DISTRIBUTION OF EIGENVALUES OF WEIGHTED, STRUCTURED MATRIX ENSEMBLES

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Abstract

Given a structured random matrix ensemble where each random variable occurs $o(N)$ times in each row and the limiting rescaled spectral measure $\widetilde{\mu}$ exists, we fix a $p \in [1/2, 1]$ and study the ensemble of signed structured matrices by multiplying the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ entries of a matrix by a randomly chosen $\epsilon_{ij} \in \{1, -1\}$, with $\text{Prob}(\epsilon_{ij} = 1) = p$ (i.e., the Hadamard product). For $p = 1/2$ the limiting signed rescaled spectral measure is the semi-circle; for other p it has bounded (resp., unbounded) support if $\widetilde{\mu}$ has bounded (resp., unbounded) support, and converges to $\widetilde{\mu}$ as $p \to 1$. The proofs are by Markov's Method of Moments, and involve the pairings of $2k$ vertices on a circle. The contribution of each pairing in the signed case is weighted by a factor depending on p and the number of vertices involved in at least one crossing. These numbers are of interest in their own right, appearing in problems in combinatorics and knot theory. The number of configurations with no vertices involved in a crossing is well-studied (the Catalan numbers). We discover and prove similar formulas for other configurations.

1. Introduction

1.1. Background

Although Random Matrix Theory began with statistical investigations by Wishart, the work of Wigner, Dyson, and others made its power and universality apparent. Wigner observed that ensembles of matrices with randomly chosen entries accurately model many nuclear phenomena. These ensembles have also emerged outside of physics, in subjects ranging from number theory [23] to random graphs [17, 30] to bus routes in Mexico [1, 25].

The study of structured sub-ensembles of real symmetric matrices, especially the density of their normalized eigenvalues, is of particular interest. Wigner's famous Semicircle Law states that the limiting spectral measure of the normalized eigenvalues of real symmetric matrices with independent entries (drawn from distributions with mean 0, variance 1, and finite higher moments) converges almost surely to the semi-circle density as the size of the matrix tends to infinity; however, very different behavior emerges for many subfamilies of structured matrices. Examples include band matrices, circulant matrices, random abelian G-circulant matrices, adjacency matrices associated to d-regular graphs, and Hankel and Toeplitz matrices, among others [3, 2, 4, 6, 7, 8, 9, 10, 13, 15, 16, 18, 24, 26, 27, 28, 29, 32]. The Toeplitz [10, 15] and singly palindromic Toeplitz ensembles [27] are particularly interesting. We concentrate on generalizing these ensembles for definiteness and ease of exposition, noting that similar arguments apply for other structured ensembles.

A real symmetric matrix is Toeplitz if it is constant along its diagonals. It is palindromic Toeplitz if, in addition, the first row is a palindrome. The limiting rescaled spectral measure for the palindromic Toeplitz ensemble is a Gaussian and the limiting rescaled spectral measure for the full Toeplitz ensemble is almost a Gaussian (the first three moments agree with the moments of a standard Gaussian, but the fourth moment is $2\frac{2}{3}$, slightly smaller than 3, the fourth moment of a standard Gaussian.) Our focus will be on other related ensembles. Before stating our results, we first quickly review some standard notation; below $\mathfrak{c}$ and r are constants depending on the system.

- A *random matrix ensemble* is a collection of $N \times N$ real symmetric matrices whose independent entries are drawn from iidrvs whose density $\mathfrak{p}$ has mean 0, variance 1 and finite higher moments. More generally, we could consider matrices that are functions of some collection of iidrvs. The associated probability measure is $\mathrm{Prob}(A)dA = \prod_{(i,j)\in\mathcal{I}_N} \mathfrak{p}(a_{ij})da_{ij}$, where $\mathcal{I}_N$ is a complete set of indices corresponding to the independent entries of the matrices, i.e., the full set of random variables underlying the ensemble.[2]

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[2] For real symmetric Toeplitz matrices $a_{ij} = a_{k\ell}$ if $|i - j| = |k - \ell|$, and thus $\mathcal{I}_N = \{a_{11}, a_{12}, \ldots, a_{1N}\}$.

