A NUMBER THEORETIC PROBLEM ON THE DISTRIBUTION OF POLYNOMIALS WITH BOUNDED ROOTS

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Abstract

Let $\mathcal{E}_d^{(s)}$ denote the set of coefficient vectors $(a_1, \ldots, a_d) \in \mathbb{R}^d$ of contractive polynomials $x^d + a_1x^{d-1} + \cdots + a_d \in \mathbb{R}[x]$ that have exactly s pairs of complex conjugate roots and let $v_d^{(s)} = \lambda_d(\mathcal{E}_d^{(s)})$ be its (d-dimensional) Lebesgue measure. We settle the instance $s = 1$ of a conjecture by Akiyama and Pethő, stating that the ratio $v_d^{(s)}/v_d^{(0)}$ is an integer for all $d \geq 2s$. Moreover we establish the surprisingly simple formula $v_d^{(1)}/v_d^{(0)} = (P_d(3) - 2d - 1)/4$, where $P_d(x)$ are the Legendre polynomials.

- Dedicated to Prof. Dominique Foata on the occasion of his 80^{th} birthday.

1. Introduction

Let $\mathcal{E}_d$ denote the set of all coefficient vectors $(a_1, \ldots, a_d) \in \mathbb{R}^d$ of polynomials $x^d + a_1x^{d-1} + \cdots + a_d$ with coefficients in $\mathbb{R}$ and all roots having absolute value less than 1, and let $\mathcal{E}_d^{(s)}$ denote the subset of the coefficient vectors of those polynomials in $\mathcal{E}_d$ that have exactly s pairs of complex conjugate roots. Let furthermore $v_d =$

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$\lambda_d(\mathcal{E}_d)$ and $v_d^{(s)} = \lambda_d(\mathcal{E}_d^{(s)})$ denote the d-dimensional Lebesgue measures of the referring sets.

The sets $\mathcal{E}_d$ have been studied by several authors in different context, compare e.g. Schur [11], Fam and Meditch [4] or Fam [3]. More recently, the regions $\mathcal{E}_d$ have become of interest in the study of "shift radix systems", since the regions where those systems have a certain periodicity property are in close connection with the regions $\mathcal{E}_d$ (compare e.g. Kirschenhofer et al. [7]). Fam [3] established the formula

$$v_d = \begin{cases} 2^{2m^2} \prod_{j=1}^m \frac{(j-1)!^4}{(2j-1)!^2} & \text{if } d = 2m, \\ 2^{2m^2+2m+1} \prod_{j=1}^m \frac{j!^2(j-1)!^2}{(2j-1)!(2j+1)!} & \text{if } d = 2m+1. \end{cases} \tag{1.1}$$

In [1] Akiyama and Pethő gave a number of results on the quantities $v_d^{(s)}$, including an integral representation for general s from which they derived an explicit formula in the instance $s = 0$ as well as a somewhat involved expression for $s = 1$ reading

$$\begin{aligned} v_d^{(0)} &= \frac{2^{d(d+1)/2}}{d!} S_d(1,1,1/2), \\ v_d^{(1)} &= 2^{(d-1)(d-2)/2-2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} B_{d-2}(d-2-k, d-2-k-j) \\ &\quad \int_{z=0}^{1} \int_{y=-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz \end{aligned} \tag{1.2}$$

for $d \geq 2$ and $0 \leq k \leq j \leq d$ where

$$S_d(1,1,1/2) := \frac{1}{\prod_{i=0}^{d-1} \binom{2i+1}{i}} \tag{1.3}$$

is a special instance of the Selberg integral $S_n(\alpha, \beta, \gamma)$ and where

$$\begin{aligned} B_d(j,k) := &\left(\prod_{i=1}^{k} \frac{2 + (d-i-1)/2}{3 + (2d-i-1)/2} \frac{\prod_{i=1}^{j} (1 + (d-i)/2) \prod_{i=1}^{k} (1 + (d-i)/2)}{\prod_{i=1}^{j+k} (2 + (2d-i-1)/2)} \right) \\ &S_d(1,1,1/2). \end{aligned} \tag{1.4}$$

is a special instance of Aomoto's generalization of the Selberg integral (compare Andrews et al. [2, Section 8] for Selberg's and Aomoto's integrals).

