# TWIST UNTANGLE AND RELATED KNOT GAMES

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**Abstract**

We investigate some classes of UNTANGLE, a combinatorial game connected to knot theory. In particular, we offer results for TWIST UNTANGLE, which is a pile subtraction game like NIM and WYTHOFF'S GAME, but which does not break easily into a sum of games.

## 1. Introduction

The game UNTANGLE, introduced in [6], is an impartial combinatorial game inspired by knot theory. It is a hard game in the sense that it does not easily break into sums to which the Sprague-Grundy algebraic theory of games can be applied.

### 1.1. Reidemeister Moves and Rules of the Game

UNTANGLE is a 2-player game played on a diagram[1] of the unknot. The players take turns reducing the number of crossings in the diagram using Reidemeister moves, which are well-known from knot theory. Given a knot diagram, the three types of Reidemeister moves are simple deformations that change the crossings but do not change the knot. They are illustrated in Figure 1, where each picture is meant to represent a portion of a larger knot diagram.

[1] A *diagram* of a knot is a regular, planar projection with extra structure indicating over- and under-crossings. See [1], [7] or [8] for the general theory.

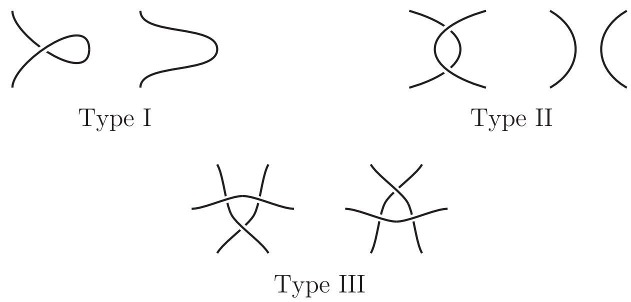


Figure 1: Reidemeister moves.

Reidemeister [9], and independently, Alexander and Briggs [3] proved that any two diagrams of a knot can be transformed from one to the other using only these moves, together with planar isotopies. In particular, any diagram of the unknot can be transformed into a circle with a finite sequence of Reidemeister moves and planar isotopies. Figure 2 shows an example of this process.

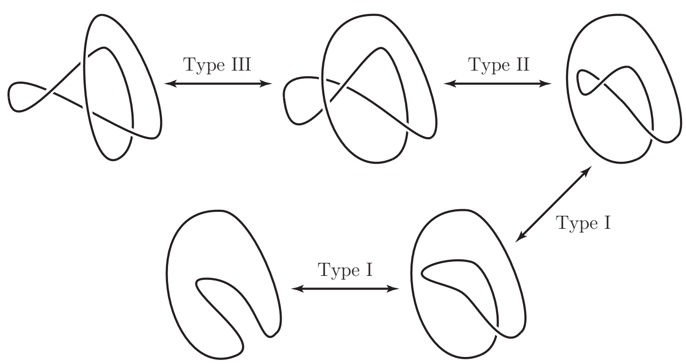


Figure 2: Transforming the unknot.

The positions of UNTANGLE are diagrams of the unknot, and players alternate moves. A move in the game consists of changing the diagram by using a sequence of Reidemeister moves (typically a single Reidemeister move), subject to the restriction that it must be a minimal reducing sequence. That is, the sequence of Reidemeister moves must reduce the number of crossings in the diagram, and if the sequence consists of $m$ Reidemeister moves, there cannot be a sequence of fewer than $m$ Reidemeister moves that would reduce the number of crossings. The game ends when there are no crossings remaining in the diagram. The winner is the last player to move, that is, the player who untangles the diagram.

The rules guarantee that the game will end in a finite number of turns. Note that, if a reducing Type I or Type II move is available, the restriction to a minimal reducing sequence forces a turn to consist of a single Reidemeister move. If both such reducing moves are available, the player may make either move. In all classes of games that we will consider in this paper, there will be reducing Reidemeister moves available from every position.

UNTANGLE is an impartial game, as both players have the same available moves from every position. So, every game of UNTANGLE is equivalent to a single-heap game of NIM via the Sprague-Grundy theory. The size of the equivalent NIM heap is called the Grundy value of the game. One can compute the Grundy value recursively: given a game position $T$ with available *options* (positions to which a player can move) $S_1, \ldots, S_k$, the Grundy value $\mathcal{G}(T) := \text{mex}\{\mathcal{G}(S_1), \ldots, \mathcal{G}(S_k)\}$. Here the mex (minimal excluded value) of a set of nonnegative integers is the smallest nonnegative integer not in the set. An impartial game is considered to be solved if there is a formula (computable in polynomial time) for the Grundy value of any given position. It is sufficient for a winning strategy to know all the positions that have Grundy value 0, i.e., the $P$-positions. These are the positions that guarantee a winning strategy for the previous player (or the second player to move). Players want to move to these positions. All other positions are called $N$-positions, since the next player to move has a winning strategy. It is always possible to move from an $N$-position to a $P$-position, and never possible to move from a $P$-position to a $P$-position. For the general theory of combinatorial games, see [2], [4] or [5].

