Since $\lambda = 3+\sqrt{8}$, it can be easily shown that the set $\mathbb{Z}[\lambda] = \{a\lambda - b : a, b \in \mathbb{Z}\}$ is a ring. Therefore, the set $\mathbb{Z}[X] = \{aX - bI : a, b \in \mathbb{Z}\}$ is also a ring. Further, the mapping $\varphi : \mathbb{Z}[\lambda] \to \mathbb{Z}[X]$ defined by $\varphi(a\lambda - b) = aX - bI$ is a ring isomorphism and by considering the facts $\varphi(\lambda) = X$ and $\varphi(C_m) = -C_m I$, we get

$$X^n = [\varphi(\lambda)]^n = \varphi(\lambda^n) = \varphi(\lambda B_n - B_{n-1}) = B_n X - B_{n-1} I.$$

□

Observe that if $S = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$, then $S^2 = 6S - I$. Using the well known identity $3B_n - B_{n-1} = C_n$ [12], the following result follows from Lemma 2.1.

Corollary 2.2. *If* $S = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$, $S^n = \begin{bmatrix} C_n & 8B_n \\ B_n & C_n \end{bmatrix}$.

As usual, let B_m and C_m be the m^{th} balancing number and m^{th} Lucas-balancing number respectively. Since $\lambda = 3+\sqrt{8}$, the following identities can be easily verified.

$$\lambda^{2m} - 2C_m\lambda^m + 1 = 0, \tag{4}$$
$$\lambda^{2m} - 2B_m\sqrt{8}\lambda^m - 1 = 0. \tag{5}$$

Moreover, as the mapping $\varphi : Z[\lambda] \to Z[S]$ defined by $\varphi(a\lambda - b) = aS - bI$ is a ring isomorphism, applying φ to the identities (4) and (5), we obtain

$$S^{2m} - 2C_mS^m + I = 0, \tag{6}$$
$$S^{2m} - 2B_mKS^m - I = 0, \tag{7}$$

where $K = \varphi(\sqrt{8}) = \varphi(\lambda - 3) = S - 3I = \begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix}$.

We are now in a position to present our main results.

Theorem 2.3. *For any* $n \in \mathbb{N}$ *and* $m, k \in \mathbb{Z}$, *we have*

$$C_{2mn+k} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j 2^j C_m^j C_{mj+k},$$
$$B_{2mn+k} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j 2^j C_m^j B_{mj+k}.$$

Proof. By (5), $S^{2m} = 2C_mS^m - I$ for each $m \in \mathbb{Z}$. This gives

$$S^{2mn} = (2C_mS^m - I)^n = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j 2^j C_m^j S^{mj}.$$

Multiplying both sides by S^k, we obtain

$$S^{2mn+k} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j 2^j C_m^j S^{mj+k}.$$

Now the results follow from Corollary 2.2. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4. *For any* $n \in \mathbb{N}$ *and* $m, k \in \mathbb{Z}$,

$$B_{2mn+k} \equiv (-1)^n B_k \ (mod\ C_m), \ \ C_{2mn+k} \equiv (-1)^n C_k \ (mod\ C_m). \tag{8}$$

Since $K = S - 3I$, it follows that $2K = S - S^{-1}$. Therefore, $S^m K = KS^m$ for every integer m. Moreover, $K^2 = 8I$ and

$$\begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 8c & 8d \\ a & b \end{bmatrix}.$$

These results are very useful for the proof of the following theorem.

Theorem 2.5. *For each $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$,*

$$C_{2mn+k} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} 8^j 2^{2j} B_m^{2j} C_{2mj+k} + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} 8^{j+1} 2^{2j+1} B_m^{2j+1} B_{2mj+m+k},$$

$$B_{2mn+k} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} 8^j 2^{2j} B_m^{2j} B_{2mj+k} + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} 8^j 2^{2j+1} B_m^{2j+1} C_{2mj+m+k}.$$