Furthermore, Akiyama and Pethő in [1] proved that the ratios $v_d^{(s)}/v_d^{(0)}$ are rational, and, motivated by extensive numerical evidence, stated the following

Conjecture 1.1. [1, Conjecture 5.1] The quotient

$$v_d^{(s)}/v_d^{(0)}$$

is an integer for all non-negative integers d, s with $d \geq 2s$.

In Section 2 of this paper we will prove this conjecture for the instance $s = 1$ and in addition give a surprisingly simple explicit formula for the quotient in this case involving the Legendre polynomials evaluated at $x = 3$. In the proof we will combine several transformations of binomial sums, one of them corresponding to a special instance of Pfaff's reflection law for hypergeometric functions. We refer the reader in particular to the standard reference [6, Section 5] for the techniques that we will apply.

In Section 3 we will use our main theorem to establish a linear recurrence for the sequence $\left(v_d^{(1)}/v_d^{(0)}\right)_{d\geq 0}$, and from its generating function will derive its asymptotic behaviour for $d \to \infty$. Combined with a result from [1], this also gives information on the asymptotic behaviour of the probability $p_d^{(1)} = v_d^{(1)}/v_d$ of a contractive polynomial of degree d to have exactly one pair of complex conjugate roots.

In the final section we discuss possible generalizations of our results.

2. Main Result

Theorem 2.1. *The quotient $v_d^{(1)}/v_d^{(0)}$ is an integer for each $d \geq 2$. Furthermore we have*

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4}, \qquad \textit{where}$$

$$P_d(x) := 2^{-d} \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \binom{d-k}{k} \binom{2d-2k}{d-k} x^{d-2k} = \sum_{k=0}^{d} \binom{d+k}{2k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k$$

are the Legendre polynomials (cf. [10, p. 66]).

Proof. In a first step we solve the double integral in identity (1.2) for $v_d^{(1)}$. Let $j \geq 0$, $k \geq 0$. Then

$$\begin{aligned}\int_{z=0}^{1}\int_{y=-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy\, dz &= \int_{y=-2}^{2}\int_{z=y^2/4}^{1} y^j (y+z+1)^k \, dz\, dy \\ &= \frac{1}{k+1}\left(\int_{-2}^{2} y^j (y+2)^{k+1}\, dy - \int_{-2}^{2} y^j (y/2+1)^{2k+2}\, dy\right) \\ &= \frac{1}{k+1}\left(2^{j+k+2}\int_{-1}^{1} y^j (y+1)^{k+1}\, dy - 2^{j+1}\int_{-1}^{1} y^j (y+1)^{2k+2}\, dy\right)\end{aligned}$$

where we performed the substitution $y/2 \to y$ in the last step. By iterated partial integration we gain now from the last expression that

$$\begin{aligned}\int_{z=0}^{1}\int_{y=-2\sqrt{z}}^{2\sqrt{z}} & y^j(y+z+1)^k\,dy\,dz \\ &= \frac{2^{j+2k+4}}{k+1}\left(\sum_{r=1}^{j+1}\frac{(-2)^{r-1}(j)_{r-1}}{(k+r+1)_r}-\sum_{r=1}^{j+1}\frac{(-2)^{r-1}(j)_{r-1}}{(2k+r+2)_r}\right)\end{aligned} \tag{2.1}$$

with $(x)_j := \prod_{i=0}^{j-1}(x-i)$.

In the following we insert (2.1) in formula (1.2) and perform stepwise a first evaluation of $v_d^{(1)}/v_d^{(0)}$ mainly as a sum of products of factorials.

