QUADRANT MARKED MESH PATTERNS IN 132-AVOIDING PERMUTATIONS II

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Abstract

Given a permutation $\sigma = \sigma_1 \ldots \sigma_n$ in the symmetric group S_n, we say that σ_i matches the marked mesh pattern $\mathrm{MMP}(a, b, c, d)$ in σ if there are at least a points to the right of σ_i in σ which are greater than σ_i, at least b points to the left of σ_i in σ which are greater than σ_i, at least c points to the left of σ_i in σ which are smaller than σ_i, and at least d points to the right of σ_i in σ which are smaller than σ_i. This paper is continuation of the systematic study of the distributions of quadrant marked mesh patterns in 132-avoiding permutations started by the present authors where we mainly studied the distribution of the number of matches of $\mathrm{MMP}(a, b, c, d)$ in 132-avoiding permutations where exactly one of a, b, c, d is greater than zero and the remaining elements are zero. In this paper, we study the distribution of the number of matches of $\mathrm{MMP}(a, b, c, d)$ in 132-avoiding permutations where exactly two of a, b, c, d are greater than zero and the remaining elements are zero. We provide explicit recurrence relations to enumerate our objects which can be used to give closed forms for the generating functions associated with such distributions. In many cases, we provide combinatorial explanations of the coefficients that appear in our generating functions. The case of quadrant marked mesh patterns $\mathrm{MMP}(a, b, c, d)$ where three or more of a, b, c, d are constrained to be greater than 0 will be studied in a future article by the present authors.

1. Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [2] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. This notion was further studied in [1, 3, 5, 6, 9, 12].

Kitaev and Remmel [6] initiated the systematic study of distributions of quadrant marked mesh patterns on permutations. The study was extended to 132-avoiding permutations by Kitaev, Remmel and Tiefenbruck in [9], and the present paper continues this line of research. Kitaev and Remmel also studied the distributions of quadrant marked mesh patterns in up-down and down-up permutations [7, 8].

Let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation written in one-line notation. Then we will consider the *graph of* σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \ldots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ and any $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, the set of all permutations of length n, we say that σ_i *matches the quadrant marked mesh pattern* $\mathrm{MMP}(a, b, c, d)$ in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are at least a points in quadrant I, at least b points in quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. For example, if $\sigma = 471569283$, the point $\sigma_4 = 5$ matches the marked mesh pattern $\mathrm{MMP}(2, 1, 2, 1)$ since in $G(\sigma)$ relative to the coordinate system with the origin at $(4, 5)$, there are 3 points in quadrant I, 1 point in quadrant II, 2 points in quadrant III, and 2 points in quadrant IV. Note that if a coordinate in $\mathrm{MMP}(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant.

In addition, we shall consider patterns $\mathrm{MMP}(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{0\} \cup \{\emptyset\}$. Here when a coordinate of $\mathrm{MMP}(a, b, c, d)$ is the empty set, then for σ_i to match $\mathrm{MMP}(a, b, c, d)$ in $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, it must be the case that there are no points in $G(\sigma)$ relative to the coordinate system with the origin at (i, σ_i) in the corresponding quadrant. For example, if $\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern $\mathrm{MMP}(4, 2, \emptyset, \emptyset)$ since in $G(\sigma)$ relative to the coordinate system with the origin at $(3, 1)$, there are 6 points in $G(\sigma)$ in quadrant I, 2 points in $G(\sigma)$ in quadrant II, no points in both quadrants III and IV. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches $\mathrm{MMP}(a, b, c, d)$ in σ.

Note how the (two-dimensional) notation of Úlfarsson [12] for marked mesh patterns corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

Given a sequence $w = w_1 \ldots w_n$ of distinct integers, let $\mathrm{red}(w)$ be the permutation found by replacing the i-th smallest integer that appears in σ by i. For example, if $\sigma = 2754$, then $\mathrm{red}(\sigma) = 1432$. Given a permutation $\tau = \tau_1 \ldots \tau_j$ in the symmetric group S_j, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ provided there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \ldots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. Let $S_n(\tau)$ denote the set of permutations in S_n which avoid τ. In the theory of permutation patterns, τ is

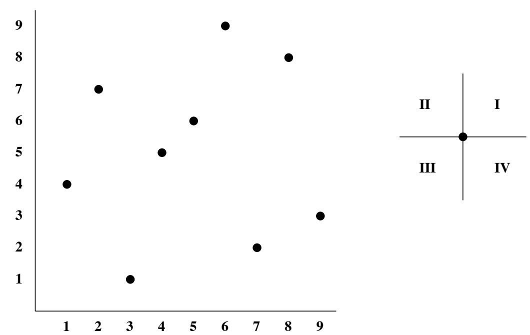

Figure 1: The graph of $\sigma = 471569283$.

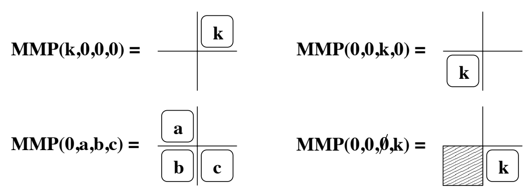

Figure 2: Úlfarsson notation for quadrant marked mesh patterns.

called a *classical pattern*. See [4] for a comprehensive introduction to the study of patterns in permutations.

It has been a rather popular direction of research in the literature on permutation patterns to study permutations avoiding a 3-letter pattern subject to extra restrictions (see [4, Subsection 6.1.5]). In [9], we started the study of the generating functions

$$Q_{132}^{(a,b,c,d)}(t,x) := 1 + \sum_{n\geq 1} Q_{n,132}^{(a,b,c,d)}(x)t^n$$

where for any $a,b,c,d \in \mathbb{N} \cup \{0\} \cup \{\emptyset\}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

For any a,b,c,d, we will write $Q_{n,132}^{(a,b,c,d)}(x)|_{x^k}$ for the coefficient of x^k in $Q_{n,132}^{(a,b,c,d)}(x)$.

For any fixed (a,b,c,d), we know that $Q_{n,132}^{(a,b,c,d)}(1)$ is the number of 132-avoiding permutations in S_n which is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus the coefficients in the polynomial $Q_{n,132}^{(a,b,c,d)}(x)$ represent a refinement of the nth Catalan number. It is then a natural question to ask whether (i) we can give explicit formulas for the coefficients that appear in $Q_{n,132}^{(a,b,c,d)}(x)$ or (ii) whether such coefficients count other interesting classes of combinatorial objects. Of course, there is an obvious answer to question (ii). That is, if one has a bijection from $S_n(132)$ to other classes of combinatorial objects which are counted by the Catalan numbers such as Dyck paths or binary trees, then one can use that bijection to give an interpretation of the pattern $\mathrm{MMP}(a,b,c,d)$ in the other setting. We shall see that in many cases, there are interesting connections with the coefficients that arise in our polynomials $Q_{n,132}^{(a,b,c,d)}(x)$ and other sets of combinatorial objects that do not just arise by such bijections.

In particular, it is natural to try to understand $Q_{n,132}^{(a,b,c,d)}(0)$ which equals the number of $\sigma \in S_n(132)$ that have no occurrences of the pattern $\mathrm{MMP}(a,b,c,d)$ as well as the coefficient of the highest power of x that occurs in $Q_{n,132}^{(a,b,c,d)}(x)$ since that coefficient equals the number of $\sigma \in S_n(132)$ that have the maximum possible number of occurrences of the pattern $\mathrm{MMP}(a,b,c,d)$. We shall see that in many cases, $Q_{n,132}^{(a,b,c,d)}(x)|_x$ and $Q_{n,132}^{(a,b,c,d)}(x)|_{x^2}$, the number of $\sigma \in S_n(132)$ with exactly one occurrence and two occurrences, respectively, of the pattern $\mathrm{MMP}(a,b,c,d)$ also have interesting combinatorics associated with them. There are many more interesting questions of this type that can be pursued, but due to space considerations, we shall mostly restrict ourselves to trying to understand the four coefficients in $Q_{n,132}^{(a,b,c,d)}(x)$ described above. We should note, however, that there is a uniform way to compute generating functions of the form

$$F_k^{(a,b,c,d)}(t) = \sum_{n\geq 0} Q_{n,132}^{(a,b,c,d)}(x)|_{x^k} t^n.$$

That is, $F_k^{(a,b,c,d)}(t)$ is just the result of taking the generating function

$$\frac{1}{k!}\frac{\partial^k}{\partial x^k} Q_{132}^{(a,b,c,d)}(t,x)$$

and then setting $x = 0$. Due to space considerations, we will not pursue the study of the functions $F_k^{(a,b,c,d)}(t)$ for $k \geq 2$ in this paper.

There is one obvious symmetry in this case which is induced by the fact that if $\sigma \in S_n(132)$, then $\sigma^{-1} \in S_n(132)$. That is, the following lemma was proved in [9].

Lemma 1. ([9]) *For any* $a,b,c,d \in \mathbb{N} \cup \{0\} \cup \{\emptyset\}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x).$$

In [9], we studied the generating functions

$$Q_{132}^{(0,k,0,0)}(t,x) = Q_{132}^{(0,0,0,k)}(t,x), Q_{132}^{(k,0,0,0)}(t,x), \text{ and } Q_{132}^{(0,0,k,0)}(t,x),$$

where k can be either the empty set or a positive integer as well as the generating functions $Q_{132}^{(k,0,\emptyset,0)}(t,x)$ and $Q_{132}^{(\emptyset,0,k,0)}(t,x)$. We also showed that sequences of the form $\{Q_{n,132}^{(a,b,c,d)}(x)|_{x^r}\}_{n\geq s}$ count a variety of combinatorial objects that appear in the *On-line Encyclopedia of Integer Sequences* (OEIS) [11]. Thus, our results gave new combinatorial interpretations of certain classical sequences such as the Fine numbers and the Fibonacci numbers as well as provided certain sequences that appear in the OEIS with a combinatorial interpretation where none had existed before. Another particular result of our studies in [9] is enumeration of permutations avoiding simultaneously the patterns 132 and 1234.

The main goal of this paper is to continue the study of $Q_{132}^{(a,b,c,d)}(t,x)$ and combinatorial interpretations of sequences of the form $\{Q_{n,132}^{(a,b,c,d)}(x)|_{x^r}\}_{n\geq s}$ in the case where $a,b,c,d \in \mathbb{N}$ and exactly two of these parameters are non-zero. The case when at least three of the parameters are non-zero will be studied in [10].

Next we list several results from [9] which we need in this paper.

Theorem 1. *([9, Theorem 3.1])*

$$Q_{132}^{(0,0,0,0)}(t,x) = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt}$$

and, for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,x)}.$$

Hence

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t}$$

and, for $k \geq 2$,

$$Q_{132}^{(k,0,0,0)}(t,0) = \frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,0)}. \tag{1}$$

Theorem 2. *([9, Theorem 4.1]) For $k \geq 1$,*

$$\begin{aligned} Q_{132}^{(0,0,k,0)}(t,x) &= \frac{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) - \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}}{2tx} \\ &= \frac{2}{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) + \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}} \end{aligned}$$

and

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1-t(C_0+C_1t+\cdots+C_{k-1}t^{k-1})}.$$

It follows from Lemma 1 that $Q_{132}^{(0,k,0,0)}(t,x) = Q_{132}^{(0,0,0,k)}(t,x)$ for all $k \geq 1$. Thus, our next theorem (obtained in [9]) gives an expression for $Q_{132}^{(0,k,0,0)}(t,x) = Q_{132}^{(0,0,0,k)}(t,x)$.

Theorem 3. *([9, Theorem 5.1])*

$$Q_{132}^{(0,1,0,0)}(t,x) = Q_{132}^{(0,0,0,1)}(t,x) = \frac{1}{1-tC(tx)}.$$

For $k > 1$,

$$Q_{132}^{(0,k,0,0)}(t,x) = Q_{132}^{(0,0,0,k)}(t,x) = \frac{1 + t\sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t,x) - C(tx))}{1 - tC(tx)}$$

and

$$Q_{132}^{(0,k,0,0)}(t,0) = Q_{132}^{(0,0,0,k)}(t,0) = \frac{1 + t\sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t,0) - 1)}{1 - t}.$$

As it was pointed out in [9], *avoidance* of a marked mesh pattern without quadrants containing the empty set can always be expressed in terms of multi-avoidance of (possibly many) classical patterns. Thus, among our results we will re-derive several known facts in permutation patterns theory. However, our main goals are more ambitious since they are aimed at finding distributions in question.

2. $Q_{n,132}^{(k,0,\ell,0)}(x)$ Where $k, \ell \geq 1$

Throughout this paper, we shall classify the 132-avoiding permutations $\sigma = \sigma_1 \ldots \sigma_n$ by the position of n in σ. That is, let $S_n^{(i)}(132)$ denote the set of $\sigma \in S_n(132)$ such that $\sigma_i = n$.