### 1.2. Games of Two or Fewer Crossings

We can get a flavor for the structure of the game UNTANGLE by looking at the positions with two or fewer crossings. Figure 3 is a directed graph of all such positions, where moves in the game follow the edges downward. We may use this diagram to notice that, for instance, the position in the center at the top of the figure is a $P$-position since all play ends after exactly two moves. The position in the very center of the diagram has Grundy value 2, since one of its options has Grundy value 0 and the other two have Grundy value 1, and $\text{mex}\{0, 1\} = 2$. The diagram also gives a sense of the vast expanse of the general game—such a diagram of positions with three or fewer crossings would be a mess.

### 1.3. Twists and Their Relatives

A **twist** is a game position created from a planar circle by performing Type I Reidemeister moves (twisting left or twisting right) on a single arc. Figure 4 shows the diagram of a twist position created with $a$ left twists followed by $b$ right twists. We label this position $L^a R^b$. In general, a twist may be denoted $L^{a_1} R^{a_2} L^{a_3} R^{a_4} \cdots L^{a_n}$, or $L^{a_1} \cdots R^{a_n}$, depending on whether $n$, the number of groupings of same-direction

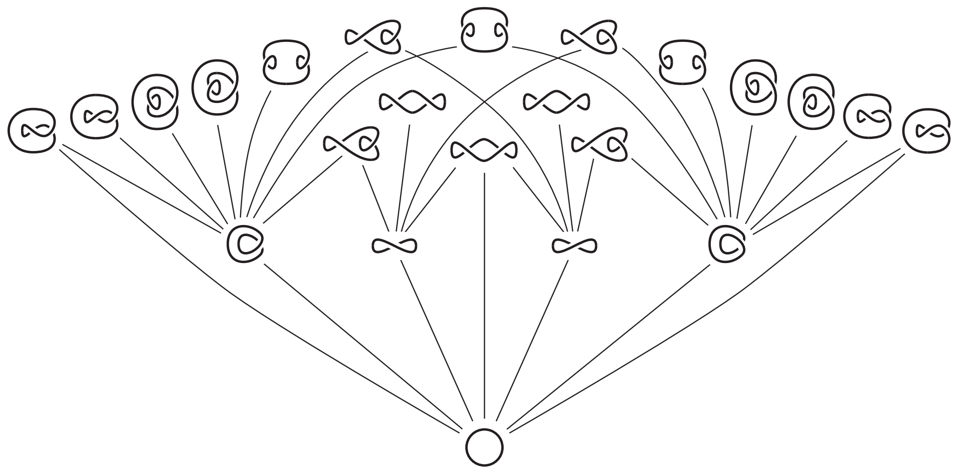

Figure 3: Positions of UNTANGLE with two or fewer crossings.

twists, is odd or even. We will also denote this twist by the vector of nonnegative integers $(a_1, a_2, a_3, \ldots, a_n)$. (Occasionally it will be useful notationally to have $a_i = 0$.) We call each instance of $L$ and $R$ in a twist a *letter*, and each expression of the form $L^k$ or $R^k$ is a *syllable* of length $k$. So, the position $(a, b) = L^a R^b$ is a twist with two syllables and $a + b$ letters. Note that all the options of a twist position are also twists. So, the game TWIST UNTANGLE is a subgame of UNTANGLE. As we will see, TWIST UNTANGLE is itself a nontrivial game.

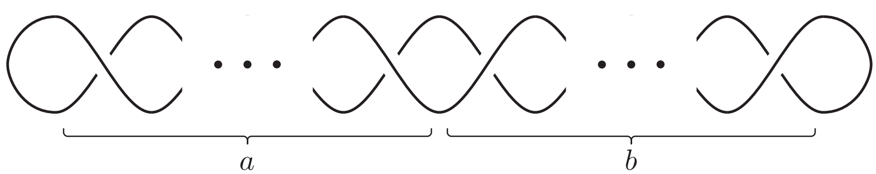


Figure 4: The $L^a R^b$ twist position.

An **inner twist** is a game position created by the same method as a twist, except that the arc is twisted inside the original circle. Note that the twist position $(a_1, a_2, a_3, \ldots, a_n)$ is equivalent to $(a_n, \ldots, a_3, a_2, a_1)$, but for an inner twist, the order is not reversible. We denote an inner twist by $(a_1, a_2, a_3, \ldots, a_n]$. Figure 5(a) shows the inner twist position $(2, 2, 1]$.