$$\begin{aligned}\frac{v_d^{(1)}}{v_d^{(0)}} &= \Bigg(2^{\frac{(d-1)(d-2)}{2}-2}\sum_{j=0}^{d-2}\sum_{k=0}^{d-2-j}\frac{(-1)^{d+k}2^{2d-2-2k-j}}{j!k!(d-2-j-k)!}\prod_{i=1}^{d-2-k-j}\frac{2+\frac{d-2-i-1}{2}}{3+\frac{2(d-2)-i-1}{2}} \\ &\quad \frac{\prod_{i=1}^{d-2-k}(1+\frac{d-2-i}{2})\prod_{i=1}^{d-2-k-j}(1+\frac{d-2-i}{2})}{\prod_{i=1}^{d-2-k+d-2-k-j}(2+\frac{2(d-2)-i-1}{2})}\frac{1}{\prod_{i=0}^{d-2-1}\binom{2i+1}{i}} \\ &\quad \frac{2^{j+2k+4}}{k+1}\left(\sum_{r=1}^{j+1}\frac{(-2)^{r-1}(j)_{r-1}}{(k+r+1)_r}-\sum_{r=1}^{j+1}\frac{(-2)^{r-1}(j)_{r-1}}{(2k+r+2)_r}\right)\Bigg)\Bigg/\left(\frac{2^{d(d+1)/2}}{d!\prod_{i=0}^{d-1}\binom{2i+1}{i}}\right) \\ &= \sum_{j=0}^{d-2}\sum_{k=0}^{d-j-2}(-1)^{d+k+1}\frac{d!}{j!(k+1)!(d-j-k-2)!}\prod_{i=1}^{d-j-k-2}\frac{d-i+1}{2d-i+1} \\ &\quad \frac{\prod_{i=1}^{d-k-2}(d-i)\prod_{i=1}^{d-j-k-2}(d-i)}{\prod_{i=1}^{2d-j-2k-4}(2d-i-1)}\frac{\prod_{i=0}^{d-1}\binom{2i+1}{i}}{\prod_{i=0}^{d-3}\binom{2i+1}{i}} \\ &\quad \left(\sum_{r=1}^{j+1}\frac{(-2)^r(j)_{r-1}}{(k+r+1)_r}-\sum_{r=1}^{j+1}\frac{(-2)^r(j)_{r-1}}{(2k+r+2)_r}\right) \\ &= \sum_{j=0}^{d-2}\sum_{k=0}^{d-j-2}(-1)^{d+k+1}\frac{d!}{j!(k+1)!(d-j-k-2)!}\frac{\frac{d!}{(j+k+2)!}}{\frac{(2d)!}{(d+j+k+2)!}}\frac{\frac{(d-1)!}{(k+1)!}\frac{(d-1)!}{(j+k+1)!}}{\frac{(2d-2)!}{(j+2k+2)!}} \\ &\quad \frac{(2d-3)!}{(d-2)!(d-1)!}\frac{(2d-1)!}{(d-1)!d!}\left(\sum_{r=1}^{j+1}\frac{(-2)^r\frac{j!}{(j-r+1)!}}{\frac{(k+r+1)!}{(k+1)!}}-\sum_{r=1}^{j+1}\frac{(-2)^r\frac{j!}{(j-r+1)!}}{\frac{(2k+r+2)!}{(2k+2)!}}\right) \\ &= \sum_{j=0}^{d-2}\sum_{k=0}^{d-j-2}(-1)^{d+k+1}\frac{(d+j+k+2)!(j+2k+2)!}{(d-j-k-2)!j!(j+k+2)!(j+k+1)!(k+1)!^2} \\ &\quad \left(\sum_{r=1}^{j+1}\frac{(-2)^{r-2}j!(k+1)!}{(j-r+1)!(k+r+1)!}-\sum_{r=1}^{j+1}\frac{(-2)^{r-2}j!(2k+2)!}{(j-r+1)!(2k+r+2)!}\right).\end{aligned}$$

In the next step we rewrite the last expression as a sum over products of binomial coefficients.

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} (-1)^{d+k+1} \binom{d}{j+k+2} \binom{d+j+k+2}{d} \frac{j+k+2}{j+2k+3}$$
$$\left(\sum_{r=1}^{j+1} (-2)^{r-2} \binom{j+2k+3}{2k+r+2} \binom{2k+r+2}{k+1} - \sum_{r=1}^{j+1} (-2)^{r-2} \binom{j+2k+3}{2k+r+2} \binom{2k+2}{k+1} \right).$$

Using the substitution $j+k+2 \to a, k+1 \to b$ the latter expression reads

$$\sum_{a=2}^{d} \sum_{b=0}^{a-1} \sum_{r=1}^{a-b} (-1)^{d+b} (-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{d} \binom{a+b}{2b+r} \binom{2b+r}{b} -$$
$$\sum_{a=2}^{d} \sum_{b=0}^{a-1} \sum_{r=1}^{a-b} (-1)^{d+b} (-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{d} \binom{a+b}{2b+r} \binom{2b}{b}$$

so that

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d} (-1)^d a \binom{d}{a} \binom{d+a}{d} \sum_{r=1}^{a} (-2)^{r-2}$$
$$\left(\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} - \sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b}{b} \right). \tag{2.2}$$

In the following we will simplify the two innermost sums.