Clearly each $\sigma \in S_n^{(i)}(132)$ has the structure pictured in Figure 3. That is, in the graph of σ, the elements to the left of n, $A_i(\sigma)$, have the structure of a 132-avoiding permutation, the elements to the right of n, $B_i(\sigma)$, have the structure of a 132-avoiding permutation, and all the elements in $A_i(\sigma)$ lie above all the elements in $B_i(\sigma)$. It is well-known that the number of 132-avoiding permutations in S_n is the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the generating function for the C_n's is given by

$$C(t) = \sum_{n \geq 0} C_n t^n = \frac{1 - \sqrt{1-4t}}{2t} = \frac{2}{1 + \sqrt{1-4t}}.$$

If $k \geq 1$, it is easy to compute a recursion for $Q_{n,132}^{(k,0,\ell,0)}(x)$ for any fixed $\ell \geq 1$. It is clear that n can never match the pattern MMP$(k,0,\ell,0)$ for $k \geq 1$ in any $\sigma \in S_n(132)$. For $i \geq 1$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(k-1,0,\ell,0)}(x)$ to $Q_{n,132}^{(k,0,\ell,0)}(x)$ since none of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern MMP$(k,0,\ell,0)$ and the presence of n ensures that an element in $A_i(\sigma)$ matches MMP$(k,0,\ell,0)$ in σ if and only if it matches MMP$(k-1,0,\ell,0)$ in $A_i(\sigma)$. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,\ell,0)}(x)$ to $Q_{n,132}^{(k,0,\ell,0)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an

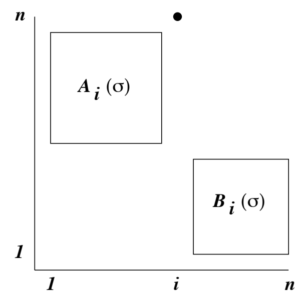

Figure 3: The structure of 132-avoiding permutations.

element in $B_i(\sigma)$ matches the pattern MMP$(k,0,\ell,0)$. Thus,

$$Q_{n,132}^{(k,0,\ell,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,\ell,0)}(x)\ Q_{n-i,132}^{(k,0,\ell,0)}(x). \tag{2}$$

Multiplying both sides of (2) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(k,0,\ell,0)}(t,x) = tQ_{132}^{(k-1,0,\ell,0)}(t,x)\ Q_{132}^{(k,0,\ell,0)}(t,x)$$

so that we have the following theorem.

Theorem 4. *For all $k, \ell \geq 1$,*

$$Q_{132}^{(k,0,\ell,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,\ell,0)}(t,x)}. \tag{3}$$

Note that by Theorem 2, we have an explicit formula for $Q_{132}^{(0,0,\ell,0)}(t,x)$ for all $\ell \geq 1$ so that we can then use the recursion (3) to compute $Q_{132}^{(k,0,\ell,0)}(t,x)$ for all $k \geq 1$.

2.1. Explicit Formulas for $Q_{n,132}^{(k,0,\ell,0)}(x)|_{x^r}$

Note that

$$Q_{132}^{(k,0,\ell,0)}(t,0) = \frac{1}{1 - tQ_{132}^{(k-1,0,\ell,0)}(t,0)}. \tag{4}$$

Since $Q_{132}^{(1,0,0,0)}(t,0) = Q_{132}^{(0,0,1,0)}(t,0) = \frac{1}{1-t}$, it follows from the recursions (1) and (4) that for all $k \geq 2$, $Q_{132}^{(k,0,0,0)}(t,0) = Q_{132}^{(k-1,0,1,0)}(t,0)$. This is easy to see directly. That is, it is clear that if in $\sigma \in S_n(132)$, σ_j matches MMP$(k-1,0,1,0)$, then there is an $i < j$ such that $\sigma_i < \sigma_j$ so that σ_i matches MMP$(k,0,0,0)$. Vice versa, suppose that in $\sigma \in S_n(132)$, σ_j matches MMP$(k,0,0,0)$ where $k \geq 2$. Because σ is 132-avoiding this means the elements in the first quadrant relative to the coordinate system with (j, σ_j) as the origin must be increasing. Thus, there exist $j < j_1 < \cdots < j_k \leq n$ such that $\sigma_j < \sigma_{j_1} < \cdots < \sigma_{j_k}$ and, hence, σ_{j_1} matches $\mathrm{MMP}(k-1,0,1,0)$. Thus, the number of $\sigma \in S_n(132)$ where $\mathrm{mmp}^{(k,0,0,0)}(\sigma) = 0$ is equal to the number of $\sigma \in S_n(132)$ where $\mathrm{mmp}^{(k-1,0,1,0)}(\sigma) = 0$ for $k \geq 2$.

In [9], we computed the generating function $Q_{132}^{(k,0,0,0)}(t,0)$ for small k. Thus, we have that

$$\begin{aligned}
Q_{132}^{(2,0,0,0)}(t,0) = Q_{132}^{(1,0,1,0)}(t,0) &= \frac{1-t}{1-2t};\\
Q_{132}^{(3,0,0,0)}(t,0) = Q_{132}^{(2,0,1,0)}(t,0) &= \frac{1-2t}{1-3t+t^2};\\
Q_{132}^{(4,0,0,0)}(t,0) = Q_{132}^{(3,0,1,0)}(t,0) &= \frac{1-3t+t^2}{1-4t+3t^2};\\
Q_{132}^{(5,0,0,0)}(t,0) = Q_{132}^{(4,0,1,0)}(t,0) &= \frac{1-4t+3t^2}{1-5t+6t^2-t^3};\\
Q_{132}^{(6,0,0,0)}(t,0) = Q_{132}^{(5,0,1,0)}(t,0) &= \frac{1-5t+6t^2-t^3}{1-6t+10t^2-4t^3}, \text{and}\\
Q_{132}^{(7,0,0,0)}(t,0) = Q_{132}^{(6,0,1,0)}(t,0) &= \frac{1-6t+10t^3-4t^3}{1-7t+15t^2-10t^3+t^4}.
\end{aligned}$$

Note that $Q_{132}^{(0,0,2,0)}(t,0) = \frac{1}{1-t-t^2}$ by Theorem 2. Thus, by (4), we can compute that

$$\begin{aligned}
Q_{132}^{(1,0,2,0)}(t,0) &= \frac{1-t-t^2}{1-2t-t^2};\\
Q_{132}^{(2,0,2,0)}(t,0) &= \frac{1-2t-t^2}{1-3t+t^3};\\
Q_{132}^{(3,0,2,0)}(t,0) &= \frac{1-3t+t^3}{1-4t+2t^2+2t^3}, \text{ and}\\
Q_{132}^{(4,0,2,0)}(t,0) &= \frac{1-4t+2t^2+2t^3}{1-5t+5t^2+2t^3-t^4}.
\end{aligned}$$

We note that $\{Q_{n,132}^{(1,0,2,0)}(0)\}_{n\geq 1}$ is the sequence of the Pell numbers which is A000129 in the OEIS. This result should be compared with a known fact [4, page 250] that the avoidance of 123, 2143 and 3214 simultaneously gives the Pell numbers (the avoidance of $\mathrm{MMP}(1,0,2,0)$ is equivalent to avoiding simultaneously 2134 and 1234).

Problem 1. Find a combinatorial explanation of the fact that the number of permutations of S_n which are (132,2134,1234)-avoiding equals the number of permutaions of S_n which are (123,2143,3214)-avoiding. Can any of the known bijections between 132-avoiding permutations and 123-avoiding permutations (see [4, Chapter 4]) be of help here?

The sequence $\{Q_{n,132}^{(2,0,2,0)}(0)\}_{n\geq 1}$ is sequence A052963 in the OEIS which has the generating function $\frac{1-t-t^2}{1-3t+t^3}$. That is, $\frac{1-2t-t^2}{1-3t+t^3}-1 = t\frac{1-t-t^2}{1-3t+t^3}$. This sequence had no listed combinatorial interpretation so that we have now given a combinatorial interpretation to this sequence.

Similarly, $Q_{132}^{(0,0,3,0)}(t,0) = \frac{1}{1-t-t^2-2t^3}$. Thus, by (4), we can compute that

$$\begin{aligned}
Q_{132}^{(1,0,3,0)}(t,0) &= \frac{1-t-t^2-2t^3}{1-2t-t^2-2t^3},\\
Q_{132}^{(2,0,3,0)}(t,0) &= \frac{1-2t-t^2-2t^3}{1-3t-t^3+2t^4},\\
Q_{132}^{(3,0,3,0)}(t,0) &= \frac{1-3t-t^3+2t^4}{1-4t+2t^2+4t^4}, \text{ and}\\
Q_{132}^{(4,0,3,0)}(t,0) &= \frac{1-4t+2t^2+4t^4}{1-5t+5t^2+5t^4-2t^5}.
\end{aligned}$$

In this case, the sequence $\{Q_{n,132}^{(1,0,3,0)}(0)\}_{n\geq 1}$ is sequence A077938 in the OEIS which has the generating function $\frac{1}{1-2t-t^2-2t^3}$. That is, $\frac{1-t-t^2-2t^3}{1-2t-t^2-2t^3}-1 = t\frac{1}{1-2t-t^2-2t^3}$. This sequence had no listed combinatorial interpretation so that we have now given a combinatorial interpretation to this sequence.

We can also find the coefficient of the highest power of x that occurs in $Q_{n,132}^{(k,0,\ell,0)}(x)$ for any $k,\ell \geq 1$. That is, it is easy to see that the maximum possible number of matches of MMP$(k,0,\ell,0)$ for a $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ occurs when $\sigma_1 \ldots \sigma_\ell$ is a 132-avoiding permutation in S_ℓ and $\sigma_{\ell+1} \ldots \sigma_n$ is an increasing sequence. Thus, we have the following theorem.

Theorem 5. *For any $k,\ell \geq 1$ and $n \geq k+\ell+1$, the highest power of x that occurs in $Q_{n,132}^{(k,0,\ell,0)}(x)$ is $x^{n-k-\ell}$ which appears with a coefficient of C_ℓ.*

We can compute the generating functions $Q_{132}^{(0,0,\ell,0)}(t,x)$ via Theorem 2. For example, one can compute that

$$\begin{aligned}
Q_{132}^{(0,0,1,0)}(t,x) &= \frac{1+t(-1+x)-\sqrt{(1+t(-1+x))^2-4tx}}{2tx},\\
Q_{132}^{(0,0,2,0)}(t,x) &= \frac{1+t(1+t)(-1+x)-\sqrt{(1+t(1+t)(-1+x))^2-4tx}}{2tx}, \text{ and}\\
Q_{132}^{(0,0,3,0)}(t,x) &= \frac{1+t\left(1+t+2t^2\right)(-1+x)-\sqrt{\left(1+t\left(1+t+2t^2\right)(-1+x)\right)^2-4tx}}{2tx}.
\end{aligned}$$

We can then use (3) to compute the functions of the form $Q_{132}^{(k,0,\ell,0)}(t,x)$ for small k and ℓ. The formulas for these functions get more and more complicated so we will not in general give explicit formulas. However, one can easily use any computer algebra system such as Mathematica or Maple to compute the following.

$$\begin{aligned}Q_{132}^{(1,0,1,0)}(t,x) &= 1+t+2t^2+(4+x)t^3+\left(8+5x+x^2\right)t^4+\left(16+17x+8x^2+x^3\right)t^5\\&+\left(32+49x+38x^2+12x^3+x^4\right)t^6+\left(64+129x+141x^2+77x^3+17x^4+x^5\right)t^7\\&+\left(128+321x+453x^2+361x^3+143x^4+23x^5+x^6\right)t^8\\&+\left(256+769x+1326x^2+1399x^3+834x^4+247x^5+30x^6+x^7\right)t^9+\cdots.\end{aligned}$$

$$\begin{aligned}Q_{132}^{(2,0,1,0)}(t,x) &= 1+t+2t^2+5t^3+(13+x)t^4+\left(34+7x+x^2\right)t^5\\&+\left(89+32x+10x^2+x^3\right)t^6+\left(233+122x+59x^2+14x^3+x^4\right)t^7+\\&+\left(610+422x+272x^2+106x^3+19x^4+x^5\right)t^8\\&+\left(1597+1376x+1090x^2+591x^3+182x^4+25x^5+x^6\right)t^9+\cdots.\end{aligned}$$

$$\begin{aligned}Q_{132}^{(3,0,1,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+(41+x)t^5+\left(122+9x+x^2\right)t^6\\&+\left(365+51x+12x^2+x^3\right)t^7+\left(1094+235x+84x^2+16x^3+x^4\right)t^8\\&+\left(3281+966x+454x^2+139x^3+21x^4+x^5\right)t^9+\cdots.\end{aligned}$$

We can explain several of the coefficients that appear in the polynomials $Q_{n,132}^{(k,0,1,0)}(x)$ for various k.