Proof. By (6), $S^{2m} = 2B_m KS^m + I$ and we have

$$\begin{aligned}
\begin{bmatrix} C_{2mn+k} & 8B_{2mn+k} \\ B_{2mn+k} & C_{2mn+k} \end{bmatrix}
&= S^{2mn+k} \\
&= (2B_m KS^m + I)^n S^k \\
&= \sum_{j=0}^{n} \binom{n}{j} 2^j K^j B_m^j S^{mj+k} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} 2^{2j} K^{2j} B_m^{2j} S^{2mj+k} + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} 2^{2j+1} K^{2j+1} B_m^{2j+1} S^{(2j+1)m+k} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} 8^j 2^{2j} B_m^{2j} S^{2mj+k} + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} 8^j 2^{2j+1} B_m^{2j+1} KS^{(2j+1)m+k} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} 8^j 2^{2j} B_m^{2j} \begin{bmatrix} C_{2mj+k} & 8B_{2mj+k} \\ B_{2mj+k} & C_{2mj+k} \end{bmatrix} \\
&\quad + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} 8^j 2^{2j+1} K^{2j+1} B_m^{2j+1} \begin{bmatrix} 8B_{2mj+m+k} & 8C_{2mj+m+k} \\ C_{2mj+m+k} & 8B_{2mj+m+k} \end{bmatrix}.
\end{aligned}$$

This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6. *For each $n \in \mathbb{N} \cup \{0\}$ and $m, k \in \mathbb{Z}$,*

$$B_{2mn+k} \equiv B_k \ (mod\ B_m), \quad C_{2mn+k} \equiv C_k \ (mod\ B_m). \tag{9}$$

The following result is an important congruence for balancing numbers.

Theorem 2.7. *For positive integers l, m and n with $l \neq m$,*

$$B_m^n B_{ln} \equiv B_l^n B_{mn} \ (mod\ B_{m-1}).$$

In order to prove Theorem 2.7, we need the following lemma. Note that the lemma's equation is an expansion of the identity $\sum_{k=0}^{n} \binom{n}{k} B_l^k B_{l-1}^{n-k} B_k = B_{ln}$.

Lemma 2.8. *For positive integers l, m and n with $l \neq m$,*

$$\sum_{k=0}^{n} \binom{n}{k} B_l^k B_{m-l}^{n-k} B_{mk} = B_m^n B_{ln}.$$

Proof. By virtue of (1) and the identities $\lambda^n = \lambda B_n - B_{n-1}$ and $B_{m-l} = B_{l+1}B_m - B_l B_{m+1}$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} B_l^k B_{m-l}^{n-k} \lambda_1^{mk} &= \sum_{k=0}^{n} \binom{n}{k} (B_l \lambda^m)^k B_{m-l}^{n-k} \\
&= (B_l \lambda^m + B_{m-l})^n \\
&= [B_l(\lambda B_m - B_{m-1}) + B_{l+1}B_m - B_l B_{m+1}]^n \\
&= [\lambda B_l B_m - 6 B_l B_m + B_{l+1} B_m]^n \\
&= B_m^n [\lambda B_l - B_{l-1}]^n = B_m^n \lambda^{ln}.
\end{aligned}$$

In a similar manner, we can get

$$\sum_{k=0}^{n} \binom{n}{k} B_l^k B_{m-l}^{n-k} \lambda^{-mk} = B_m^n \lambda^{-ln}.$$

Consequently,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} B_l^k B_{m-l}^{n-k} B_{mk} &= \sum_{k=0}^{n} \binom{n}{k} B_l^k B_{m-l}^{n-k} \frac{\lambda^{mk} - \lambda^{-mk}}{\lambda - \lambda^{-1}} \\
&= B_m^n \frac{\lambda^{ln} - \lambda^{-ln}}{\lambda - \lambda^{-1}} = B_m^n B_{ln}.
\end{aligned}$$

This ends the proof. □

Now we are in a position to prove Theorem 2.7.

Proof of Theorem 2.7. By virtue of Lemma 2.8 and the fact that B_{m-l} divides B_{m-l}^{n-k}, we have

$$B_m^n B_{ln} - B_l^n B_{mn} = \sum_{k=0}^{n-1} \binom{n}{k} B_l^k B_{m-l}^{n-k} B_{mk} \equiv 0 \ (mod\ B_{m-l}),$$

from which Theorem 2.7 follows. □

3. Divisibility Properties of Balancing and Lucas-Balancing Numbers

The oldest non-trivial example of a divisibility sequence is probably the Fibonacci sequence. Since then many examples of divisibility sequences such as Lucas sequence, Mersenne sequence, generalized Mersenne sequences, balancing and Lucas-balancing sequences were studied by different authors [5, 12, 17]. In [3], J.P. Bézivin et al. characterized the totality of the divisibility properties of such sequences.