An **outer flower** is a game position created from a circle by performing Type I Reidemeister moves, twisting outward from the circle on separate arcs. As with

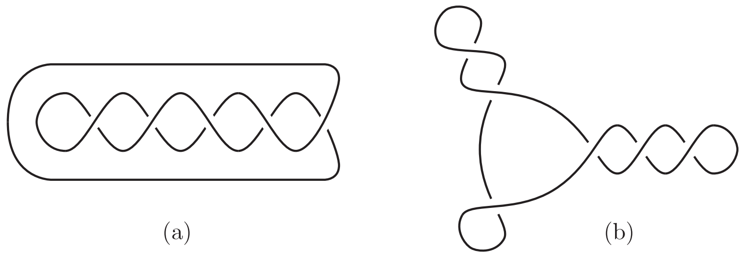


Figure 5: Inner twist and outer flower.

twists, we may denote an outer flower by counting left and right twists as we trace the diagram in one direction. Figure 5(b) shows the outer flower $\{L^1; R^2; R^3\}$, which has three *petals* of one syllable each.

An **inner flower** is created the same way, except all twists are inward, into the original circle. Because there are never Type II moves available that reduce two petals at once, every inner flower is equivalent to the sum of the inner twist games for each petal, so all analysis of inner flowers reduces to the computation of Grundy values of inner twists.

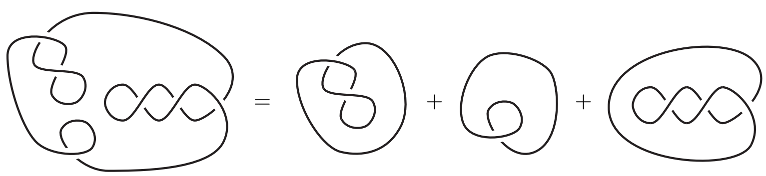


Figure 6: An inner flower is a sum of inner twists.

Note that these are all special cases of UNTANGLE that can be obtained from the circle by Type I moves only. This more general class of all such cases appears quite difficult to analyze.

## 2. Twists

### 2.1. General Facts

Recall that a position in TWIST UNTANGLE is represented by $(a_1, a_2, \cdots, a_n)$ where each $a_i$ represents a syllable of $a_i$ letters, or twists, all of the same type (either right or left twists). The available moves on these positions are reducing type I and type II Reidemeister moves. A type I Reidemeister move will reduce either $a_1$ or $a_n$

by one, by untwisting the last twist on either the left or right end of the twist. If, for example, $a_1 = 1$ and we remove that one twist, then we end up with the new position $(a_2, \ldots, a_n)$. A type II Reidemeister move will reduce a pair of consecutive $a_i$ by one each, by removing either $LR$ or $RL$ in the middle of the twist. If both of these $a_i$ are reduced from one to zero, then both syllables are removed and the remaining syllables are concatenated. For example, a type II move on the second and third syllables of $(2, 1, 1, 5, 6)$ leads to the position $(2, 5, 6)$. If exactly one of the $a_i$ is reduced to zero, then that syllable is removed and the two syllables surrounding it are combined. For example, a type II move on the second and third syllables of $(2, 3, 1, 5, 6)$ results in the position $(2, 7, 6)$. Our notation here makes TWIST UNTANGLE look like a NIM-type game, and in some cases (see Theorem 2 below) it is. The fact that one can reduce neighboring syllables in one move reminds us of WYTHOFF'S GAME. As with many interesting games, this one does not break into sums of games, as the above examples of concatenating and combining syllables show, and so the typical algebraic analysis does not easily apply. Since each move only reduces syllables by one letter, we have the following general result.

**Theorem 1.** *A twist $T = (a_1, a_2, a_3, \ldots, a_n)$, with all $a_i$ even, is a $P$-position.*

*Proof.* We will prove this by induction. Let $T$ be a game position $(a_1, a_2, \ldots, a_n)$ with all $a_i$ even. Suppose that the above statement is true for all potential positions obtained from $T$. Player 1's move will affect two adjacent letters (if he makes a type 2 Reidemeister move in somewhere in the center), or will affect the single letter $a_1$ or $a_n$ (if he makes a type 1 Reidemeister move on one of the ends). Player 2 can then respond by mimicking Player 1, affecting the same letters as Player 1. Notice that the resulting position contains only syllables with even values, which is a $P$-position by the induction hypothesis. Thus, no matter what move Player 1 makes on $T$, the result is an $N$-position. Therefore, $T$ is a $P$-position. The base case $T = (0)$ is a $P$-position as well. □

The game contains a simple subtraction game, when all syllables have one letter.