Theorem 6. *$Q_{n,132}^{(1,0,1,0)}(0) = 2^{n-1}$ for $n \geq 1$.*

Proof. This follows immediately from the fact that $Q_{132}^{(1,0,1,0)}(t,0) = \frac{1-t}{1-2t}$. We can also give a simple inductive proof of this fact.

Clearly $Q_{1,132}^{(1,0,1,0)}(0) = 1$. Assume that $Q_{k,132}^{(1,0,1,0)}(0) = 2^{k-1}$ for $k < n$. Then suppose that $\mathrm{mmp}^{(1,0,1,0)}(\sigma) = 0$ and $\sigma_i = n$. Then it must be the case that the elements to the left of σ_i are decreasing so that $\sigma_1 \ldots \sigma_{i-1} = (n-1)(n-2)\ldots(n-(i-1))$. But then the elements to the right of σ_i must form a 132-avoiding permutation of S_{n-1} which has no occurrence of the pattern MMP$(1,0,1,0)$. Thus, if $i = n$, we only have one such σ and if $i < n$, we have 2^{n-i-1} choices for $\sigma_{i+1}\ldots\sigma_n$ by induction. It follows that

$$Q_{n,132}^{(1,0,1,0)}(0) = 1 + \sum_{i=1}^{n-1} 2^{i-1} = 2^{n-1}.$$

□

The sequence $\{Q_{n,132}^{(1,0,1,0)}(x)|_x\}_{n\geq 3}$ is the sequence A000337 in the OEIS which has the formula $a(n) = (n-1)2^n + 1$, and the following theorem confirms this fact.

Theorem 7. *For $n \geq 3$,*

$$Q_{n,132}^{(1,0,1,0)}(x)|_x = (n-3)2^{n-2} + 1. \tag{5}$$

Proof. To prove (5), we classify the $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$ such that $\mathrm{mmp}^{(1,0,1,0)}(\sigma) = 1$ according to whether the σ_i which matches $\mathrm{MMP}(1,0,1,0)$ occurs to the left or right of position of n in σ.

First, suppose that $\sigma_i = n$ and the σ_s matching $\mathrm{MMP}(1,0,1,0)$ in σ is such that $s < i$. It follows that $\mathrm{red}(\sigma_1 \dots \sigma_{i-1})$ is an element of $S_{i-1}(132)$ such that $\mathrm{mmp}^{(0,0,1,0)} = 1$. We proved in [9, Theorem 4.3] that $Q^{(0,0,1,0)}_{n,132}(x)|_x = \binom{n}{2}$ so that we have $\binom{i-1}{2}$ choices for $\sigma_1 \dots \sigma_{i-1}$. It must be the case that $\mathrm{mmp}^{(1,0,1,0)}(\sigma_{i+1} \dots \sigma_n) = 0$ so that we have 2^{n-i-1} choices for $\sigma_{i+1} \dots \sigma_n$ by Theorem 6. It follows that there are $\binom{n-1}{2} + \sum_{i=3}^{n-1} \binom{i-1}{2} 2^{n-i-1}$ permutations $\sigma \in S_n(132)$ where the unique element which matches $\mathrm{MMP}(1,0,1,0)$ occurs to the left of the position of n in σ.

Next suppose that $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$, $\mathrm{mmp}^{(1,0,1,0)}(\sigma) = 1$, $\sigma_i = n$ and the σ_s matching $\mathrm{MMP}(1,0,1,0)$ is such that $s > i$. Then the elements to the left of σ_i in σ must be decreasing and the elements to the right of σ_i in σ must be such that $\mathrm{mmp}^{(1,0,1,0)}(\sigma_{i+1} \dots \sigma_n) = 1$. Thus, we have $1 + (n-i-3)2^{n-i-2}$ choices for $\sigma_{i+1} \dots \sigma_n$ by induction. It follows that there are

$$\sum_{i=1}^{n-3} (1 + (n-i-3)2^{n-i-2}) = (n-3) + \sum_{j=1}^{n-4} j 2^{j+1}$$

permutations $\sigma \in S_n(132)$ where the unique element which matches $\mathrm{MMP}(1,0,1,0)$ occurs to the right of the position of n in σ. Thus,

$$\begin{aligned} Q^{(1,0,1,0)}_{n,132}(x)|_x &= (n-3) + \sum_{j=1}^{n-4} j 2^{j+1} + \binom{n-1}{2} + \sum_{i=3}^{n-1} \binom{i-1}{2} 2^{n-i-1} \\ &= (n-3)2^{n-2} + 1. \end{aligned}$$

Here the last equality can easily be proved by induction or be verified by Mathematica. $\square$

We also can find explicit formulas for the second highest coefficient of x in $Q^{(k,0,1,0)}_n(x)$ for $k \geq 1$.

Theorem 8. *We have*

$$Q^{(k,0,1,0)}_{n,132}(x)|_{x^{n-2-k}} = 2k + \binom{n-k}{2} \tag{6}$$

for all $n \geq k+3$.

Proof. We proceed by induction on k.

First we shall prove that $Q^{(1,0,1,0)}_{n,132}(x)|_{x^{n-3}} = 2 + \binom{n-1}{2}$ for $n \geq 4$. That is, suppose that $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$ and $\mathrm{mmp}^{(1,0,1,0)}(\sigma) = n-3$. If $\sigma_1 = n$, then $\sigma_2 \dots \sigma_n$ must be strictly increasing. Similarly, if $\sigma_{n-1} = n$ so that $\sigma_n = 1$, then $\sigma_1 \dots \sigma_{n-1}$ must be strictly increasing. It cannot be that $\sigma_i = n$ where $1 < i < n-1$ because in that case the most $\mathrm{MMP}(1,0,1,0)$-matches that we can have in σ occurs when

$\sigma_1 \ldots \sigma_i$ is an increasing sequence and $\sigma_{i+1} \ldots \sigma_n$ is an increasing sequence which would give us a total of $i-2+n-i-2 = n-4$ matches of MMP$(1,0,1,0)$. Thus, the only other possibility is if $\sigma_n = n$ in which case $\mathrm{mmp}^{(0,0,1,0)}(\sigma_1 \ldots \sigma_{n-1}) = n-3$. We proved in [9, Theorem 4.3] that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$. Thus, if $\sigma_n = n$, we have $\binom{n-1}{2}$ choices for $\sigma_1 \ldots \sigma_{n-1}$. It follows that $Q_{n,132}^{(1,0,1,0)}(x)|_{x^{n-3}} = 2 + \binom{n-1}{2}$ for $n \geq 4$.

Assume that $k \geq 2$ we have established (6) for $k-1$. We know that the highest power of x that occurs in $Q_{n,132}^{(k,0,1,0)}(x)$ is x^{n-1-k} which occurs with a coefficient of 1 for $n \geq k+2$. Now

$$Q_{n,132}^{(k,0,1,0)}(x)|_{x^{n-2-k}} = \sum_{i=1}^{n} (Q_{i-1,132}^{(k-1,0,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x))|_{x^{n-2-k}}.$$

Since the highest power of x that occurs in $Q_{i-1,132}^{(k-1,0,1,0)}(x)$ is $x^{\max\{i-1-k,0\}}$ and the highest power of x that occurs in $Q_{n-i,132}^{(k,0,1,0)}(x)$ is $x^{\max\{n-i-1-k,0\}}$,

$$(Q_{i-1,132}^{(k-1,0,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x))|_{x^{n-2-k}} = 0$$

unless $i \in \{1, n-1, n\}$. Thus, we have 3 cases.

`Case 1.` $i = 1$. In this case,

$$(Q_{i-1,132}^{(k-1,0,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x))|_{x^{n-2-k}} = Q_{n-1,132}^{(k,0,1,0)}(x)|_{x^{n-2-k}} = 1.$$

`Case 2.` $i = n-1$. In this case, we are considering permutations of the form $\sigma = \sigma_1 \ldots \sigma_{n-2} n 1$. Then we must have $\mathrm{mmp}^{(k-1,0,1,0)}(\mathrm{red}(\sigma_1 \ldots \sigma_{n-2})) = n-k-2 = (n-2)-1-(k-1)$ so that there is only one choice for $\sigma_1 \ldots \sigma_{n-2}$. Thus, in this case,

$$(Q_{i-1,132}^{(k-1,0,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x))|_{x^{n-2-k}} = Q_{n-2,132}^{(k-1,0,1,0)}(x))|_{x^{n-2-k}} = 1.$$

`Case 3.` $i = n$. In this case,

$$\begin{aligned}(Q_{i-1,132}^{(k-1,0,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x))|_{x^{n-2-k}} &= Q_{n-1,132}^{(k-1,0,1,0)}(x))|_{x^{n-2-k}} \\ &= 2(k-1) + \binom{n-1-(k-1)}{2} \\ &= 2(k-1) + \binom{n-k}{2}\end{aligned}$$

for $n-1 \geq k-1+3$.

Thus, it follows that $Q_{n,132}^{(k,0,1,0)}(x)|_{x^{n-2-k}} = 2k + \binom{n-k}{2}$ for $n \geq k+3$. $\square$

Similarly, we have computed the following.

$$
\begin{aligned}
Q_{132}^{(1,0,2,0)}(t,x) &= 1+t+2t^2+5t^3+(12+2x)t^4+\left(29+11x+2x^2\right)t^5 \\
&+\left(70+45x+15x^2+2x^3\right)t^6+\left(169+158x+81x^2+19x^3+2x^4\right)t^7 \\
&+\left(408+509x+359x^2+129x^3+23x^4+2x^5\right)t^8 \\
&+\left(985+1550x+1409x^2+700x^3+189x^4+27x^5+2x^6\right)t^9+\cdots.
\end{aligned}
$$

$$
\begin{aligned}
Q_{132}^{(2,0,2,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+(40+2x)t^5+\left(115+15x+2x^2\right)t^6 \\
&+\left(331+77x+19x^2+2x^3\right)t^7+\left(953+331x+121x^2+23x^3+2x^4\right)t^8 \\
&+\left(2744+1288x+624x^2+177x^3+27x^4+2x^5\right)t^9+\cdots.
\end{aligned}
$$

In this case, the sequence $(Q_{n,132}^{(2,0,2,0)}(0))_{n\geq 1}$ is A052963 in the OEIS which satisfies the recursion $a(n)=3a(n-1)-a(n-3)$ with $a(0)=1$, $a(1)=2$ and $a(2)=5$, and has the generating function $\frac{1-t-t^2}{1-3t+t^3}$.

$$
\begin{aligned}
Q_{132}^{(3,0,2,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+42t^5+(130+2x)t^6 \\
&+\left(408+19x+2x^2\right)t^7+\left(1288+117x+23x^2+2x^3\right)t^8 \\
&+\left(4076+588x+169x^2+27x^3+2x^4\right)t^9+\cdots.
\end{aligned}
$$

We have also computed the following.

$$
\begin{aligned}
Q_{132}^{(1,0,3,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+(37+5x)t^5+\left(98+29x+5x^2\right)t^6 \\
&+\left(261+124x+39x^2+5x^3\right)t^7+\left(694+475x+207x^2+49x^3+5x^4\right)t^8 \\
&+\left(1845+1680x+963x^2+310x^3+59x^4+5x^5\right)t^9 \\
&+\left(4906+5635x+4056x^2+1692x^3+433x^4+69x^5+5x^6\right)t^{10}+\cdots.
\end{aligned}
$$

$$
\begin{aligned}
Q_{132}^{(2,0,3,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+42t^5+(127+5x)t^6 \\
&+\left(385+39x+5x^2\right)t^7+\left(1169+207x+49x^2+5x^3\right)t^8 \\
&+\left(3550+938x+310x^2+59x^3+5x^4\right)t^9 \\
&+\left(10781+3866x+1642x^2+433x^3+69x^4+5x^5\right)t^{10}+\cdots.
\end{aligned}
$$

$$
\begin{aligned}
Q_{132}^{(3,0,3,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+42t^5+132t^6+(424+5x)t^7 \\
&+\left(1376+49x+5x^2\right)t^8+\left(4488+310x+59x^2+5x^3\right)t^9 \\
&+\left(14672+1617x+433x^2+69x^3+5x^4\right)t^{10}+\cdots.
\end{aligned}
$$

We can also find a formula for the second highest coefficient in $Q_{n,132}^{(k,0,m,0)}(x)$ for $m \geq 2$.