We start with the first sum. If $r = a$ the sum trivially equals $\frac{1}{a}$. Let us assume $1 \le r \le a-1$ now. Then we have

$$\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} = \frac{1}{a-r} \sum_{b=0}^{a-r} (-1)^b \binom{a-r}{b} \binom{a+b-1}{b+r}$$
$$= \frac{(-1)^r}{a-r} \sum_{b=0}^{a-r} \binom{a-r}{b} \binom{r-a}{b+r} = \frac{(-1)^r}{a-r} \binom{0}{a} = 0,$$

where we used

$$(-1)^k \binom{k-n-1}{k} = \binom{n}{k} \qquad (n \in \mathbb{Z}, k \ge 0)$$

for the second identity, and Vandermonde's identity

$$\sum_{k=0}^{n} \binom{n}{k} \binom{s}{k+t} = \sum_{k=0}^{n} \binom{n}{k} \binom{s}{n+t-k} = \binom{n+s}{n+t} \qquad (s \in \mathbb{Z}, n, t \ge 0)$$

for the third one. Altogether we have established

$$\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} = \frac{1}{a}\delta_{r,a} \qquad (1 \le r \le a), \tag{2.3}$$

where $\delta_{r,a}$ denotes the Kronecker symbol.

Now we turn to the second sum in question. Since this is a sum reminiscent of a sum treated in [6, Section 5.2, Problem 7] we first try to adopt the strategy followed there and use [6, Section 5.1, identity 5.26]

$$\binom{l+q+1}{m+n+1} = \sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n} \quad (l,m \ge 0, n \ge q \ge 0). \tag{2.4}$$

With $l = a+b-1, q = 0, m = 2b, n = r-1$ and $k = s$ we get

$$\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b} = \sum_{b=0}^{a-r}\sum_{s=0}^{a+b-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{s}{r-1}\binom{2b}{b}$$

which by a change of summations yields

$$\begin{aligned} &= \sum_{s=r-1}^{2a-r-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b} \\ &= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}. \end{aligned} \tag{2.5}$$

Now we are ready to apply sum S_m from [6, Section 5.2, Problem 8]

$$S_m = \sum_{k=0}^{n}(-1)^k \frac{1}{k+m+1}\binom{n+k}{2k}\binom{2k}{k} = (-1)^n \frac{m!n!}{(m+n+1)!}\binom{m}{n} \quad (m,n \ge 0). \tag{2.6}$$

With $m = a-1, n = a-s-1$ and $k = b$ we find that (2.5) from above equals

$$\begin{aligned} &\sum_{s=r-1}^{a-1}\binom{s}{r-1}\frac{(-1)^{a+s+1}(a-1)!(a-s-1)!}{(2a-s-1)!}\binom{a-1}{a-s-1} \\ &= \frac{(-1)^{a+1}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=r-1}^{a-1}(-1)^s\binom{2a-r}{s-r+1} \\ &= \frac{(-1)^{a+r}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=0}^{a-r}(-1)^s\binom{2a-r}{s} \end{aligned} \tag{2.7}$$

(where we applied the substitution $s-r+1 \to s$ in the last step). Using the basic identity

$$\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k} \qquad (n,k \geq 0)$$

to evaluate the last sum in (2.7) we finally get

$$\begin{aligned}\sum_{b=0}^{a-r}(-1)^b\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b} &= \frac{(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\binom{2a-r-1}{a-r}\\ &= \frac{1}{2a-r}\binom{a-1}{a-r}.\end{aligned} \tag{2.8}$$