**Theorem 2.** *A twist in which every syllable has one letter has Grundy value equal to the residue of the number of syllables modulo three. In particular, it is a $P$-position iff the number of syllables is divisible by three.*

*Proof.* Let T be a game position $(1, 1, 1, \ldots, 1)$ of arbitrary length $n$. Notice that any Type I Reidemeister move on the endpoints will result in a game position $(1, 1, 1, \ldots, 1)$ of length $n - 1$. Furthermore, any Type II Reidemeister move will result in a game position $(1, 1, 1, \ldots, 1)$ of length $n - 2$. So, this game is equivalent to the single pile subtraction game where a move consists of taking either one or two counters, for which the Grundy values are well known. □

For further results on twists of arbitrary length, see Theorem 6.

We now consider the positions with three or fewer syllables. Clearly, a twist with one syllable ends in the same number of moves as the number of letters, and so is a $P$-position if and only if it has an even number of letters.

**Theorem 3.** *A twist $(a, b)$ with $a \geq b$ has Grundy value 0 if $a$ and $b$ are both even, 2 if $a$ and $b$ are both odd, 1 if $a$ is odd and $b$ is even, and 3 if $a$ is even and $b$ is odd.*

**Corollary 4.** *A twist with two syllables is a $P$-position iff each syllable is even (i.e., has an even number of letters).*

*Proof.* Let $T$ be a twist, $T = (a, b)$. If $a$ and $b$ are both even, then the position has Grundy value 0 by Theorem 1. We use induction to cover the remaining cases. Suppose that the theorem statement is true for all potential positions from $T$:

1. Suppose $a$ and $b$ have opposite parity. Then one Type I move results in Grundy value 0 while the other results in Grundy value 2. A Type II move results in a position $(a', b')$, where $a'$ and $b'$ have opposite parity but have switched roles. Thus, if $a$ is even, then $(a', b')$ has Grundy value 1 and $(a, b)$ has Grundy value 3. If $a$ is odd, then $(a', b')$ has Grundy value 3 and $(a, b)$ has Grundy value 1.

2. Suppose $a$ and $b$ are both odd. Then a Type I move on $b$ results in Grundy value 1, while a Type I move on $a$ results in Grundy value 3 (or 1 if $a = b$). A Type II move results in Grundy value 0 (even when $a$ or $b$ or both are equal to one). Thus $T$ has Grundy value 2.

Therefore, in all cases the theorem statement is true for $T$. This completes the proof □

With similar methods, we may compute the Grundy values for positions with three syllables.

**Theorem 5.** *Suppose the twist $T = (a, b, c)$ has $a \geq c$. Then the Grundy value of $T$ is*

$$\mathcal{G}(T) = \begin{cases} 0 & \text{if } a \equiv c \pmod 2 \\ 2 & \text{if } a \not\equiv c \pmod 2,\ c = 1,\ \text{and } b > a \\ 1 & \text{otherwise} \end{cases}$$

Here we show only that $T$ is a $P$-position if and only if $a \equiv c \pmod 2$.

Suppose $a \equiv c \pmod 2$. Then a Type I move will lead to a three-syllable position with $a \not\equiv c \pmod 2$, unless it removes the entire syllable. Note that if it removes the syllable $a$ or $c$, that syllable is 1 and the other is odd. In that case, the move results in a two-syllable position, one of whose syllables is odd, which is an $N$-position by Corollary 4. A Type II move will lead to a position where $a \not\equiv c \pmod 2$ unless it removes one or two syllables. If it removes an end syllable, the result is a two-syllable position, at least one of whose syllables is odd. If it removes the center syllable, the result is a one-syllable position with an odd number of letters. If it removes two syllables, the result again is a one-syllable position with an odd number of letters. Thus, every possible move results in an $N$-position, and so $T$ is a $P$-position.

Now suppose $a \not\equiv c \pmod 2$. Then either $a > 1$ or $c > 1$, and so there is a Type I move that results in a three-syllable position with $a \equiv c \pmod 2$, a $P$-position. Thus, $T$ is an $N$-position.

**Theorem 6.** *Suppose $T = (a_1, 1, a_2, 1, \ldots, a_n, 1)$, $S = (a_1, 1, a_2, 1, \ldots, a_n)$, and $R = (a_1, 1, a_2, 1, \ldots, a_n, 1, 1)$, with all $a_i > 1$. If $a_i$ is odd for an even number of $i$, then $\mathcal{G}(T) = 3$, $\mathcal{G}(S) = 0$ and $\mathcal{G}(R) = 1$. If $a_i$ is odd for an odd number of $i$, then $\mathcal{G}(T) = 2$, $\mathcal{G}(S) = 1$ and $\mathcal{G}(R) = 0$.*

*Proof.* In most cases, the options of these positions are of the same type as those positions covered in the theorem. Note that a type II move typically results in the sequence $a_i, 1, a_{i+1}$ being replaced by the single number $a_i + a_{i+1} - 1$, thus changing the total number of odd $a_i$ by 1 and keeping all $a_i > 1$. As we proceed by induction, the base cases are those where the options may not be covered by the statement of the theorem: those where the position has three or fewer syllables (which are cases of the theorems above), and those where $a_1 = 2$, $a_n = 2$, or both (cases in which an option can have some $a_i = 1$).