Theorem 9. *For all $k \geq 1$, $m \geq 2$ and $n \geq m+k+2$,*

$$Q_{n,132}^{(k,0,m,0)}(x)|_{x^{n-m-k-1}} = C_{m+1} + (2k+1)C_m + 2C_m(n-k-m-2).$$

Proof. First we establish the base case which is when $k=1$ and $m \geq 2$. In this case,

$$Q_{n,132}^{(1,0,m,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,0,m,0)}(x) Q_{n-i,132}^{(1,0,m,0)}(x).$$

Since the highest power of x that can appear in $Q_{n,132}^{(0,0,m,0)}(x)$ is x^{n-m} for $n > m$ and the highest power of x that can appear in $Q_{n,132}^{(1,0,m,0)}(x)$ is x^{n-m-1} for $n > m+1$, it follows that the highest power of x that appears in $Q_{i-1,132}^{(0,0,m,0)}(x) Q_{n-i,132}^{(1,0,m,0)}(x)$ will be less than x^{n-m-2} for $i = 2, \ldots, n-2$. Thus, we have three cases to consider.

`Case 1.` $i=1$. In this case, $Q_{i-1,132}^{(0,0,m,0)}(x) Q_{n-i,132}^{(1,0,m,0)}(x) = Q_{n-1,132}^{(1,0,m,0)}(x)$ and we know by Theorem 5 that

$$Q_{n-1,132}^{(1,0,m,0)}(x)|_{x^{n-m-2}} = C_m \text{ for } n \geq m+2.$$

`Case 2.` $i=n-1$. In this case, $Q_{i-1,132}^{(0,0,m,0)}(x) Q_{n-i,132}^{(1,0,m,0)}(x) = Q_{n-2,132}^{(0,0,m,0)}(x)$ and it was proved in [9, Theorem 4.2] that

$$Q_{n-2,132}^{(0,0,m,0)}(x)|_{x^{n-m-2}} = C_m \text{ for } n \geq m+2.$$

`Case 3.` $i=n$. In this case, $Q_{i-1,132}^{(0,0,m,0)}(x) Q_{n-i,132}^{(1,0,m,0)}(x) = Q_{n-1,132}^{(0,0,m,0)}(x)$ and it was proved in [9, Theorem 4.2] that

$$Q_{n-1,132}^{(0,0,m,0)}(x)|_{x^{n-m-2}} = C_{m+1} - C_m + 2C_m(n-2-m) \text{ for } n \geq m+3.$$

Thus, it follows that

$$\begin{aligned} Q_{n,132}^{(1,0,m,0)}(x)|_{x^{n-m-2}} &= C_{m+1} + C_m + 2C_m(n-2-m) \\ &= C_{m+1} + 3C_m + 2C_m(n-3-m) \text{ for } n \geq m+3. \end{aligned}$$

For example, for $m=2$, we get that

$$Q_{n,132}^{(1,0,2,0)}(x)|_{x^{n-4}} = 11 + 4(n-5) \text{ for } n \geq 5$$

and, for $m=3$, we get that

$$Q_{n,132}^{(1,0,3,0)}(x)|_{x^{n-5}} = 29 + 10(n-6) \text{ for } n \geq 6,$$

which agrees with the series that we computed.

Now assume that $k > 1$ and we have proved the theorem for $k-1$ and all $m \geq 2$. Then

$$Q_{n,132}^{(k,0,m,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x).$$

Since the highest power of x that can appear in $Q_{n,132}^{(k-1,0,m,0)}(x)$ is $x^{n-m-(k-1)}$ for $n \geq m+k$ and the highest power of x that can appear in $Q_{n,132}^{(k,0,m,0)}(x)$ is x^{n-m-k} for $n > m+k$, it follows that the highest power of x that appears in $Q_{i-1,132}^{(k-1,0,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x)$ will be less than $x^{n-m-k-1}$ for $i = 2, \ldots, n-2$. Thus, we have three cases to consider.

`Case 1.` $i = 1$. In this case, $Q_{i-1,132}^{(k-1,0,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x) = Q_{n-1,132}^{(k,0,m,0)}(x)$ and we know by Theorem 5 that

$$Q_{n-1,132}^{(k,0,m,0)}(x)|_{x^{n-m-k-1}} = C_m \text{ for } n \geq m+k+2.$$

`Case 2.` $i = n-1$. In this case, $Q_{i-1,132}^{(k-1,0,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x) = Q_{n-2,132}^{(k-1,0,m,0)}(x)$ and we know by Theorem 5 that

$$Q_{n-2,132}^{(k-1,0,m,0)}(x)|_{x^{n-m-k-1}} = C_m \text{ for } n \geq m+k+2.$$

`Case 3.` $i = n$. In this case, $Q_{i-1,132}^{(k-1,0,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x) = Q_{n-1,132}^{(k-1,0,m,0)}(x)$ and we know by induction that

$$Q_{n-1,132}^{(k-1,0,m,0)}(x)|_{x^{n-m-k-1}} = C_{m+1} + (2(k-1)+1)C_m + 2C_m(n-m-(k-1)-1)$$

for $n \geq m+k+2$. Thus, it follows that

$$Q_{n,132}^{(k,0,m,0)}(x)|_{x^{n-m-k-1}} = C_{m+1} + (2k+1)C_m + 2C_m(n-m-k-2) \text{ for } n \geq m+k+2.$$

□

3. $Q_{n,132}^{(k,0,0,\ell)}(x) = Q_{n,132}^{(k,\ell,0,0)}(x)$ Where $k, \ell \geq 1$

By Lemma 1, we know that $Q_{n,132}^{(k,0,0,\ell)}(x) = Q_{n,132}^{(k,\ell,0,0)}(x)$. Thus, we will only consider $Q_{n,132}^{(k,0,0,\ell)}(x)$ in this section.

Suppose that $n \geq \ell+1$. It is clear that, for $k \geq 1$, n can never match the pattern $\mathrm{MMP}(k,0,0,\ell)$ in any $\sigma \in S_n(132)$. For $i \leq n-\ell$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(k-1,0,0,0)}(x)$ to $Q_{n,132}^{(k,0,0,\ell)}(x)$. That is, all the elements of $A_i(\sigma)$ have the elements in $B_i(\sigma)$ in their fourth quadrant, and $B_i(\sigma)$ consists of at least ℓ elements, so that the presence of n ensures that an element in $A_i(\sigma)$ matches MMP$(k,0,0,\ell)$ in σ if and only if it matches MMP$(k-1,0,0,0)$ in $A_i(\sigma)$. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,0,\ell)}(x)$ to $Q_{n,132}^{(k,0,0,\ell)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches MMP$(k,0,0,\ell)$.

Now suppose $i > n-\ell$ and $j = n-i$. In this case, $B_i(\sigma)$ consists of j elements. In this situation, an element of $A_i(\sigma)$ matches MMP$(k,0,0,\ell)$ in σ if and only if it matches MMP$(k-1,0,0,\ell-j)$ in $A_i(\sigma)$. Thus, our choices for $A_i(\sigma)$ contribute a factor of $Q_{i-1,132}^{(k-1,0,0,\ell-j)}(x) = Q_{n-j-1,132}^{(k-1,0,0,\ell-j)}(x)$ to $Q_{n,132}^{(k,0,0,\ell)}(x)$. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,0,\ell)}(x)$ to $Q_{n,132}^{(k,0,0,\ell)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern MMP$(k,0,0,\ell)$. Note that since $j < \ell$, we know that $Q_{n-i,132}^{(k,0,0,\ell)}(x) = C_j$.

It follows that for $n \geq \ell+1$,

$$\begin{aligned} Q_{n,132}^{(k,0,0,\ell)}(x) &= \sum_{i=1}^{n-\ell} Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,\ell)}(x) + \\ & \sum_{j=0}^{\ell-1} C_j Q_{n-j-1,132}^{(k-1,0,0,\ell-j)}(x). \end{aligned} \tag{7}$$

Multiplying both sides of (7) by t^n, summing for $n \geq \ell+1$ and observing that $Q_{j,132}^{(k,0,0,\ell)}(x) = C_j$ for $j \leq \ell$, we see that for $k,\ell \geq 1$,

$$\begin{aligned} Q_{132}^{(k,0,0,\ell)}(t,x) - \sum_{j=0}^{\ell} C_j t^j &= tQ_{132}^{(k-1,0,0,0)}(t,x)\left(Q_{132}^{(k,0,0,\ell)}(t,x) - \sum_{j=0}^{\ell-1} C_j t^j\right) + \\ & t\sum_{j=0}^{\ell-1} C_j t^j \left(Q_{132}^{(k-1,0,0,\ell-j)}(t,x) - \sum_{s=0}^{\ell-j-1} C_s t^s\right). \end{aligned}$$

Thus, we have the following theorem.

Theorem 10. *For all $k,\ell \geq 1$,*

$$Q_{132}^{(k,0,0,\ell)}(t,x) = \frac{C_\ell t^\ell + \sum_{j=0}^{\ell-1} C_j t^j (1 - tQ_{132}^{(k-1,0,0,0)}(t,x) + t(Q_{132}^{(k-1,0,0,\ell-j)}(t,x) - \sum_{s=0}^{\ell-j-1} C_s t^s))}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}. \tag{8}$$

Note that we can compute generating functions of the form $Q_{132}^{(k,0,0,0)}(t,x)$ by Theorem 1 and generating functions of the form $Q_{132}^{(0,0,0,\ell)}(t,x)$ by Theorem 3 so that we can use (8) to compute $Q_{132}^{(k,0,0,\ell)}(t,x)$ for any $k,\ell \geq 0$.

3.1. Explicit Formulas for $Q_{n,132}^{(k,0,0,\ell)}(x)|_{x^r}$

By Theorem 10, we have that

$$
\begin{aligned}
Q_{132}^{(k,0,0,1)}(t,x) &= \frac{t + (1 - tQ_{132}^{(k-1,0,0,0)}(t,x)) + t(Q_{132}^{(k-1,0,0,1)}(t,x) - 1)}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)} \\
&= \frac{1 - tQ_{132}^{(k-1,0,0,0)}(t,x) + tQ_{132}^{(k-1,0,0,1)}(t,x)}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}. \qquad (9)
\end{aligned}
$$

We note that $Q_{132}^{(0,0,0,0)}(t,x) = C(tx)$ so that $Q_{132}^{(0,0,0,0)}(t,0) = 1$. As described in the previous section, we have computed $Q_{132}^{(k,0,0,0)}(t,0)$ for small values of k in [9]. Plugging those generating functions into (9), one can compute that

$$
\begin{aligned}
Q_{132}^{(1,0,0,1)}(t,0) &= \frac{1 - t + t^2}{(1-t)^2}, \\
Q_{132}^{(2,0,0,1)}(t,0) &= \frac{1 - 2t + t^2 + t^3}{1 - 3t + 2t^2}, \\
Q_{132}^{(3,0,0,1)}(t,0) &= \frac{1 - 3t + 2t^2 + t^4}{1 - 4t + 4t^2 - t^3}, \\
Q_{132}^{(4,0,0,1)}(t,0) &= \frac{1 - 4t + 4t^2 - t^3 + t^5}{1 - 5t + 7t^2 - 3t^3}, \text{ and} \\
Q_{132}^{(5,0,0,1)}(t,0) &= \frac{1 - 5t + 7t^2 - 3t^3 + t^6}{1 - 6t + 11t^2 - 7t^3 + t^4}.
\end{aligned}
$$

It is easy to see that the maximum number of MMP$(1,0,0,1)$-matches occurs when either σ ends with $1n$ or $n1$. It follows that for $n \geq 3$, the highest power of x in $Q_{n,132}^{(1,0,0,1)}(x)$ is x^{n-2} and its coefficient is $2C_{n-2}$. More generally, it is easy to see that the maximum number of MMP$(k,0,0,1)$-matches occurs when $\sigma \in S_n(132)$ ends with a shuffle of 1 with $(n-k+1)(n-k)\ldots n$. Thus, we have the following theorem.

Theorem 11. *For $n \geq k+1$, the highest power of x in $Q_{n,132}^{(k,0,0,1)}(x)$ is x^{n-k-1} and its coefficient is $(k+1)C_{n-k-1}$.*

In general, the formulas for $Q_{132}^{(k,0,0,\ell)}(x,t)$ become increasingly complex as ℓ increases. For $\ell = 1$, the formulas are reasonable. For example, using (9) and Theorem 3, we have computed that

$$
\begin{aligned}
Q_{132}^{(1,0,0,1)}(x,t) &= \frac{1 - t + 2t^2 - 2tx - 2t^2x + (1-t)\sqrt{1-4tx}}{1 - 2t + 2t^2 - 2tx + (1-2t)\sqrt{1-4tx}}, \\
Q_{132}^{(2,0,0,1)}(x,t) &= \frac{1 - 2t + 2t^2 + 2t^3 - 2tx - 2t^3x + (1-2t)\sqrt{1-4tx}}{1 - 3t + 3t^2 - 2tx + 2t^2x + (1-3t+t^2)\sqrt{1-4tx}}, \text{ and} \\
Q_{132}^{(3,0,0,1)}(x,t) &= \frac{1 - 3t + 3t^2 + 2t^4 - 2tx + 2t^2x - 2t^4x + (1-3t+t^2)\sqrt{1-4tx}}{1 - 4t + 5t^2 - 2t^3 - 2tx + 4t^2x + (1-4t+3t^2)\sqrt{1-4tx}}.
\end{aligned}
$$

We have also computed that

$$Q_{132}^{(1,0,0,1)}(t,x) = 1 + t + 2t^2 + (3+2x)t^3 + (4+6x+4x^2)t^4$$
$$+(5+12x+15x^2+10x^3)t^5 + (6+20x+36x^2+42x^3+28x^4)t^6$$
$$+(7+30x+70x^2+112x^3+126x^4+84x^5)t^7$$
$$+(8+42x+120x^2+240x^3+360x^4+396x^5+264x^6)t^8$$
$$+(9+56x+189x^2+450x^3+825x^4+1188x^5+1287x^6+858x^7)t^9+\cdots.$$

It is easy to explain some of these coefficients. That is, we have the following theorem.