In this section, we prove some known and new results concerning the divisibility properties of balancing and Lucas-balancing numbers with the help of the congruences given in Corollary 2.4 and Corollary 2.6. Before proving the results, we first present some identities involving balancing and Lucas-balancing numbers which will be needed subsequently. The proofs of these identities are omitted as they can be easily obtained from the Binet's formulas (3).

Lemma 3.1. *For any integers m and k,*

$$C_{m+k} = B_k C_{m+1} - C_m B_{k-1}, \tag{10}$$

$$8B_{m-k} = C_m C_{k-1} - C_k C_{m-1}, \tag{11}$$

$$B_{m+k} = B_{m+1} B_k - B_m B_{k-1}. \tag{12}$$

Theorem 3.2. *If m and n are any integers with $m \geq 1$, then $C_m | C_n$ if and only if $m|n$ and $\frac{n}{m}$ is an odd integer.*

Proof. Assume that $C_m|C_n$ and $n = mq + k$, where $0 \leq k < m$. if q is an even integer, then $q = 2t$ for some $t \in \mathbb{Z}$. So using (7), we obtain

$$C_n = C_{2mt+k} \equiv (-1)^t C_k \ (mod\ C_m).$$

It follows that $C_m|C_k$. This is impossible because $k < m$, implies that $C_k < C_m$. Therefore q must be an odd integer. Let $q = 2t + 1$ for some $t \in \mathbb{Z}$. Now by (7), we have

$$C_n = C_{2mt+m+k} \equiv (-1)^t C_{m+k} \ (mod\ C_m).$$

Thus, $C_m|C_n$ implies that $C_m|C_{m+k}$. If $k > 0$, in view of (10), $C_m|B_k C_{m+1}$. Since $\gcd(C_m, C_{m+1}) = 1$, $C_m|B_k$, which is impossible because $k < m$ and hence $B_k \leq B_m < C_m$. Thus, $k = 0$ and hence $n = mq$ where q is an odd integer.

Conversely, suppose $m|n$ and $\frac{n}{m}$ is an odd integer. Let $n = (2t + 1)m$ for some $t \in \mathbb{Z}$. By (7), we have $C_n = C_{2mt+m} \equiv (-1)^t C_m \ (mod\ C_m)$; it follows that $C_m|C_n$. □

Theorem 3.3. *If m and n are any integers with $m \geq 1$, then $C_m|B_n$ if and only if $m|n$ and $\frac{n}{m}$ is an even integer.*

Proof. Assume that $C_m|C_n$ and $n = mq + k$ where $0 \leq k < m$ and $m \geq 1$. If q is an odd integer, then $q = 2t - 1$ for some $t \in \mathbb{Z}$. Now by virtue of (7) and the well known identity $B_{-n} = -B_n$, we obtain

$$B_n = B_{2mt-m+k} \equiv (-1)^t B_{-m+k} \ (mod\ C_m) = (-1)^{t+1} B_{m-k} \ (mod\ C_m).$$

Now $C_m|B_n$, we have $C_m|B_{m-k}$ which implies that $C_m|8B_{m-k}$. By (11), $C_m|C_k C_{m-1}$. Since $\gcd(C_m, C_{m-1}) = 1$, $C_m|C_k$ which is impossible since $k < m$. Thus, q must be an even integer. Putting $q = 2t$ and using (7), we get

$$B_n = B_{2mt+k} \equiv (-1)^t B_k \ (mod\ C_m).$$

Now $C_m|B_n$ implies $C_m|B_k$ which is impossible since $k < m$. Therefore, we must have $k = 0$. Thus, $n = mq$ where q is an even integer.