For now we assume neither $a_1$ nor $a_n$ is 2. We say a position is $T$-type if it fits the hypotheses of $T$ above, and analogously for $S$-type and $R$-type. For any such position $Q$, we let $m(Q)$ be the residue of the number of odd $a_i$ modulo 2. In the inductive proof, we will say an option $Q'$ of $Q$ has the *same* $m$ as $Q$ if $m(Q') = m(Q)$. Otherwise, we say it has the *opposite* $m$. By induction, these options will have the Grundy values claimed in the statement of the theorem for their type and parity $m$.

The options of $T$ are $(a_1 - 1, 1, \ldots, a_n, 1)$, which is $T$-type with opposite $m$ from that of $T$, $(a_1, 1, a_2, \ldots, a_n)$, which is $S$-type with the same $m$, $(a_1, 1, \ldots, 1, a_i + a_{i+1} - 1, 1, \ldots, a_n, 1)$, which is $T$-type with opposite $m$, and $(a_1, 1, \ldots, 1, a_n - 1)$, which is $S$-type with opposite $m$. If $m(T)$ is even, then by induction these options have Grundy values 2, 1, 2, and 0 respectively, and so $\mathcal{G}(T) = \text{mex}\{0, 1, 2\} = 3$. If $m(T)$ is odd, then by induction these options have Grundy values 3, 0, 3, and 1 respectively, and so $\mathcal{G}(T) = \text{mex}\{0, 1, 3\} = 2$.

Similarly, any type I or type II move on $S$ will result in an $S$-type position with opposite $m$. So, by induction, if $m(S)$ is even, then $\mathcal{G}(S) = \text{mex}\{1\} = 0$, and if $m(S)$ is odd, then $\mathcal{G}(S) = \text{mex}\{0\} = 1$.

With the exception of moves involving the two rightmost syllables, any type I or type II move on $R$ will result in an $R$-type position with opposite $m$. The other three options are an $S$-type position with the same $m$ and a $T$-type position with the same $m$. Thus, by induction, if $m(R)$ is even, then the options of $R$ have Grundy values 0 and 3, and so $\mathcal{G}(R) = 1$. If $m(R)$ is odd, then the options of $R$ have Grundy values 1 and 2, and so $\mathcal{G}(R) = 0$.

Now for the base cases $a_1 = 2$ or $a_n = 2$, with the positions having at least four syllables. In case $a_n = 2$, the last of the options of $T$ listed above becomes $(a_1, 1, \ldots, a_{n-1}, 1, 1)$, which is $R$-type with the same $m$, and thus has the same Grundy value as an $S$-type position with opposite $m$. Thus, in this case $\mathcal{G}(T)$ will be as claimed. If $a_n = 2$, $S$ will also have the additional option $(a_1, 1, \ldots, a_{n-1}, 1, 1)$, which has the same Grundy value as an $S$-type position with opposite $m$, and so $\mathcal{G}(S)$ will remain as claimed. The options of $R$ are unchanged in case $a_n = 2$.

In case $a_1 = 2$, the first option of $T$ becomes $T' = (1, 1, \ldots, a_n, 1)$, which itself has the option $(a_2, 1, \ldots, a_n, 1)$, which is $T$-type with the same $m$. Thus by the mex rule, $T'$ does not have the claimed Grundy value of $T$. Since the remaining options of $T$ cover the same types of options as in the case $a_1 > 2$, $\mathcal{G}(T)$ will be as claimed. If $a_1 = 2$, a type II move on $S$ gives a position of $S$-type and opposite $m$, but $S$ has the additional option $(1, 1, a_2, \ldots, a_n)$. By symmetry, this position is again of $R$-type with the same $m$ as $S$, and thus has the same Grundy value as an $S$-type position with opposite $m$. Thus, $S$ has the claimed Grundy value. If $a_1 = 2$, $R$ has the options listed above with the exception of the first option, which will be $R' = (1, 1, a_2, \ldots, a_n, 1, 1)$. But, $R'$ has the option $(a_2, 1, \ldots, a_n, 1, 1)$, which is $R$-type with the same $m$ as $R$, and thus $R'$ cannot have the same Grundy value as an $R$-type position with the same $m$ as $R$. Since $R$-type Grundy values are only 0 or 1, $\mathcal{G}(R)$ will be as claimed. □

One might hope to find a similar result when the 1's in the even positions are replaced by general odd numbers. The situation is a bit more complicated, as seen in the theorem on twists with four syllables below. Furthermore, the position $(2, 3, 2, 1, 2, 3, 4, 8, 4)$ has Grundy value 1 and $(2, 3, 2, 1, 2, 3, 4, 9, 4)$ has Grundy value 3, which would rule out a similar result for the $S$-type positions, even when the $a_i$ are all even.