Theorem 12. *We have*

(i) $Q_{n,132}^{(1,0,0,1)}(0) = n$ *for all* $n \geq 1$,

(ii) $Q_{n,132}^{(1,0,0,1)}(x)|_x = (n-1)(n-2)$ *for all* $n \geq 3$, *and*

(iii) $Q_{n,132}^{(1,0,0,1)}(x)|_{x^{n-3}} = 3C_{n-2}$ *for all* $n \geq 3$.

Proof. To see that $Q_{n,132}^{(1,0,0,1)}(0) = n$ for $n \geq 1$ note that the only permutations $\sigma \in S_n(132)$ that have no MMP$(1,0,0,1)$-matches are the identity $12\ldots n$ plus the permutations of the form $n(n-1)\ldots(n-k)12\ldots(n-k-1)$ for $k = 0,\ldots,n-1$.

For $n \geq 3$, we claim that

$$a(n) = Q_{n,132}^{(1,0,0,1)}(x)|_x = (n-1)(n-2).$$

This is easy to see by induction. That is, there are three ways to have a $\sigma \in S_n(132)$ with $\mathrm{mmp}^{(1,0,0,1)}(\sigma) = 1$. That is, σ can start with n in which case we have $a(n-1) = (n-2)(n-3)$ ways to arrange $\sigma_2\ldots\sigma_n$ or σ can start with $(n-1)n$ in which case there can be no MMP$(1,0,0,1)$ matches in $\sigma_3\ldots\sigma_n$ which means that we have $(n-2)$ choices to arrange $\sigma_3\ldots\sigma_n$ or σ can end with n in which case $\sigma_1\ldots\sigma_{n-1}$ must have exactly one MMP$(0,0,0,1)$-match so that [9, Theorems 3.1, 3.3, and 5.1], we have $n-2$ ways to arrange $\sigma_1\ldots\sigma_n$. Thus, $a(n) = (n-2)(n-3)+2(n-2) = (n-1)(n-2)$.

For $Q_{n,132}^{(1,0,0,1)}(x)|_{x^{n-3}}$, we note that

$$\begin{aligned} Q_{n,132}^{(1,0,0,1)}(x) &= Q_{n-1,132}^{(0,0,0,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,0,0,0)}(x) Q_{n-i,132}^{(1,0,0,1)}(x) \\ &= Q_{n-1,132}^{(0,0,0,1)}(x) + \sum_{i=1}^{n-1} C_{i-1} x^{i-1} Q_{n-i,132}^{(1,0,0,1)}(x). \end{aligned}$$

Thus,

$$Q_{n,132}^{(1,0,0,1)}(x)|_{x^{n-3}} = Q_{n-1,132}^{(0,0,0,1)}(x)|_{x^{n-3}} + \sum_{i=1}^{n-2} C_{i-1} Q_{n-i,132}^{(1,0,0,1)}(x)|_{x^{n-i-2}}.$$

It was proved in [9, Theorem 5.2] that $Q_{n,132}^{(0,0,0,1)}(x)|_{x^{n-2}} = C_{n-1}$ for $n \geq 2$ and, by Theorem 11,
$Q_{n,132}^{(1,0,0,1)}(x)|_{x^{n-2}} = 2C_{n-2}$ for $n \geq 2$. Thus, for $n \geq 3$,

$$\begin{aligned} Q_{n,132}^{(1,0,0,1)}(x)|_{x^{n-3}} &= C_{n-2} + \sum_{i=1}^{n-2} C_{i-1} 2C_{n-i-2} \\ &= C_{n-2} + 2\sum_{i=1}^{n-2} C_{i-1}C_{n-i-2} = C_{n-2} + 2C_{n-2} = 3C_{n-2}. \end{aligned}$$

□

One can also compute that

$$\begin{aligned} Q_{132}^{(2,0,0,1)}(t,x) &= 1 + t + 2t^2 + 5t^3 + (11+3x)t^4 + (23+13x+6x^2)t^5 \\ &+ (47+40x+30x^2+15x^3)t^6 + (95+107x+104x^2+81x^3+42x^4)t^7 \\ &+ (191+266x+308x^2+301x^3+238x^4+126x^5)t^8 \\ &+ (383+633x+837x^2+949x^3+926x^4+738x^5+396x^6)t^9 + \cdots \end{aligned}$$

and

$$\begin{aligned} Q_{132}^{(3,0,0,1)}(t,x) &= 1 + t + 2t^2 + 5t^3 + 14t^4 + (38+4x)t^5 + (101+23x+8x^2)t^6 \\ &+ (266+92x+51x^2+20x^3)t^7 + (698+320x+221x^2+135x^3+56x^4)t^8 \\ &+ (1829+1038x+821x^2+614x^3+392x^4+168x^5)t^9 + \cdots. \end{aligned}$$

Here the sequence $(Q_{n,132}^{(2,0,0,1)}(0))_{n\geq 1}$ which starts out $1, 2, 5, 11, 23, 47, 95, 191, \ldots$ is the sequence A083329 from the OEIS which counts the number of set partitions π of $\{1, \ldots, n\}$, which when written in *increasing form*, is such that the permutation flatten(π) avoids the permutations 213 and 312. For the increasing form of a set partition π, one write the parts in increasing order separated by backslashes where the parts are written so that minimal elements in the parts increase. Then flatten(π) is just the permutation that results by removing the backslashes. For example, $\pi = 13/257/468$ is written in increasing form and flatten(π) = 13257468.

Problem 2. Find a bijection between the $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(2,0,0,1)}(\sigma) = 0$ and the set partitions π of n such that flatten(π) avoid 231 and 312.

None of the sequences $\{Q_{n,132}^{(k,0,0,1)}(0)\}_{n\geq 1}$ for $k = 3, 4, 5$ appear in the OEIS.
Similarly, one can compute that

$$Q_{132}^{(k,0,0,2)}(t,x) = \frac{1 - (t+t^2)Q_{132}^{(k-1,0,0,0)}(t,x) + tQ_{132}^{(k-1,0,0,2)}(t,x) + t^2 Q_{132}^{(k-1,0,0,1)}(t,x)}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}.$$

Then one can use this formula to compute that

$$\begin{aligned}Q_{132}^{(1,0,0,2)}(t,x) &= 1+t+2t^2+5t^3+(9+5x)t^4+(14+18x+10x^2)t^5\\&+(20+42x+45x^2+25x^3)t^6+(27+80x+126x^2+126x^3+70x^4)t^7\\&+(35+135x+280x^2+392x^3+378x^4+210x^5)t^8\\&+(44+210x+540x^2+960x^3+1260x^4+1088x^5+660x^6)t^9+\cdots.\end{aligned}$$

It is easy to see that permutations $\sigma \in S_n(132)$ which have the maximum number of MMP$(1,0,0,2)$-matches in σ are those permutations that end in $n12$, $n12$, $21n$, $2n1$ or $n21$. Thus, the highest power of x that occurs in $Q_{n,132}^{(1,0,0,2)}(x)$ is x^{n-3}, which has a coefficient of $5C_{n-3}$.

Also,

$$\begin{aligned}Q_{132}^{(2,0,0,2)}(t,x) &= 1+t+2t^2+5t^3+14t^4+(33+9x)t^5+(72+42x+18x^2)t^6\\&+(151+135x+98x^2+45x^3)t^7+(310+370x+358x^2+266x^3+126x^4)t^8\\&+(629+931x+1093x^2+1047x^3+784x^4+378x^5)t^9+\cdots.\end{aligned}$$

It is easy to see that permutations $\sigma \in S_n(132)$ which have the maximum number of MMP$(2,0,0,2)$-matches in σ are those permutations that end in either a shuffle of 21 and $(n-1)n$ or $(n-1)n12$, $(n-1)12n$, and $12(n-1)n$. Thus, the highest power of x that occurs in $Q_{n,132}^{(2,0,0,2)}(x)$ is x^{n-4} for $n \geq 5$, which has a coefficient of $9C_{n-4}$.

We also have

$$\begin{aligned}Q_{132}^{(3,0,0,2)}(t,x) &= 1+t+2t^2+5t^3+14t^4+42t^5+(118+14x)t^6\\&+(319+82x+28x^2)t^7+(847+329x+184x^2+70x^3)t^8\\&+(2231+1138x+807x^2+490x^3+196x^4)t^9+\cdots.\end{aligned}$$

It is easy to see that permutations $\sigma \in S_n(132)$ which have the maximum number of MMP$(3,0,0,2)$-matches in σ are those permutations that end in either a shuffle of 21 and $(n-2)(n-1)n$ or $(n-2)(n-1)n12$, $(n-2)(n-1)12n$, $(n-2)12(n-1)n$, and $12(n-2)(n-1)n$. Thus, the highest power of x that occurs in $Q_{n,132}^{(3,0,0,2)}(x)$ is x^{n-5} for $n \geq 6$, which has a coefficient of $14C_{n-5}$. More generally, the maximum number of MMP$(k,0,0,2)$-matches in σ are those permutations that end in either a shuffle of 21 and $(n-k+1)\ldots(n-1)n$ or

$$(n-k+1)\ldots(n-1)n12,(n-k+1)\ldots(n-1)12n,\ldots,12(n-k+1)\ldots(n-1)n.$$

Thus, the highest power of x that occurs in $Q_{n,132}^{(k,0,0,2)}(x)$ is x^{n-k-2} for $n \geq k+3$, which has a coefficient of $\left(\binom{k+2}{2}+k+1\right)C_{n-k-2} = \frac{1}{2}(k+4)(k+1)C_{n-k-2}$. None of the series $\{Q_{n,132}^{(k,0,0,2)}(0)\}_{n\geq 1}$ for $k=1,2,3$ appear in the OEIS.

4. $Q_{n,132}^{(0,k,\ell,0)}(x) = Q_{n,132}^{(0,0,\ell,k)}(x)$ Where $k,\ell \geq 1$

By Lemma 1, we know that $Q_{n,132}^{(0,k,\ell,0)}(x) = Q_{n,132}^{(0,0,\ell,k)}(x)$. Thus, we will only consider $Q_{n,132}^{(0,k,\ell,0)}(x)$ in this section.

Suppose that $n \geq k$. It is clear that n can never match the pattern $\mathrm{MMP}(0,k,\ell,0)$ for $k \geq 1$ in any $\sigma \in S_n(132)$. For $i \geq k$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,\ell,0)}(x)$ to $Q_{n,132}^{(0,k,\ell,0)}(x)$ since none of the elements to the right of $A_i(\sigma)$ have any effect on whether an element of $A_i(\sigma)$ matches $\mathrm{MMP}(0,k,\ell,0)$. The presence of n and the elements of $A_i(\sigma)$ ensures that an element in $B_i(\sigma)$ matches $\mathrm{MMP}(0,k,\ell,0)$ in σ if and only if it matches $\mathrm{MMP}(0,0,\ell,0)$ in $B_i(\sigma)$. Thus, our choices for $B_i(\sigma)$ contribute a factor of $Q_{n-i,132}^{(0,0,\ell,0)}(x)$ to $Q_{n,132}^{(0,k,\ell,0)}(x)$.

Now suppose $i < k$ and $j = n-i$. In this case, $A_i(\sigma)$ consists of $i-1$ elements. In this situation, an element of $B_i(\sigma)$ matches $\mathrm{MMP}(0,k,\ell,0)$ in σ if and only if it matches $\mathrm{MMP}(0,k-i,\ell,0)$ in $B_i(\sigma)$. Thus, our choices for $B_i(\sigma)$ contribute a factor of $Q_{n-i,132}^{(0,k-i,\ell,0)}(x)$ to $Q_{n,132}^{(0,k,\ell,0)}(x)$. As before, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,\ell,0)}(x)$ to $Q_{n,132}^{(0,k,\ell,0)}(x)$ but in such a situation $Q_{i-1,132}^{(0,k,\ell,0)}(x) = C_{i-1}$.