- The *empirical spectral measure* of an $N \times N$ real symmetric matrix A is

$$\mu_A(x) := \frac{1}{N} \sum_{k=1}^{N} \delta(x - \lambda_k(A)), \qquad (1.1)$$

with $\delta(x)$ the Dirac delta functional and the $\lambda_k(A)$'s are the eigenvalues of A. The *rescaled empirical spectral measure* is

$$\widetilde{\mu}_A(x) := \frac{1}{N} \sum_{k=1}^{N} \delta\left(x - \frac{\lambda_k(A)}{\mathfrak{c}N^r}\right) = \mu_{A/\mathfrak{c}N^r}(x), \qquad (1.2)$$

and the *normalized empirical spectral distribution* F^{A_N} is defined by

$$F^{A_N}(x) = \frac{1}{N} \# \left\{ k \le N : \frac{\lambda_k(A)}{\mathfrak{c}N^r} \le x \right\}. \qquad (1.3)$$

- Given real symmetric $N \times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, their *Hadamard product* $A \circ B$ is the matrix whose $(i,j)^{\text{th}}$ entry is $a_{ij}b_{ij}$; its empirical spectral measure is $\mu_{A \circ B}(x)$.

- If the limit of the sequence of average moments of an ensemble exists and uniquely determines a measure, that measure is called the *limiting spectral measure of the ensemble*.

- Let $p \in [1/2, 1]$ and consider the random matrix ensemble of real symmetric matrices $\mathcal{E} = (\epsilon_{ij})$ with independent entries iidrvs that are 1 with probability p and -1 with probability $1 - p$. Given a random matrix ensemble with matrices A, consider the signed random matrix ensemble with matrices $A \circ \mathcal{E}$. The ensemble has measure

$$\left(\prod_{i \le j} p^{(1+\epsilon_{ij})/2}(1-p)^{(1-\epsilon_{ij})/2}\right) \text{Prob}(A)dA. \qquad (1.4)$$

We rescale the eigenvalues of the Hadamard product by the same factor used for the unsigned matrices, obtaining the *expected empirical spectral measure* $\widetilde{\mu}_{A \circ \mathcal{E}}(x) = \mu_{(A/\mathfrak{c}N^r) \circ \mathcal{E}}(x)$. The average k^{th} moment is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \le i \le j \le N} \sum_{\epsilon_{ij} \in \{-1,1\}} \int_{x=-\infty}^{\infty} x^k \widetilde{\mu}_{A \circ \mathcal{E}}(x)$$
$$p^{(1+\epsilon_{ij})/2}(1-p)^{(1-\epsilon_{ij})/2}\text{Prob}(A)dx \ dA. \qquad (1.5)$$

Definition 1. *Let* $\mathfrak{p}$ *be a density with mean 0, variance 1 and finite higher moments. For fixed* n*, an* $N \times N$ *real symmetric Toeplitz matrix* A *is* (degree n) *highly palindromic if the first row is* 2^n *copies of a palindrome, where the independent entries of the matrices are independently drawn with density* $\mathfrak{p}$*. If* $n = 0$ *we say* A *is singly palindromic. If* a_{ij} *is the entry in the* i^{th} *row and* j^{th} *column of* A*, then we set* $b_{|i-j|} = a_{ij}$*.*

Remark 1. *We assumed in the above definition that* N *is a multiple of* 2^n *so that each element occurs exactly* 2^{n+1} *times in the first row. We also often omit 'real symmetric' as that is understood. Entries are constant along diagonals, and entries on two diagonals* $N/2^n$ *diagonals apart from each other, or symmetric within a palindrome, are equal. If the ensemble is at least doubly palindromic, then the* b*'s are not distinct and satisfy additional relations due to the palindromicity.*

1.2. Results

The Eigenvalue Trace Lemma implies the kth moment of $\widetilde{\mu}_A$ is

$$M_{k;N}(A) \;=\; \int_{-\infty}^{\infty} x^k \widetilde{\mu}_A(x) dx \;=\; \frac{\text{Trace}(A^k)}{\mathfrak{c}^k N^{rk+1}}. \qquad (1.6)$$

In §2, we show that expanding $\text{Trace}(A^k)$ as a sum of products of k-tuples of entries of A, a standard degree of freedom argument shows that in the limit, the sum is unchanged when we restrict to products consisting of "matched pairs." (All other terms taken together contribute zero in the limit.) The difficulty is figuring out the contribution of each of these pairings, and the answer greatly depends on the structure of the matrix. The odd moments trivially vanish, and for even moments the only contribution in the limit comes when the indices are matched in pairs with opposite orientation. We may view these terms as pairings of $2k$ vertices, $\{1, 2, \ldots, 2k\}$, on a circle.