Using Theorem 6 and similar techniques, we can also prove

**Theorem 7.** *Suppose* $U = (1, a_1, 1, a_2, 1, \ldots, a_n, 1)$, $V = (1, a_1, 1, a_2, 1, \ldots, a_n, 1, 1)$, *and* $W = (1, 1, a_1, 1, a_2, 1, \ldots, a_n, 1, 1)$, *with all* $a_i > 1$. *If* $a_i$ *is odd for an even number of* $i$, *then* $\mathcal{G}(V) = 2$ *and* $\mathcal{G}(W) = 0$. *If* $a_i$ *is odd for an odd number of* $i$, *then* $\mathcal{G}(V) = 3$ *and* $\mathcal{G}(W) = 1$. $\mathcal{G}(U)$ *is* 0 *if* $n$ *is odd and* 1 *if* $n$ *is even.*

### 2.2. Twists with Four Syllables

Our theorem on four-syllable twists illustrates some of the complexity of TWIST UNTANGLE.

**Theorem 8.** *Let $T = (a,b,c,d)$ be a twist with four letters, reflected so that $(a,b,c,d) \mod 2$ is smaller in the dictionary order. $T$ is a $P$-position if and only if it satisfies one of the following.*

1. $(a,b,c,d) \equiv (0,0,0,0) \pmod 2$
2. $(a,b,c,d) \equiv (0,0,1,0) \pmod 2$, $b-a > c-d$ *and* $b > a$
3. $(a,b,c,d) \equiv (0,1,0,1) \pmod 2$, $a > b$ *and* $c < d$
4. $(a,b,c,d) \equiv (0,1,0,1) \pmod 2$, $a < b$, $b-a > c-d$ *and* $d > 1$
5. $(a,b,c,d) \equiv (0,1,1,1) \pmod 2$, $a > b$ *and* $d \geq c$
6. $(a,b,c,d) \equiv (0,1,1,1) \pmod 2$, $a < b$, $b-a > c-d$ *and* $d > 1$
7. $(a,b,c,d) \equiv (1,0,0,1) \pmod 2$, $a < b$, $b-a = c-d$, $a > 1$ *and* $d > 1$

Our proof of this theorem is omitted, but uses the same ideas as those used to prove the special case of Theorem 5 above.

### 2.3. Palindromes

A twist is a **palindrome** if it can be read the same way backwards and forwards: for example, the twist $(5,4,2,6,2,4,5)$ is a palindrome. Palindromes are somewhat easier to analyze due to their symmetry.

**Theorem 9.** *Let $S$ be a palindrome with $2n-1$ letters. If the $n$th (i.e., central) syllable of $S$ is even, then $S$ is a $P$-position.*

*Proof.* We will prove this by induction. Let $S$ be the palindrome described above, and suppose that the above statement is true for all potential positions of $S$. Then no matter where Player 1 moves, Player 2 can make a symmetrical move. Notice that the resulting twist will be a palindrome with an even center. By induction, this is a $P$-position, making $T$ a $P$-position as well. Note that our base case $T = (0)$ is a $P$ position. □

The case when the central syllable is odd is interesting. Such twists with three syllables are $P$-positions, as we can see easily from Theorem 5. Those with five syllables are $N$-positions, since they have an option of the type covered in the following theorem.

**Theorem 10.** *Let $S = (a_1, a_2, a_3, a_2 - 1, a_1)$, where $a_3 \geq 0$ is even and $a_1, a_2 > 0$. Then $S$ is a $P$-position.*

*Proof.* We note that if $a_3 = 0$ or $a_2 = 1$, the position resolves to a three syllable position where the ends have the same parity, and is thus a $P$-position. Assuming $a_3 > 0$ and $a_2 > 1$, we suppose that the statement is true of all potential positions of $S$. In most cases, Player 2 can make a symmetrical move to Player 1's move and obtain the same type of position, which is a $P$-position by induction. The cases not covered are when $a_1 = 1$ and Player 1 moves to either $(a_2, a_3, a_2 - 1, 1)$, $(a_2 - 1, a_3, a_2 - 1, 1)$, $(1, a_2, a_3, a_2 - 2)$, or $(1, a_2, a_3, a_2 - 1)$. These positions have one of the options $(a_2, a_3, a_2 - 2)$, or $(a_2 - 1, a_3, a_2 - 1)$, which are both $P$-positions, even if $a_2 = 2$. Thus, the theorem is proved. □

Seven-syllables palindromes with central syllable odd can be either. For example, $(1, 2, 3, 1, 3, 2, 1)$ is a $P$-position while $(1, 2, 3, 3, 3, 2, 1)$ is an $N$-position. We have the following conjectures based on data for seven syllable palindromes.