Now we go on plugging the results (2.3) and (2.8) from above in (2.2) and find

$$\begin{aligned}\frac{v_d^{(1)}}{v_d^{(0)}} &= \sum_{a=2}^{d}(-1)^d a\binom{d}{a}\binom{d+a}{d}\left(\frac{(-2)^{a-2}}{a} - \sum_{r=1}^{a}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r}\right)\\ &= \sum_{a=2}^{d}(-1)^{d+1}a\binom{d}{a}\binom{d+a}{d}\left(\sum_{r=0}^{a-1}(-2)^{r-1}\frac{1}{2a-r-1}\binom{a-1}{r} - (-2)^{a-2}\frac{1}{a}\right).\end{aligned} \tag{2.9}$$

In order to get rid of the inner sum we use an identity that may be proved as an application of the classical reflection law

$$\frac{1}{(1-z)^a}F\left(\left.{a,b \atop c}\right|\frac{-z}{1-z}\right) = F\left(\left.{a,c-b \atop c}\right| z\right) \tag{2.10}$$

for hypergeometric functions by J.F. Pfaff [8], namely

$$\sum_{k=0}^{m}(-2)^k\frac{2m+1}{2m-k+1}\binom{m}{k} = \frac{(-1)^m 2^{2m}}{\binom{2m}{m}} \qquad (m \geq 0), \tag{2.11}$$

cf. [6, identity (5.104)]. In this way we find

$$\begin{aligned}\frac{v_d^{(1)}}{v_d^{(0)}} &= \sum_{a=2}^{d}(-1)^{d+a+1}a\binom{d}{a}\binom{d+a}{d}\left(2^{2a-3}\frac{1}{2a-1}\frac{1}{\binom{2a-2}{a-1}} - 2^{a-2}\frac{1}{a}\right)\\ &= \sum_{a=2}^{d}(-1)^{d+a}\binom{d}{a}\binom{d+a}{d}\left(2^{a-2} - 2^{2a-2}\frac{1}{\binom{2a}{a}}\right),\end{aligned}$$

i.e.

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d}(-1)^{d+a}2^{a-2}\binom{d+a}{2a}\left(\binom{2a}{a} - 2^a\right), \tag{2.12}$$

so that we have proved that the ratio $v_d^{(1)}/v_d^{(0)}$ is an integer.

In the last step of the proof we establish the explicit formula for the ratios. Recall the Legendre polynomials $P_d(x)$, as defined in the theorem, and let

$$\rho_d(x) := \sum_{k=0}^{d} \binom{d+k}{d-k} x^k \tag{2.13}$$

denote the associated Legendre polynomials (cf. [10, p. 66]). Then (2) yields

$$\frac{v_d^{(1)}}{v_d^{(0)}} = (-1)^d \frac{P_d(-3) - \rho_d(-4)}{4}. \tag{2.14}$$

Now (cf. [9, p. 158])

$$P_d(-x) = (-1)^d P_d(x). \tag{2.15}$$

Furthermore ρ_d satisfies the recursive formula

$$\begin{aligned} &\rho_d(x) = (x+2)\rho_{d-1}(x) - \rho_{d-2}(x) \\ &\rho_0(x) = 0, \rho_1(x) = x+1 \end{aligned} \tag{2.16}$$

(cf. [10, p. 66]) so that $(-1)^d \rho_d(-4) = 2d+1$, which completes the proof of (2.1). □

3. Recurrence, Asymptotic Behaviour, and Probabilities

In this section we apply Theorem 2.1 in order to establish a recurrence for the quotients $v_d^{(1)}/v_d^{(0)}$ as well as to establish the asymptotic behaviour of this sequence for $d \to \infty$ and its consequence on the probabilities $v_d^{(1)}/v_d$.

Since the Legendre polynomials satisfy the recursive formula

$$\begin{aligned} &dP_d(x) - (2d-1)xP_{d-1}(x) + (d-1)P_{d-2}(x) = 0 \quad (d \geq 2), \\ &P_0(x) = 1, P_1(x) = x \end{aligned} \tag{3.1}$$

(cf. [9, p. 160]) we get the following second order linear recurrence for $v_d^{(1)}/v_d^{(0)}$.