We determine the contribution of these pairings in §3. We focus on Toeplitz and related ensembles for ease of presentation. Similar general results hold in other cases, although connections between Toeplitz ensembles and knot theory allow us to provide closed form results for the ensembles that we study. We show that the depression of the contribution of a pairing c compared to the unsigned Toeplitz and singly palindromic Toeplitz ensembles depends only on $e(c)$, where $e(c)$ is the number of vertices in crossing pairs in the pairing. When $p = 1/2$, as in the real symmetric case, the limiting rescaled spectral measure is the semi-circle distribution. All crossing configurations contribute 0, and all non-crossing configurations contribute 1. Thus the $2k^{\text{th}}$ moment equals the kth Catalan number, which is both the number of non-crossing pairings of $2k$ objects and the $2k^{\text{th}}$ moment of the semicircle density.[3] By contrast, when $p = 1$ the contribution of each pairing is the same

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[3] The normalized semi-circular density is $f_{\text{sc}}(x) = \frac{1}{\pi} \sqrt{1 - \left(\frac{x}{2}\right)^2}$ if $|x| \leq 2$ and 0 otherwise, and

as in the unsigned case. Intermediate p interpolate between these extremes. Moreover, any distribution that had unbounded or bounded support before weighting still has unbounded or bounded, respectively, support after weighting.

Before stating the theorem below we quickly define the almost sure convergence, and comment on one of the assumptions. Our outcome space is $\Omega_\mathbb{N} = \{b_0, b_1, \dots\}$, where the b_i's are iidrv with density $\mathfrak{p}$, and if $\omega = (\omega_0, \omega_1, \dots) \in \Omega_\mathbb{N}$ then $\mathrm{Prob}(\omega_i \in [\alpha_i, \beta_i]) = \int_{\alpha_i}^{\beta_i} \mathfrak{p}(x_i) dx_i$. We denote elements of $\Omega_\mathbb{N}$ by A to emphasize the correspondence with matrices, and we set A_N to be the structured real symmetric matrix obtained by truncating $A = (b_0, b_1, \dots)$ to give us the requisite number of independent entries to generate an $N \times N$ matrix in our ensemble. We denote the probability space by $(\Omega_\mathbb{N}, \mathcal{F}_\mathbb{N}, \mathbb{P}_\mathbb{N})$. To each integer $m \geq 0$ we define the random variable $X_{m;N}$ on $\Omega_\mathbb{N}$ by $X_{m;N}(A) = \int_{-\infty}^{\infty} x^m dF^{A_N}(x)$; note this is the mth moment of the measure μ_{A_N}. For each m, $X_{m;N} \to X_m$ almost surely if $\mathbb{P}_\mathbb{N}(\{A \in \Omega_\mathbb{N} : X_{m;N}(A) \to X_m(A) \text{ as } N \to \infty\}) = 1$.

If independent random variables in our ensemble occurred order N times then degenerate behavior could happen. Restricting to the case where each occurs $o(N)$ times precludes some highly structured matrix ensembles, such as block matrices that are a fixed scalar times the matrix of all 1's, or the "right angle" family where $a_{ij} = b_{\min(i,j)}$.

Theorem 1. *Consider any ensemble of $N \times N$ real-symmetric structured matrices, where the independent entries are drawn from a distribution $\mathfrak{p}$ with mean 0, variance 1 and finite higher moments. We assume the following about our random matrix ensemble.*

1. *As $N \to \infty$ the associated rescaled empirical spectral measures converge to a measure, $\widetilde{\mu}$.*

2. *Each of the independent random variables occurs $o(N)$ times in each row of the matrices for this ensemble.*

Fix a $p \in [1/2, 1]$ and consider the Hadamard product of our ensemble and real symmetric signed matrices (ϵ_{ij}), where the entries are independently chosen from $\{-1, 1\}$ with $\mathrm{Prob}(\epsilon_{ij} = 1) = p$.

If $p = 1/2$ the expected empirical spectral measures converge to the semi-circle, while for all other p the limiting spectral measure has bounded (resp. unbounded) support if the original ensemble's limiting rescaled spectral measure has bounded (resp. unbounded) support. If the density $\mathfrak{p}$ is an even function then the measures converge almost surely.

Remark 2. *Theorem 1 holds for real symmetric Toeplitz and singly palindromic Toeplitz matrices. See Theorem 3 for an explicit, closed form expression for the depression of the moments of these ensembles as $p \to 1/2$.*

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the even moments are the Catalan numbers.