It follows that for $n \geq k$,

$$Q_{n,132}^{(0,k,\ell,0)}(x) = \sum_{i=k}^{n} Q_{i-1,132}^{(0,k,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,0)}(x) + \sum_{j=1}^{k-1} C_{j-1} Q_{n-j,132}^{(0,k-j,\ell,0)}(x). \quad (10)$$

Multiplying both sides of (10) by t^n, summing for $n \geq k$ and observing that $Q_{j,132}^{(0,k,\ell,0)}(x) = C_j$ for $j \leq k+\ell$, we see that for $k,\ell \geq 1$,

$$\begin{aligned} Q_{132}^{(0,k,\ell,0)}(t,x) - \sum_{j=0}^{k-1} C_j t^j &= tQ_{132}^{(0,0,\ell,0)}(t,x)\left(Q_{132}^{(0,k,\ell,0)}(t,x) - \sum_{s=0}^{k-2} C_s t^s\right) \\ &\quad + t\sum_{i=0}^{k-2} C_i t^i \left(Q_{132}^{(0,k-i-1,\ell,0)}(t,x) - \sum_{s=0}^{k-i-2} C_s t^s\right). \end{aligned}$$

It follows that we have the following theorem.

Theorem 13. *For all $k,\ell \geq 1$,*

$$Q_{132}^{(0,k,\ell,0)}(t,x) = \frac{C_{k-1}t^{k-1} + \sum_{j=0}^{k-2} C_j t^j \left(1 - tQ_{132}^{(0,0,\ell,0)}(t,x) + t\left(Q_{132}^{(0,k-j-1,\ell,0)}(t,x) - \sum_{s=0}^{k-j-2} C_s t^s\right)\right)}{1 - tQ_{132}^{(0,0,\ell,0)}(t,x)}. \quad (11)$$

Since we can compute $Q_{132}^{(0,0,\ell,0)}(t,x)$ by Theorem 2, we can use (11) to compute $Q_{132}^{(0,k,\ell,0)}(t,x)$ for all $k,\ell \geq 1$.

4.1. Explicit Formulas for $Q_{n,132}^{(0,k,\ell,0)}(x)|_{x^r}$

Our first observation is that we have the following theorem.

Theorem 14. *For all $\ell \geq 0$,*

$$Q_{132}^{(1,0,\ell,0)}(t,x) = Q_{132}^{(0,1,\ell,0)}(t,x). \tag{12}$$

Proof. By Theorems 1 and 3, $Q_{132}^{(1,0,0,0)}(t,x) = Q_{132}^{(0,1,0,0)}(t,x) = \frac{1}{1-tC(xt)}$. By (2), we have that

$$Q_{n,132}^{(1,0,\ell,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,0,\ell,0)}(x)\ Q_{n-i,132}^{(1,0,\ell,0)}(x). \tag{13}$$

On the other hand, by (10), we have that

$$Q_{n,132}^{(0,1,\ell,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,0)}(x) \tag{14}$$

or, equivalently,

$$Q_{n,132}^{(0,1,\ell,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,0,\ell,0)}(x) Q_{n-i,132}^{(0,1,\ell,0)}(x). \tag{15}$$

Comparing (13) and (15), we see that we can easily prove by induction that $Q_{n,132}^{(0,1,\ell,0)}(x) = Q_{n,132}^{(1,0,\ell,0)}(x)$ for all $n \geq 0$ for any $\ell \geq 1$.

In fact, one can recursively construct a bijection $\Theta_n : S_n(132) \to S_n(132)$ such that for all $\sigma \in S_n(132)$,

$$\mathrm{mmp}^{(1,0,\ell,0)}(\sigma) = \mathrm{mmp}^{(0,1,\ell,0)}(\Theta_n(\sigma)).$$

For $n \leq 1 + \ell$, we simply let Θ_n be the identity map. Then if $n > 1 + \ell$, we inductively define Θ_n as follows. First, for any permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ and $i \geq 1$, we let $\uparrow_i (\sigma) = (\sigma_1 + i) \ldots (\sigma_n + i)$. Similarly, if $\gamma = \gamma_1 \ldots \gamma_n$ is some rearrangement of $\{i+1, \ldots, i+n\}$, then we let $\downarrow_i (\gamma) = (\gamma_1 - i) \ldots (\gamma_n - i)$.

Then if $\sigma \in S_n^{(i)}(132)$, we can write $\sigma = A_i(\sigma) n B_i(\sigma)$ as in Figure 2. Thus $B_i(\sigma) \in S_{n-i}(132)$ and $A_i(\sigma)$ is a rearrangement of $\{n-i+1, \ldots, n-1\}$ such that $\mathrm{red}(A_i(\sigma)) \in S_{i-1}(132)$. Then we let

$$\Theta_n(\sigma) = \uparrow_{n-i} (\Theta_{n-i}(B_i(\sigma)))\ n\ \downarrow_{n-i} (A_i(\sigma)). \tag{16}$$

It is then easy to check from our proofs of (13) and (14) that Θ_n is the desired bijection. $\square$

We note that it is not true that $Q_{132}^{(2,0,\ell,0)}(t,x) = Q_{132}^{(0,2,\ell,0)}(t,x)$. For example, we have computed that

$$\begin{aligned} Q_{132}^{(2,0,1,0)}(t,x) &= \frac{1-t+2x+tx-\sqrt{1+(1-x)^2t-4xt}}{1-t-2x+3tx-\sqrt{1+(1-x)^2t-4xt}} \text{ and} \\ Q_{132}^{(0,2,1,0)}(t,x) &= 1 + \frac{4tx^2}{\left(1-t+2x+tx-\sqrt{1+(1-x)^2t-4xt}\right)^2}. \end{aligned}$$

It follows from Theorem 13 and Theorem 2 that

$$\begin{aligned} Q_{132}^{(0,1,\ell,0)}(t,0) &= \frac{1}{1-tQ_{132}^{(0,0,\ell,0)}(t,0)} \\ &= \frac{1}{1-t\frac{1}{1-t(C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1})}} \\ &= \frac{1-t(C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1})}{1-t(1+C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1})}. \end{aligned}$$

Thus, one can compute that

$$\begin{aligned} Q_{132}^{(0,1,1,0)}(t,0) &= \frac{1-t}{1-2t}; \\ Q_{132}^{(0,1,2,0)}(t,0) &= \frac{1-t-t^2}{1-2t-t^2}; \\ Q_{132}^{(0,1,3,0)}(t,0) &= \frac{1-t-t^2-2t^3}{1-2t-t^2-2t^3}, \text{ and} \\ Q_{132}^{(0,1,4,0)}(t,0) &= \frac{1-t-t^2-2t^3-5t^4}{1-2t-t^2-2t^3-5t^4}. \end{aligned}$$

Similarly, one can compute

$$Q_{132}^{(0,2,\ell,0)}(t,x) = \frac{1-tQ_{132}^{(0,0,\ell,0)}(t,x)+tQ_{132}^{(0,1,\ell,0)}(t,x)}{1-tQ_{132}^{(0,0,\ell,0)}(t,x)} = 1+\frac{tQ_{132}^{(0,1,\ell,0)}(t,x)}{1-tQ_{132}^{(0,0,\ell,0)}(t,x)}.$$

Note that

$$\begin{aligned} Q_{132}^{(0,2,\ell,0)}(t,0) &= 1+\frac{t\frac{1-t(C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1})}{1-t(1+C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1})}}{1-t\frac{1}{1-t(C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1})}} \\ &= 1+\frac{t(1-t(C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1}))^2}{(1-t(1+C_0+C_1t+\cdots+C_{\ell-1}t^{\ell-1}))^2}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} Q_{132}^{(0,2,1,0)}(t,0) &= 1+t\left(\frac{1-t}{1-2t}\right)^2; \\ Q_{132}^{(0,2,2,0)}(t,0) &= 1+t\left(\frac{1-t-t^2}{1-2t-t^2}\right)^2; \\ Q_{132}^{(0,2,3,0)}(t,0) &= 1+t\left(\frac{1-t-t^2-2t^3}{1-2t-t^2-2t^3}\right)^2, \text{ and} \\ Q_{132}^{(0,2,4,0)}(t,0) &= 1+t\left(\frac{1-t-t^2-2t^3-5t^4}{1-2t-t^2-2t^3-5t^4}\right)^2. \end{aligned}$$

We can use our previous computations of $Q_{132}^{(0,0,\ell,0)}(t,x)$ and (11) to compute $Q_{132}^{(0,2,\ell,0)}(t,x)$ for all $\ell \geq 1$. For example, we have computed

$$\begin{aligned} Q_{132}^{(0,2,1,0)}(t,x) &= 1+t+2t^2+5t^3+(12+2x)t^4+(28+12x+2x^2)t^5+ \\ &+(64+48x+18x^2+2x^3)t^6+(144+160x+97x^2+26x^3+2x^4)t^7 \\ &+(320+480x+408x^2+184x^3+36x^4+2x^5)t^8 \\ &+(704+1344x+1479x^2+958x^3+327x^4+48x^5+2x^6)t^9+\cdots. \end{aligned}$$

$$\begin{aligned} Q_{132}^{(0,2,2,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+(38+4x)t^5+(102+26x+4x^2)t^6 \\ &+(271+120x+34x^2+4x^3)t^7+(714+470x+200x^2+42x^3+4x^4)t^8 \\ &+(1868+1672x+964x^2+304x^3+50x^4+4x^5)t^9+\cdots. \end{aligned}$$

$$\begin{aligned} Q_{132}^{(0,2,3,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+42t^5+(122+10x)t^6 \\ &+(351+68x+10x^2)t^7+(1006+326x+88x^2+10x^3)t^8 \\ &+(2168+1364x+512x^2+108x^3+10x^4)t^9+\cdots. \end{aligned}$$

$$\begin{aligned} Q_{132}^{(0,2,4,0)}(t,x) &= 1+t+2t^2+5t^3+14t^4+42t^5+132t^6 \\ &+(401+28x)t^7+(1206+196x+28x^2)t^8+(3618+964x+252x^2+28x^3)t^9+\cdots. \end{aligned}$$

We note that the sequence $(Q_{n,132}^{(0,2,1,0)}(0))_{n\geq 1}$ is sequence A045623 in the OEIS. The nth term of this series counts the number of 1s in all compositions of $n+1$. Moreover, using the fact that $\frac{1}{(1-2t)^2} = \sum_{n\geq 0}(n+1)2^n t^n$, it follows that for $n \geq 3$,

$$\begin{aligned} Q_{n,132}^{(0,2,1,0)}(0) &= \frac{1-2t-t^2}{(1-2t)^2}\Big|_{t^{n-1}} \\ &= n2^{n-1} - 2((n-1)2^{n-2} + (n-2)2^{n-3} \\ &= 2^{n-3}(4n-4(n-1)+n-2)) = (n+2)2^{n-3}. \end{aligned}$$

Since $2 = (2+2)2^{2-3}$, we see that we have the following theorem.

Theorem 15. *For all $n \geq 2$, $Q_{n,132}^{(0,2,1,0)}(0) = (n+2)2^{n-3}$.*

In this case, we can explicitly calculate the highest and second highest coefficients that appear in $Q_{n,132}^{(0,2,\ell,0)}(x)$ for sufficiently large n. That is, we have the following theorem.

Theorem 16. *The following hold.*

(i) For all $\ell \geq 1$ and $n \geq 3+\ell$, the highest power of x that appears in $Q_{n,132}^{(0,2,\ell,0)}(x)$ is $x^{n-2-\ell}$ which appears with a coefficient of $2C_\ell$.

(ii) For all $n \geq 5$, $Q_{n,132}^{(0,2,1,0)}(x)|_{x^{n-4}} = 6 + 2\binom{n-2}{2}$.

(iii) For all $\ell \geq 2$ and $n \geq 4+\ell$, $Q_{n,132}^{(0,2,\ell,0)}(x)|_{x^{n-3-\ell}} = 2C_{\ell+1} + 8C_\ell + 4C_\ell(n-4-\ell)$.

Proof. For (i), it is easy to see that the maximum number of MMP$(0,2,\ell,0)$-matches occurs for a $\sigma \in S_n(132)$ if σ starts with $(n-1)n$ or $n(n-1)$ followed by any permutation of $S_\ell(132)$ followed by $\ell+1, \ell+2, \ldots, n-2$ in increasing order. Thus, the highest power of x in $Q_{132}^{(0,2,\ell,0)}(t,x)$ is $x^{n-\ell-2}$ and its coefficient is $2C_\ell$.