**Conjecture 11.** *For a five syllable palindrome vector $S$ and $a > 0$, if $(a, S, a)$ is a $P$-position, then $(a + k, S, a + k)$ is a $P$-position for all $k \geq 0$.*

**Conjecture 12.** *The only seven syllable palindrome $N$-positions are* $(1, 2, 1, 1, 1, 2, 1)$ *and for $k \geq 1$,* $(1, 1, 1, k, 1, 1, 1)$, $(2, 1, 1, k, 1, 1, 2)$, *and* $(1, 2, k+2, 2k+1, k+2, 2, 1)$.

## 3. Relatives of Twists

### 3.1. Inner Twists

Intuitively, inner twists are slightly easier to analyze than full twists, since there is one fewer option from every position. For an inner twist $(a_1, \ldots, a_n]$, we call $a_1$ the *open end* syllable and $a_n$ the *closed end* syllable. First, we have an analogue of Theorem 1 with the same proof.

**Theorem 13.** *An inner twist with all even syllables is a $P$-position.*

We can also analyze the cases with a small number of syllables, as before.

**Theorem 14.** *The inner twist $(a, b]$ has Grundy value 0 if $a$ is even, 2 if $a = 1$, and 1 otherwise.*

*Proof.* The two options from the position $(a, b]$ are $(a - 1, b]$ and $(a - 1, b - 1]$. If $a = 1$, these options are single syllables of opposite parity, which have Grundy values 0 and 1, and thus $(a, b]$ has Grundy value 2. If $a$ is even and $b > 1$, these options are two syllables with the open end being odd or 1, and so by induction have Grundy value 1 or 2. If $a$ is even and $b = 1$, the second option is a single odd syllable, and thus has Grundy value 1. By the mex rule, the Grundy value of $(a, b]$ is 0. If $a > 1$ is odd, these options are two syllables with the open end being even, and so by induction have Grundy value 0. Thus, $(a, b]$ has Grundy value 1 in this case. □

**Theorem 15.** *The inner twist $(a, b, c]$ is a $P$-position if and only if either $a > 1$ has the same parity as $c$ or $a = 1$, $b = 1$, and $c$ is odd.*

*Proof.* Suppose $a > 1$ has the same parity as $c$. Any move will change the parity of $a$ or $c$, but not both, and so by induction will lead to an $N$-position. If $a = 1$, $b = 1$, and $c$ is odd, then the options are $(1, c]$, $(1, c-1]$, and $(c]$, all of which are $N$-positions, even if $c = 1$. Now suppose $(a, b, c]$ does not satisfy the condition in the statement of the theorem, so that it falls into one of the following four cases, each of which we will show has a $P$-position option. Suppose $a = 1$, $b = 1$, and $c$ is even. Then one option is $(c]$, which is a $P$-position. Suppose $a = 1$ and $b > 1$. Then two options are $(b, c]$ and $(b-1, c]$, one of which is a $P$-position by Theorem 14. Suppose $a > 1$ is odd and $c$ is even. Then the option $(a-1, b, c]$ is a $P$-position by induction. Lastly, suppose $a$ is even and $c$ is odd. Then the option $(a, b-1, c-1]$ is a $P$-position. We know this by induction if it is a three syllable position, and by previous results if it is a one or two syllable position. □

We have an analogue of Theorem 6 for inner twists.

**Theorem 16.** *Consider the inner twist position $T = (a_1, 1, a_2, 1, \ldots, a_n]$, or $T = (a_1, 1, a_2, 1, \ldots, a_n, 1]$, where $a_i > 1$ for all $i$. If $a_i$ is odd for an even number of $i$, then $\mathcal{G}(T) = 0$. If $a_i$ is odd for an odd number of $i$, then $\mathcal{G}(T) = 1$.*