The controlling factor in the real symmetric Toeplitz and singly palindromic Toeplitz cases (and in a limited manner the highly palindromic Toeplitz cases) lurking in Remark 2 is the number of vertices involved in a crossing; we make this precise in §3. Our problem thus reduces to one in combinatorics, which turns out to be related to knot theory. This connection provides additional motivation for and applications of this work; see for example [11, 19, 21, 22, 12, 31, 33]). In the course of our investigations, we prove several interesting combinatorial results. Many of the coefficients have been previously tabulated on the OEIS; see for example Remark 5. We isolate several below.

Consider all $(2k-1)!!$ pairings of $2k$ vertices on a circle. Let $\mathrm{Cr}_{2k,2m}$ denote the number of these pairings where exactly $2m$ vertices are involved in a crossing, and let C_k denote the kth Catalan number, $\frac{1}{k+1}\binom{2k}{k}$. We obtain exact formulas for $\mathrm{Cr}_{2k,2m}$ in some cases, and for large k we prove the limiting behavior of the expected value and variance of the number of vertices involved in at least one crossing.

Theorem 2. *Notation as above, we have:*

- *For $m \leq 10$ we have $\mathrm{Cr}_{2k,0} = C_k$, $\mathrm{Cr}_{2k,2} = 0$, $\mathrm{Cr}_{2k,4} = \binom{2k}{k-2}$, $\mathrm{Cr}_{2k,6} = 4\binom{2k}{k-3}$, $\mathrm{Cr}_{2k,8} = 31\binom{2k}{k-4} + \sum_{d=1}^{k-4}\binom{2k}{k-4-d}(4+d)$, and $\mathrm{Cr}_{2k,10} = 288\binom{2k}{k-5} + 8\sum_{d=1}^{k-5}\binom{2k}{k-5-d}(5+d)$.*

- *Taking all pairings to be equally likely, the expected number of vertices in crossings is*

$$\frac{2k}{2k-1}\left(2k-2-\frac{{}_2F_1(1,\frac{3}{2},\frac{5}{2}-k;-1)}{2k-3}-(2k-1)\,{}_2F_1(1,\frac{1}{2}+k,\frac{3}{2};-1)\right),$$
$$(1.7)$$

which is $2k-2-\frac{2}{k}+O\left(\frac{1}{k^2}\right)$ as $k \to \infty$; here ${}_2F_1$ is the hypergeometric function. Further, the variance of the number of vertices involved in a crossing converges to 4.

We prove our results on the limiting measure via Markov's Method of Moments (see [5, 34]) by showing the average moments converge to the moments of a distribution. Controlling the variance and the rate of convergence via a counting argument and applying Chebyshev's inequality and the Borel-Cantelli lemma completes the analysis.

2. Moment Preliminaries

For ease of exposition we consider (real symmetric) Toeplitz ensembles below, though simple modifications of our arguments would yield similar results for other ensembles. We take $(\mathfrak{c}, r)$ to be $(1, 1/2)$. We summarize the needed expansions from

previous work. For a fixed $N \times N$ matrix A drawn from a Toeplitz ensemble, the kth moment of its rescaled empirical spectral measure is

$$M_{k,N}(A) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1, \ldots, i_k \leq N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}. \tag{2.8}$$

For the signed Toeplitz and palindromic Toeplitz ensembles this is

$$M_{k,N}(A) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1, \ldots, i_k \leq N} \epsilon_{i_1 i_2} b_{|i_1-i_2|} \epsilon_{i_2 i_3} b_{|i_2-i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k-i_1|}. \tag{2.9}$$

We define the expected moment as:

$$\begin{aligned} M_k &:= \lim_{N \to \infty} \mathbb{E}\left(M_{k,N}(A)\right) \\ &= \lim_{N \to \infty} \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1, \ldots, i_k \leq N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1-i_2|} \epsilon_{i_2 i_3} b_{|i_2-i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k-i_1|}\right). \end{aligned} \tag{2.10}$$

For c a partition of $\{1, 2, \ldots, k\}$, let S_c be the set of permutations $(i_1, \ldots, i_k) \in \{1, 2, \ldots, N\}^k$ such that $b_{|i_j-i_{j+1}|} = b_{|i_\ell-i_{\ell+1}|}$ if and only if j and ℓ are in the same block of c.