Conversely, suppose that $m|n$ and $\frac{n}{m}$ is an even integer. Let $n = 2tm$ for some $t \in \mathbb{Z}$. Using (7), we get

$$B_n = B_{2mt} \equiv (-1)^t B_0 \ (mod\ C_m),$$

from which it follows that $C_m|B_n$. □

Theorem 3.4. *If m and n are any natural numbers with $m \geq 1$, then $B_m|B_n$ if and only if $m|n$.*

Proof. Suppose that $B_m|B_n$, and if possible assume that $m \nmid n$. Let $n = mq + r$ where $0 < r < m$ and $m \geq 1$. If q is an even integer, $q = 2t$ for some $t \in \mathbb{Z}$. Using (9) and the fact that $B_{-n} = -B_n$, we obtain $B_n = B_{2mt+r} \equiv B_r \ (mod\ B_m)$. Again $B_m|B_n$ implies that $B_m|B_r$. This is a contradiction since $r < m$. If q is odd, setting $q = 2t + 1$ for some $t \in \mathbb{Z}$, we obtain $B_n = B_{2mt+m+r} \equiv B_{m+r} \ (mod\ B_m)$. Since $B_m|B_n$, it follows that $B_m|B_{m+r}$. So by virtue of (12), $B_m|B_{m+1}B_r$. As $\gcd(B_m, B_{m+1}) = 1$, we have $B_m|B_r$. Thus, we must have $r = 0$. This implies that $n = mq$ and consequently $m|n$.

Conversely, suppose that $m|n$. Then $n = mq$ for some $q \in \mathbb{N}$. Therefore by equation (10) of [15], we get

$$B_{mq} = \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} B_m^j B_{m-1}^{q-j} B_j,$$

from which it follows that $B_m|B_n$. □

Acknowledgement The author wishes to thank the anonymous referees for their valuable comments and suggestions which resulted in an improved presentation of the paper.

References

[1] A. Behera and G.K. Panda, *On the square roots of triangular numbers*, Fibonacci Quart. **37** (1999), 98-105.

[2] A. Bèrczes, K. Liptai and I. Pink, *On generalized balancing numbers*, Fibonacci Quart. **48** (2010), 121-128.

[3] J.P. Bèzivin, A. Petho and A.J. van der Poorten, *A full characterization of divisibility sequences*, Amer. J. Math. **112** (1990), 985-1001.

[4] R. Keskin and O. Karaatly, *Some new properties of balancing numbers and square triangular numbers*, J. Integer Seq. **15** (2012), Article 12.1.4.

[5] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. **34** (1933), 461-479.

[6] K. Liptai, *Fibonacci balancing numbers*, Fibonacci Quart. **42** (2004), 330-340.

[7] K. Liptai, *Lucas balancing numbers*, Acta Math. Univ. Ostrav. **14** (2006), 43-47.

[8] K. Liptai, F. Luca, A. Pinter and L. Szalay, *Generalized balancing numbers*, Indag. Math. (N. S.), **20** (2009), 87-100.

[9] P. Olajos, *Properties of balancing, cobalancing and generalized balancing numbers*, Ann. Math. Inform. **37** (2010), 125-138.

[10] G.K. Panda and P.K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bull. Inst. Math. Acad. Sin. (N. S.), **6** (2011), 41-72.

[11] G.K. Panda and P.K. Ray, *Cobalancing numbers and cobalancers*, Int. J. of Math. Math. Sci. **8** (2005), 1189-1200.

[12] G.K. Panda, *Some fascinating properties of balancing numbers*, Proceeding of the Eleventh International Conference on Fibonacci Numbers and Their Applications, Congr. Numer. **194** (2009), 185-189.

[13] P.K. Ray, *Application of Chybeshev polynomials in factorization of balancing and Lucas-balancing numbers*, Bol. Soc. Parana. Mat. **30** (2012), 49-56.

[14] P.K. Ray, *Factorization of negatively subscripted balancing and Lucas-balancing numbers*, Bol. Soc. Parana. Mat. **31** (2013), 161-173.

[15] P.K. Ray, *Curious congruences for balancing numbers*, Int. J. Contemp. Math. Sci. **7** (2012), 881-889.

[16] L. Szalay, *On the resolution of simultaneous Pell equations*, Ann. Math. Inform. **34** (2007), 77-87.

[17] S.S. Wagstaff, *Divisors of Mersenne numbers*, Math. Comp. **40** (1983), 385-397.