*Proof.* The type I move creates $T' = (a_1 - 1, 1, a_2, \ldots]$. Type II moves replace the sequence $a_i, 1, a_{i+1}$ with the single number $a_i + a_{i+1} - 1$, or replace $(\ldots, a_n, 1]$ with $(\ldots, a_n - 1]$. As long as $a_1, a_n > 2$, all options of $T$ are covered by the theorem statement, and so by induction they have the Grundy value claimed by the theorem. All of the options have one more or one fewer odd $a_i$ than $T$, and so have Grundy value 0 in the case $\mathcal{G}(T)$ is claimed to be 1, and vice versa. By the mex rule, $T$ has the Grundy value as claimed in the theorem. One exception occurs when $a_1 = 2$, in which case $T' = (1, 1, a_2, 1, \ldots]$ is not covered in the theorem statement. This position itself has the option $(a_2, 1, \ldots]$, a position covered by the theorem with the same parity of odd $a_i$ as $(2, 1, a_2, \ldots]$. Thus, $T'$ cannot have the Grundy value we claim for $T$. Since $T$ also has the option $T'' = (2 + a_2 - 1, 1, \ldots]$, which is of the same type with one more or one fewer odd $a_i$ as before, the theorem remains true for $T$. The exception $a_n = 2$ is handled similarly: $T = (\ldots, a_{n-1}, 1, 2, 1]$ has option $T' = (\ldots, a_{n-1}, 1, 1]$, which is not covered by the theorem statement. But this $T'$ has an option $(\ldots, a_{n-1}]$. Thus $T'$ cannot have the Grundy value claimed for $T$. But since $T$ also has the option $T'' = (\ldots, a_{n-1} + 1, 1]$ (which has opposite parity of odd $a_i$), the theorem is true for $T$. The only remaining base case is $T = (2]$, which has Grundy value 0 as claimed. □

As with twists, we have evidence for conjectures that extend Theorems 13 and 16.

**Conjecture 17.** *The inner twist* $(a, b, c, d]$ *is a* $P$*-position if* $a > 2$ *and* $a$ *and* $c$ *have the same parity.*

**Conjecture 18.** *Let* $S = (a_1, \ldots, a_k]$ *an inner twist. If for all* $i$ *odd,* $a_i$ *is even and greater than* 2*, then* $S$ *is a* $P$*-position.*

### 3.2. Flowers

As stated in the introduction, inner flowers are sums of the inner twist positions that make up each of their petals, and so computing Grundy values for inner twists is useful for determining the outcome of games of UNTANGLE on inner flowers. Theorem 14 then leads to the following

**Corollary 19.** *Suppose an inner flower has petals with at most two syllables. It is a* $P$*-position if and only if it has an even number of two syllable petals with the open end syllable of length* 1*, and an even total number of two syllable petals with the open end syllable* $a > 1$ *odd and one syllable petals with odd length.*

Outer flowers are interesting because towards the end of the game the petals can interact. However, in the case of outer flowers with all one-syllable petals, the effect is nearly the same as if they did not interact. Of course, an outer flower with one or two petals is a game of TWIST UNTANGLE.

**Theorem 20.** *Suppose an outer flower has all one-syllable petals. In case there are more than two petals, it is a* $P$*-position if and only if the number of odd petals is even.*

*Proof.* The only available move in a position with more than two petals is a type I move to reduce a petal by one twist. Unless it removes a petal entirely, such a move will always change the parity of the number of odd petals. By induction, the options of a position with an even number of odd petals are $N$-positions, and the options of a position with an odd number of odd petals are $P$-positions, as long as the options have at least three petals. The only case we need to check is that of a three-petal outer flower, one of whose petals has one letter. Call this position $F$. In case the only odd petal of $F$ is the single letter, we may remove it with a type I move, and the result will be a position of TWIST UNTANGLE with all (one or two) even syllables, a $P$-position. So, in this case $F$ is indeed an $N$-position. In case the other two petals are opposite parity, removing the single letter petal results in a position of TWIST UNTANGLE with at most two syllables and an odd number of total letters, which is an $N$-position. The remaining options of $F$ were already seen to be $N$-positions, and so $F$ is a $P$-position. Lastly, suppose $F$ has an odd number of letters in each petal. If any of the petals has more than one letter, we may reduce that petal to an even number of letters, which is a $P$-position by induction. If all the petals have one letter, we consider $F$ to be the result of performing one type

I twist on each of three arcs of a circle. At least one pair of these twists will have had the same orientation outward from the circle (both left twists or both right twists). Removing the third petal reduces to the one-syllable position (2) in TWIST UNTANGLE, which is a $P$-position, and thus $F$ is an $N$-position. □

**Question 21.** *Suppose an outer flower has more than two petals. Can we determine if it is a $P$-position from the Grundy values of the petals (considered as inner twists)?*

## 4. Nim Dimension

We conclude with a question regarding large Grundy values found in positions of TWIST UNTANGLE, inspired by [10].

**Question 22.** *Does* TWIST UNTANGLE *have infinite* NIM *dimension? That is, given any $n$, is there a* TWIST UNTANGLE *position $T$ such that $\mathcal{G}(T) \geq n$?*

Our current record-holder is $\mathcal{G}((1, 2, 3, 4, 5, 6, 3, 4, 3, 5)) = 11$.

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