For parts (ii) and (iii), we use the fact that

$$Q_{n,132}^{(0,2,\ell,0)}(x) = Q_{n-1,132}^{(0,1,\ell,0)}(x) + \sum_{i=2}^{n} Q_{i-1,132}^{(0,2,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,0)}(x).$$

It was proved in [9, Theorem 4.2] that for $n > \ell$, the highest power of x that occurs in $Q_{n,132}^{(0,0,\ell,0)}(x)$ is $x^{n-\ell}$ and its coefficient is C_ℓ. Moreover, it was proved in [9, Theorem 4.3] that

$$Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2} \text{ for } n \geq 2$$

and in [9, Theorem 4.2] that, for $\ell \geq 2$,

$$Q_{n,132}^{(0,0,\ell,0)}(x)|_{x^{n-1-\ell}} = C_{\ell+1} - C_\ell + 2C_\ell(n-1-\ell) \text{ for } n \geq 1+\ell.$$

It follows that for $4 \leq i \leq n-1$, the highest power of x that appears in $Q_{i-1,132}^{(0,2,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,0)}(x)$ is less than $n-\ell-3$. Thus, we have four cases to consider when computing $Q_{n,132}^{(0,2,1,0)}(x)|_{x^{n-4}}$.

`Case 1.` $Q_{n-1,132}^{(0,1,1,0)}(x)|_{x^{n-4}}$. In this case, by Theorems 6 and 14, we have that,

$$Q_{n-1,132}^{(0,1,1,0)}(x)|_{x^{n-4}} = 2 + \binom{n-2}{2} \text{ for } n \geq 5.$$

`Case 2.` $i=2$. In this case $Q_{i-1,132}^{(0,2,1,0)}(x) Q_{n-i,132}^{(0,0,1,0)}(x) = Q_{n-2,132}^{(0,0,1,0)}(x)$ and

$$Q_{n-2,132}^{(0,0,1,0)}(x)|_{x^{n-4}} = \binom{n-2}{2} \text{ for } n \geq 4.$$

`Case 3.` $i=3$. In this case $Q_{i-1,132}^{(0,2,1,0)}(x) Q_{n-i,132}^{(0,0,1,0)}(x) = 2Q_{n-3,132}^{(0,0,1,0)}(x)$ so that we obtain a contribution with $2Q_{n-3,132}^{(0,0,1,0)}(x)|_{x^{n-4}} = 2C_1 = 2$ for $n \geq 5$.

`Case 4.` $i = n$. In this case $Q^{(0,2,1,0)}_{i-1,132}(x)Q^{(0,0,1,0)}_{n-i,132}(x) = Q^{(0,2,1,0)}_{n-1,132}(x)$ so that we obtain a contribution with $Q^{(0,2,1,0)}_{n-1,132}(x)|_{x^{n-4}} = 2C_1 = 2$ for $n \geq 5$.

Thus, it follows that

$$Q^{(0,2,1,0)}_{n,132}(x)|_{x^{n-4}} = 6 + 2\binom{n-2}{2} \text{ for } n \geq 5.$$

Similarly, we have four cases to consider when computing $Q^{(0,2,\ell,0)}_{n,132}(x)|_{x^{n-3-\ell}}$ for $\ell \geq 2$.

`Case 1.` $Q^{(0,1,\ell,0)}_{n-1,132}(x)|_{x^{n-3-\ell}}$. In this case, by Theorems 6 and 14, we have that

$$Q^{(0,1,\ell,0)}_{n-1,132}(x)|_{x^{n-3-\ell}} = C_{\ell+1} + 3C_\ell + 2C_\ell(n-4-\ell) \text{ for } n \geq 4+\ell.$$

`Case 2.` $i = 2$. In this case $Q^{(0,2,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,0)}_{n-i,132}(x) = Q^{(0,0,\ell,0)}_{n-2,132}(x)$ and

$$Q^{(0,0,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}} = C_{\ell+1} - C_\ell + 2C_\ell(n-3-\ell) \text{ for } n \geq 3+\ell.$$

`Case 3.` $i = 3$. In this case $Q^{(0,2,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,0)}_{n-i,132}(x) = 2Q^{(0,0,\ell,0)}_{n-3,132}(x)$ so that we obtain a contribution with $2Q^{(0,0,\ell,0)}_{n-3,132}(x)|_{x^{n-3-\ell}} = 2C_\ell$ for $n \geq 4+\ell$.

`Case 4.` $i = n$. In this case $Q^{(0,2,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,0)}_{n-i,132}(x) = Q^{(0,2,\ell,0)}_{n-1,132}(x)$ so that we obtain a contribution with $Q^{(0,2,1,0)}_{n-1,132}(x)|_{x^{n-3-\ell}} = 2C_\ell$ for $n \geq 4+\ell$.

Thus, it follows that for $n \geq 4+\ell$,

$$\begin{aligned} Q^{(0,2,\ell,0)}_{n,132}(x)|_{x^{n-3-\ell}} &= 2C_{\ell+1} + 4C_\ell + 4C_\ell(n-3-\ell) \\ &= 2C_{\ell+1} + 8C_\ell + 4C_\ell(n-4-\ell). \end{aligned}$$

For example, when $\ell = 2$, we obtain that

$$Q^{(0,2,2,0)}_{n,132}(x)|_{x^{n-5}} = 26 + 8(n-6) \text{ for } n \geq 6$$

and, when $\ell = 3$, we obtain that

$$Q^{(0,2,3,0)}_{n,132}(x)|_{x^{n-6}} = 68 + 20(n-7) \text{ for } n \geq 7$$

which agrees with the series that we computed. □

5. $Q^{(0,k,0,\ell)}_{n,132}(x)$ Where $k, \ell \geq 1$

Suppose that $n \geq k+\ell$. It is clear that n can never match the pattern $\mathrm{MMP}(0, k, 0, \ell)$ for $k \geq 1$ in any $\sigma \in S_n(132)$. There are three cases that we have to consider when dealing with the contribution of the permutations of $S^{(i)}_n(132)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$.

`Case 1.` $i \leq k-1$. It is easy to see that as we sum over all the permutations σ in $S^{(i)}_n(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q^{(0,k,0,\ell)}_{n,132}(x)$ since no element in $A_i(\sigma)$ can match $\mathrm{MMP}(0, k, 0, \ell)$. The presence of n plus the elements in $A_i(\sigma)$ ensure that an element in $B_i(\sigma)$ matches $\mathrm{MMP}(0, k, 0, \ell)$ in σ if and only if it matches $\mathrm{MMP}(0, k-i, 0, \ell)$ in $B_i(\sigma)$. Hence our choices for $B_i(\sigma)$ contribute a factor of $Q^{(0,k-i,0,\ell)}_{n-i,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$. Thus, in this case, the elements of $S^{(i)}_n(132)$ contribute $C_{i-1}Q^{(0,k-i,0,\ell)}_{n-i,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$.

`Case 2.` $k \leq i \leq n-\ell$. Note that in this case, there are at least k elements in $A_i(\sigma) \cup \{n\}$ and at least ℓ elements in $B_i(\sigma)$. The presence of the elements in $B_i(\sigma)$ ensures that an element in $A_i(\sigma)$ matches $\mathrm{MMP}(0, k, 0, \ell)$ in σ if and only if it matches $\mathrm{MMP}(0, k, 0, 0)$ in $A_i(\sigma)$. Hence our choices for $A_i(\sigma)$ contribute a factor of $Q^{(0,k,0,0)}_{i-1,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$.

The presence of n plus the elements in $A_i(\sigma)$ ensures that an element in $B_i(\sigma)$ matches $\mathrm{MMP}(0, k, 0, \ell)$ in σ if and only if it matches $\mathrm{MMP}(0, 0, 0, \ell)$ in $B_i(\sigma)$. Thus, our choices for $B_i(\sigma)$ contribute a factor of $Q^{(0,0,0,\ell)}_{n-i,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$. Thus, in this case, the elements of $S^{(i)}_n(132)$ contribute $Q^{(0,k,0,0)}_{i-1,132}(x)Q^{(0,0,0,\ell)}_{n-i,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$.

`Case 3.` $i > n-\ell$. Let $j = n-i$ so that $j < \ell$. It is easy to see that as we sum over all the permutations σ in $S^{(i)}_n(132)$, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_j to $Q^{(0,k,0,\ell)}_{n,132}(x)$ since no element in $B_i(\sigma)$ can match $\mathrm{MMP}(0, k, 0, \ell)$. The presence of the elements in $B_i(\sigma)$ ensures that an element in $A_i(\sigma)$ matches $\mathrm{MMP}(0, k, 0, \ell)$ in σ if and only if it matches $\mathrm{MMP}(0, k, 0, \ell-j)$ in $A_i(\sigma)$. Hence our choices for $A_i(\sigma)$ contribute a factor of $Q^{(0,k,0,\ell-j)}_{n-j-1,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$. Thus, in this case, the elements of $S^{(i)}_n(132)$ contribute $C_jQ^{(0,k,0,\ell-j)}_{n-j-1,132}(x)$ to $Q^{(0,k,0,\ell)}_{n,132}(x)$.

It follows that for $n \geq k+\ell$,

$$Q^{(0,k,0,\ell)}_{n,132}(x) = \sum_{i=1}^{k-1} C_{i-1}Q^{(0,k-i,0,\ell)}_{n-i,132}(x) + \sum_{i=k}^{n-\ell} Q^{(0,k,0,0)}_{i-1,132}(x)Q^{(0,0,0,\ell)}_{n-i,132}(x) + \sum_{j=0}^{\ell-1} C_jQ^{(0,k,0,\ell-j)}_{n-j-1,132}(x). \quad (17)$$

Multiplying both sides of (17) by t^n and summing, we see that

$$Q_{132}^{(0,k,0,\ell)}(t,x) - \sum_{j=0}^{k+\ell-1} C_j t^j = t\left(\sum_{j=0}^{k-2} C_j t^j \left(Q_{132}^{(0,k-j-1,0,\ell)}(t,x) - \sum_{s=0}^{k+\ell-j-2} C_s t^s\right)\right)$$
$$+ t\left(Q_{132}^{(0,k,0,0)}(t,x) - \sum_{u=0}^{k-2} C_u t^u\right)\left(Q_{132}^{(0,0,0,\ell)}(t,x) - \sum_{v=0}^{\ell-1} C_v t^v\right)$$
$$+ t\left(\sum_{j=0}^{\ell-1} C_j t^j \left(Q_{132}^{(0,k,0,\ell-j)}(t,x) - \sum_{s=0}^{k+\ell-j-2} C_s t^s\right)\right).$$

Thus

$$\begin{aligned} &Q_{132}^{(0,k,0,\ell)}(t,x) = \\ &\quad \sum_{j=0}^{k+\ell-1} C_j t^j + t\left(\sum_{j=0}^{k-2} C_j t^j \left(Q_{132}^{(0,k-j-1,0,\ell)}(t,x) - \sum_{s=0}^{k+\ell-j-2} C_s t^s\right)\right) \\ &\quad + t\left(Q_{132}^{(0,k,0,0)}(t,x) - \sum_{u=0}^{k-2} C_u t^u\right)\left(Q_{132}^{(0,0,0,\ell)}(t,x) - \sum_{v=0}^{\ell-1} C_v t^v\right) \\ &\quad + t\left(\sum_{j=0}^{\ell-1} C_j t^j \left(Q_{132}^{(0,k,0,\ell-j)}(t,x) - \sum_{w=0}^{k+\ell-j-2} C_w t^w\right)\right). \end{aligned} \tag{18}$$

Note the first term of the last term on the right-hand side of (18) is $t(Q_{132}^{(0,k,0,\ell)}(t,x) - \sum_{w=0}^{k+\ell-2} C_w t^w)$ so that we can bring the term $tQ_{132}^{(0,k,0,\ell)}(t,x)$ to the other side and solve $Q_{132}^{(0,k,0,\ell)}(t,x)$ to obtain the following theorem.

Theorem 17. *For all $k, \ell \geq 1$,*

$$Q_{132}^{(0,k,0,\ell)}(t,x) = \frac{\Phi_{k,\ell}(t,x)}{1-t} \tag{19}$$

where

$$\begin{aligned} \Phi_{k,\ell}(t,x) = &\sum_{j=0}^{k+\ell-1} C_j t^j - \sum_{j=0}^{k+\ell-2} C_j t^{j+1} \\ &+ t\left(\sum_{j=0}^{k-2} C_j t^j \left(Q_{132}^{(0,k-j-1,0,\ell)}(t,x) - \sum_{s=0}^{k-\ell-j-2} C_s t^s\right)\right) \\ &+ t\left(Q_{132}^{(0,k,0,0)}(t,x) - \sum_{u=0}^{k-2} C_u t^u\right)\left(Q_{132}^{(0,0,0,\ell)}(t,x) - \sum_{v=0}^{\ell-1} C_v t^v\right) \\ &+ t\left(\sum_{j=1}^{\ell-1} C_j t^j \left(Q_{132}^{(0,k,0,\ell-j)}(t,x) - \sum_{w=0}^{k+\ell-j-2} C_w t^w\right)\right). \end{aligned}$$

Note that we can compute $Q_{132}^{(0,k,0,0)}(t,x)$ and $Q_{132}^{(0,0,0,\ell)}(t,x)$ by Theorem 3 so that we can use (19) to compute $Q_{132}^{(0,k,0,\ell)}(t,x)$ for all $k, \ell \geq 1$.