Corollary 3.1. *We have*

$$d\frac{v_d^{(1)}}{v_d^{(0)}} - 3(2d-1)\frac{v_{d-1}^{(1)}}{v_{d-1}^{(0)}} + (d-1)\frac{v_{d-2}^{(1)}}{v_{d-2}^{(0)}} = 2d(d-1) \textit{ for } d \geq 2, \frac{v_0^{(1)}}{v_0^{(0)}} = \frac{v_1^{(1)}}{v_1^{(0)}} = 0.$$

We turn our attention now to the asymptotic behaviour of the ratios for $d \to \infty$ and start by their generating function. The generating function of the Legendre polynomials is given by ([10, p. 78])

$$\sum_{d \geq 0} P_d(x) z^d = \frac{1}{\sqrt{1 - 2xz + z^2}}, \tag{3.2}$$

so that the generating function of our ratios reads

Corollary 3.2. *We have*

$$V_1(z) := \sum_{d\geq 0} \frac{v_d^{(1)}}{v_d^{(0)}} z^d = \frac{1}{4}\left(\frac{1}{\sqrt{1-6z+z^2}} - \frac{1+z}{(1-z)^2}\right).$$

Performing singularity analysis the latter result allows to establish the asymptotic behaviour of the ratios for $d \to \infty$ as follows.

Proposition 3.3. *For $d \to \infty$*

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{1}{8\sqrt[4]{2}\sqrt{\pi d}}(3+2\sqrt{2})^{d+\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{d}\right)\right).$$

Proof. We adopt the usual technique of singularity analysis of generating functions, compare e.g. [5, Chapter IV] or [12, Chapter 8]. The dominating singularity of the generating function $V_1(z)$ is given by the zero $3-2\sqrt{2}$ of $1-6z+z^2$ closest to the origin, whereas the other zero of $1-6z+z^2$ as well as the term $\frac{1+z}{(1-z)^2}$ will give a contribution that is exponentially smaller than the contribution of the main term. The local expansion of $V_1(z)$ about the dominating singularity reads

$$V_1(z) = \frac{1}{8\sqrt[4]{2}\sqrt{3-2\sqrt{2}}}\left(1-\frac{z}{3-2\sqrt{2}}\right)^{-1/2}\left(1+\mathcal{O}\left(1-\frac{z}{3-2\sqrt{2}}\right)\right)$$
$$\text{for } z \to 3-2\sqrt{2},$$

from which the asymptotics is immediate. □

In [1] Akiyama and Pethő also discussed the probabilities

$$p_d^{(s)} := v_d^{(s)}/v_d \tag{3.3}$$

for a contractive normed polynomial of degree d in $\mathbb{R}[x]$ to have s pairs of complex conjugate roots. In particular they derived (cf. [1, Theorem 6.1])

$$\log p_d^{(0)} = -\frac{\log 2}{2}d^2 + \frac{1}{8}\log d + \mathcal{O}(1), \quad \text{for } d \to \infty, \tag{3.4}$$

for the probability of totally real polynomials and, by numerical evidence for $d \leq 100$, conjectured that

$$\log p_d^{(1)} \leq -\frac{\log 2}{2}d^2 + d\log q \tag{3.5}$$

for some constant q. Now, obviously, $p_d^{(1)} = \frac{v_d^{(1)}}{v_d^{(0)}} p_d^{(0)}$, so that from (3.4) and our Proposition 3.3 we gain

Corollary 3.4. *The probability $p_d^{(1)}$ for a contractive normed polynomial of degree d in $\mathbb{R}[x]$ to have exactly one pair of complex conjugate roots fulfills*

$$\log p_d^{(1)} = -\frac{\log 2}{2} d^2 + d \log(3 + 2\sqrt{2}) + \mathcal{O}(\log d) \quad \textit{for } d \to \infty.$$

4. Concluding Remarks

In this paper we were able to settle the instance $s = 1$ of Conjecture 1.1. The question arises, whether our methods could be used to prove the conjecture for additional instances of $s \geq 2$ or even for general $s \geq 1$. A crucial point for a possible application of our method would be to establish a generalization of the Selberg-Aomoto integral for integrands that will occur with the evaluation of $v_d^{(s)}$ for $s \geq 2$ similar to formula (1.2) in the instance $s = 1$. Work is in progress on this question, but even the explicit evaluation of the integrals that appear in instance $s = 2$ seems to be very hard.

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