5.1. Explicit Formulas for $Q_{n,132}^{(0,k,0,\ell)}(x)|_{x^r}$

It follows from Theorem 17 that

$$Q_{132}^{(0,1,0,1)}(t,x) = \frac{1 + tQ_{132}^{(0,1,0,0)}(t,x)(Q_{132}^{(0,0,0,1)}(t,x) - 1)}{1-t},$$

and

$$Q_{132}^{(0,2,0,1)}(t,x) = \frac{1 + tQ_{132}^{(0,1,0,1)}(t,x) + tQ_{132}^{(0,2,0,0)}(t,x)Q_{132}^{(0,0,0,1)}(t,x) - tQ_{132}^{(0,2,0,0)}(t,x) - tQ_{132}^{(0,0,0,1)}(t,x)}{1-t}. \tag{20}$$

Similarly, using the fact that

$$Q_{132}^{(0,2,0,0)}(t,x) = Q_{132}^{(0,0,0,2)}(t,x) \text{ and } Q_{132}^{(0,2,0,1)}(t,x) = Q_{132}^{(0,1,0,2)}(t,x),$$

one can show that

$$Q_{132}^{(0,2,0,2)}(t,x) = \frac{1 + (t+t^2)Q_{132}^{(0,2,0,1)}(t,x) + t(Q_{132}^{(0,2,0,0)}(t,x))^2 - (2t+t^2)Q_{132}^{(0,2,0,0)}(t,x)}{1-t}. \tag{21}$$

One can then compute that

$$\begin{aligned}
Q_{132}^{(0,1,0,1)}(t,x) &= \frac{R^{(0,1,0,1)}(t,x) + S^{(0,1,0,1)}(t,x)\sqrt{1-4tx}}{(1-t)(1-2t+\sqrt{1-4tx})^2}, \\
Q_{132}^{(0,2,0,1)}(t,x) &= \frac{R^{(0,2,0,1)}(t,x) + S^{(0,2,0,1)}(t,x)\sqrt{1-4tx}}{(1-t)^2(1-2t+\sqrt{1-4tx})^3}, \text{ and} \\
Q_{132}^{(0,2,0,2)}(t,x) &= \frac{R^{(0,2,0,2)}(t,x) + S^{(0,2,0,2)}(t,x)\sqrt{1-4tx}}{(1-t)^3(1-2t+\sqrt{1-4tx})^4}.
\end{aligned}$$

where

$$\begin{aligned}
R^{(0,1,0,1)}(t,x) &= 2 - 4t + 6t^2, \\
S^{(0,1,0,1)}(t,x) &= 2 - 4t + 2t^2, \\
R^{(0,2,0,1)}(t,x) &= 4 - 16t + 28t^2 - 24t^3 + 12t^4 - 8t^5 - 12tx + 36t^2x - 36t^3x \\
&\quad +8t^4x + 8t^5x \\
S^{(0,2,0,1)}(t,x) &= 4 - 16t + 28t^2 - 16t^3 + 4t^4 - 4tx + 4t^2x - 4t^3x, \\
R^{(0,2,0,2)}(t,x) &= 8 - 48t + 128t^2 - 184t^3 + 176t^4 - 104t^5 + 16t^6 + 24t^7 \\
&\quad -32tx + 160t^2x - 352t^3x + 352t^4x - 192t^5x + 40t^6x \\
&\quad -24t^7x + 16t^2x^2 - 32t^3x^2 + 32t^4x^2 + 16t^5x^2, \\
S^{(0,2,0,2)}(t,x) &= 8 - 48t + 128t^2 - 184t^3 + 160t^4 - 72t^5 \\
&\quad -16t^6 + 8t^7 - 16tx + 64t^2x - 96t^3x + 48t^4x + 24t^6x - 8t^7x.
\end{aligned}$$

Here are the first few terms of these series.

$$\begin{aligned}
Q_{132}^{(0,1,0,1)}(t,x) &= 1+t+2t^2+(4+x)t^3+(7+5x+2x^2)t^4+(11+14x+12x^2+5x^3)t^5\\
&+(16+30x+39x^2+33x^3+14x^4)t^6+(22+55x+95x^2+117x^3+98x^4+42x^5)t^7\\
&+(29+91x+195x^2+309x^3+36x^4+306x^5+132x^6)t^8\\
&+(37+140x+357x^2+684x^3+1028x^4+1197x^5+990x^6+429x^7)t^9+\cdots.
\end{aligned}$$

$$\begin{aligned}
Q_{132}^{(0,2,0,1)}(t,x) &= 1+t+2t^2+5t^3+(12+2x)t^4+(25+13x+4x^2)t^5\\
&+(46+45x+31x^2+10x^3)t^6+(77+115x+124x^2+85x^3+28x^4)t^7\\
&+(120+245x+359x^2+370x^3+252x^4+84x^5)t^8\\
&+(177+462x+854x^2+1159x^3+1160x^4+786x^5+264x^6)t^9+\cdots.
\end{aligned}$$

$$\begin{aligned}
Q_{132}^{(0,2,0,2)}(t,x) &= 1+t+2t^2+5t^3+14t^4+(38+4x)t^5+(91+33x+8x^2)t^6\\
&+(192+139x+78x^2+20x^3)t^7+(365+419x+377x^2+213x^3+56x^4)t^8\\
&+(639+1029x+1280x^2+1116x^3+630x^4+168x^5)t^9+\cdots.
\end{aligned}$$

It is easy to find the coefficients of the highest power of x in $Q_{n,132}^{(0,k,0,\ell)}(x)$. That is, we have the following theorem.

Theorem 18. *For $n \geq k+\ell+1$, the highest power of x that occurs in $Q_{n,132}^{(0,k,0,\ell)}(x)$ is $x^{n-k-\ell}$ which occurs with a coefficient of $C_kC_\ell C_{n-k-\ell}$.*

Proof. It is easy to see that the maximum number of $\text{MMP}(0,k,0,\ell)$-matches occurs for a $\sigma \in S_n(132)$ if σ starts with some 132-avoiding rearrangement of $n, n-1, \ldots, n-k+1$ and ends with some 132-avoiding rearrangement of $1,2,\ldots,\ell$. In the middle of such a permutation, we can choose any 132-avoiding permutation of $\ell+1,\ldots,n-k$. It follows that the highest power of x which occurs in $Q_{n,132}^{(0,k,0,\ell)}(x)$ is $x^{n-k-\ell}$ which occurs with a coefficient of $C_kC_\ell C_{n-k-\ell}$. ☐

We can also find an explicit formula for a coefficient of the second highest power of x that occurs in $Q_{n,132}^{(0,1,0,1)}(x)$.

Theorem 19. *For $n \geq 4$,*

$$Q_{n,132}^{(0,1,0,1)}(x)|_{n-3} = 2C_{n-2}+C_{n-3}.$$

Proof. In this case, for $n \geq 3$,

$$Q_{n,132}^{(0,1,0,1)}(x) = Q_{n-1,132}^{(0,1,0,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,0,0)}(x) Q_{n-i,132}^{(0,0,0,1)}(x). \tag{22}$$

We have already observed that $Q_{n,132}^{(1,0,0,0)}(x) = Q_{n,132}^{(0,1,0,0)}(x)$ so that for all $n \geq 0$, $Q_{n,132}^{(1,0,0,0)}(x) = Q_{n,132}^{(0,1,0,0)}(x) = Q_{n,132}^{(0,0,0,1)}(x)$. In addition, we proved in [9, Theorem

3.3] that for $n \geq 1$, the highest power of x that occurs in $Q^{(1,0,0,0)}_{n,132}(x)$ is x^{n-1} and $Q^{(1,0,0,0)}_{n,132}(x)|_{x^{n-1}} = C_{n-1}$ and that for $n \geq 2$, $Q^{(1,0,0,0)}_{n,132}(x)|_{x^{n-2}} = C_{n-1}$. It follows that for $n \geq 4$,

$$\begin{aligned} Q^{(0,1,0,1)}_{n,132}(x)|_{x^{n-3}} &= Q^{(0,1,0,1)}_{n-1,132}(x)|_{x^{n-3}} + Q^{(0,0,0,1)}_{n-2,132}(x)|_{x^{n-3}} + \\ &\quad \sum_{i=2}^{n-1} Q^{(0,1,0,0)}_{i-1,132}(x)|_{x^{i-2}} Q^{(0,0,0,1)}_{n-i,132}(x)|_{x^{n-i-1}} \\ &= C_{n-3} + C_{n-2} + \sum_{i-2}^{n-1} C_{i-2}C_{n-i-1} \\ &= C_{n-3} + C_{n-2} + C_{n-2} = 2C_{n-2} + C_{n-3}. \end{aligned}$$

□

We can also get explicit formulas for $Q^{(0,1,0,1)}_{132}(t,0)$, $Q^{(0,2,0,1)}_{132}(t,0)$, and $Q^{(0,2,0,2)}_{132}(t,0)$ based on the fact that we know that

$$\begin{aligned} Q^{(0,1,0,0)}_{132}(t,0) &= Q^{(0,0,0,1)}_{132}(t,0) = \frac{1}{1-t} \text{ and} \\ Q^{(0,2,0,0)}_{132}(t,0) &= Q^{(0,0,0,2)}_{132}(t,0) = \frac{1-t+t^2}{(1-t)^2}. \end{aligned}$$

Then one can use the above formulas to compute that

$$\begin{aligned} Q^{(0,1,0,1)}_{132}(t,0) &= \frac{1-2t+2t^2}{(1-t)^3}; \\ Q^{(0,2,0,1)}_{132}(t,0) &= \frac{1-3t+4t^2-t^3+t^4}{(1-t)^4}, \text{ and} \\ Q^{(0,2,0,2)}_{132}(t,0) &= \frac{1-4t+7t^2-5t^3+4t^4+2t^5}{(1-t)^5}. \end{aligned}$$

These generating functions allows us to prove the following results.

Theorem 20.

$$\begin{aligned} Q^{(0,1,0,1)}_{n,132}(0) &= 1 + \binom{n}{2} \text{ for } n \geq 2, \\ Q^{(0,2,0,1)}_{n,132}(0) &= \frac{n^2-3n+5}{3} \text{ for } n \geq 3, \text{ and} \\ Q^{(0,2,0,2)}_{n,132}(0) &= \frac{5n^4-34n^4+103n^2-122n+72}{24} \text{ for } n \geq 4. \end{aligned}$$

Proof. Note that for any $k \geq 1$, $\frac{1}{(1-t)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} t^n$ by Newton's binomial theorem. Thus for $n \geq 2$,

$$Q^{(0,1,0,1)}_{n,132}(0) = \binom{n+2}{2} - 2\binom{n+1}{2} + 2\binom{n}{2}.$$

For $n \geq 3$,

$$Q_{n,132}^{(0,2,0,1)}(0) = \binom{n+3}{3} - 3\binom{n+2}{3} + 4\binom{n+1}{3} - \binom{n}{3} + \binom{n-1}{3}.$$

For $n \geq 4$,

$$Q_{n,132}^{(0,2,0,2)}(0) = \binom{n+4}{4} - 4\binom{n+3}{4} + 7\binom{n+2}{4} - 5\binom{n+1}{4} + 4\binom{n}{4} + 2\binom{n-1}{4}.$$

One can then use Mathematica to simplify these formulas to obtain the results stated in theorem. $\square$

The fact that $Q_{n,132}^{(0,1,0,1)}(0) = 1 + \binom{n}{2}$ for $n \geq 2$ is a known fact [4, Table 6.1] since avoidance of the pattern MMP$(0,1,0,1)$ is equivalent to avoiding the (classical) pattern 321 so that we are dealing with avoidance of 132 and 321.

The sequence $\{Q_{n,132}^{(0,2,0,1)}(0)\}_{n\geq 1}$ is A116731 in the OEIS counting the number of permutations of length n which avoid the patterns 321, 2143, and 3142. Thus we get an alternative combinatorial interpretation of this sequence which is number of permutations that avoid 132, 3421, 4321.

Problem 3. Find a combinatorial explanation of the fact that in S_n, the number of $(132, 4321, 3421)$-avoiding permutations is the same as the number of $(321, 2143, 3142)$-avoiding permutations.

Similarly, the sequence $\{Q_{n,132}^{(0,2,0,2)}(0)\}_{n\geq 1}$ is the number of permutations of length n which avoid the patterns 132, 54312, 45312, 45321, and 54321.

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