Set

$$M_{k,c} := \lim_{N \to \infty} \frac{1}{N^{\frac{k}{2}+1}} \sum_{(i_1, \ldots, i_k) \in S_c} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1-i_2|} \epsilon_{i_2 i_3} b_{|i_2-i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k-i_1|}\right). \tag{2.11}$$

Taking $\mathcal{P}(k)$ to be the set of partitions of $\{1, \ldots, k\}$, $M_k = \sum_{c \in \mathcal{P}(k)} M_{k,c}$. We call $M_{k,c}$ the *contribution* of the partition c to M_k. Let $S_c' \subset S_c$ be the subset such that for $j \neq \ell$ in the same block of c, $i_j - i_{j+1} = -(i_\ell - i_{\ell+1}) + \mathcal{C}_{r_j \ell}$, where $\mathcal{C}_{r_j \ell}$ is belongs to a well-specified set of size depending on the level of palindromicity of the ensemble and the moment being computed, but independent of N. Define $M_{k,c}'$ analogously to $M_{k,c}$ with S_c' replacing S_c.

The following lemma shows that most partitions contribute zero to M_k as $N \to \infty$.

Lemma 1. *Let k be an integer and consider any Toeplitz ensemble. If $M_{k,c} \neq 0$, then c is a pairing, i.e., c consists of $k/2$ blocks of size 2. For these terms, b_α appears in the product exactly twice. The odd moments of the limiting rescaled spectral measure vanish. To compute $M_{2k,c}$, for c a pairing, it suffices to sum over terms with b's matched in exactly pairs with a minus sign in each of the k equations of the form*

$$i_j - i_{j+1} = \pm(i_\ell - i_{\ell+1}). \tag{2.12}$$

See $[15, 16]^4$ for a proof. We can rephrase the lemma as follows: $M_{k,c} = M'_{k,c}$, and both terms are zero if c has any blocks of magnitude different from 2. Lemma 1 motivates the following definition.

Definition 2. *A* pairing *is a matching of the indices* $1, 2, \ldots, 2k$ *such that the indices are matched exactly in pairs, and with a negative sign in* (2.12). *There are* $(2k-1)!!$ *pairings of the* $2k$ *vertices.*

Note these pairings correspond to $O\left(N^{k+1}\right)$ terms in the sum in (2.10) for the $2k^{\text{th}}$ moment.

As pairings that are the same up to a rotation of the vertices contribute equally to the moments, we make the following definition.

Definition 3. *Two pairings* $\{(a_1, a_2), (a_3, a_4), \ldots, (a_{2k-1}, a_{2k})\}$ *and* $\{(b_1, b_2), (b_3, b_4), \ldots, (b_{2k-1}, b_{2k})\}$ *are in the same* configuration *if they are equivalent up to a relabeling by rotating the vertices; i.e., there is some constant* l *such that* $b_j = a_j + l$ mod $2k$.

We display the five distinct configurations for the sixth moment in Figure 1. To determine the moments it suffices to calculate for each configuration both the contribution of a pairing with that configuration to the sum in (2.10), and the number of pairings with that configuration.

3. Determining the Moments

We first introduce a convenient notation.

Definition 4. *Fix an integer* $2k$ *and consider the circle with* $2k$ *vertices spaced uniformly, labeled 1, 2, ..., $2k$. If a, b and x are three of these vertices, by $a < x < b$ we mean the* vertex ordering *that we pass through vertex x as we travel clockwise about the circle from vertex a to vertex b.*

By Lemma 1, for the rest of the paper we may assume the vertices are matched in exactly pairs. We distinguish between three types of vertices in these pairings.

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[4]After correcting equations (2.7) and (2.8) of [16] to fix an omission and to take $\mathcal{C}_1 \in \{(-\lfloor \frac{|i_l - i_{l+1}|}{N/2^n} \rfloor + k - 1)\frac{N}{2^n} : k \in \{1, \ldots, 2^n\}\}$ and $\mathcal{C}_2 \in \{(\lfloor \frac{|i_l - i_{l+1}|}{N/2^n} \rfloor + k)\frac{N}{2^n} - 1 : k \in \{1, \ldots, 2^n\}\}$ into account, we have that $b_{|i_j i_{j+1}|}$ is paired with $b_{|i_k - i_{k+1}|}$ if and only if $i_j - i_{j+1} = \pm(i_k - i_{k+1}) + \mathcal{C}_{r_{jk}}$. For singly palindromic Toeplitz matrices, the only possible values are $\mathcal{C}_{r_{jk}}$ equals $\pm(N-1)$ or 0. The number of possible values for each $\mathcal{C}_{r_{jk}}$ depends on the moment m being computed and on the level n of palindromicity of the ensemble, but is independent of N, a fact which will be crucially important in later proofs.
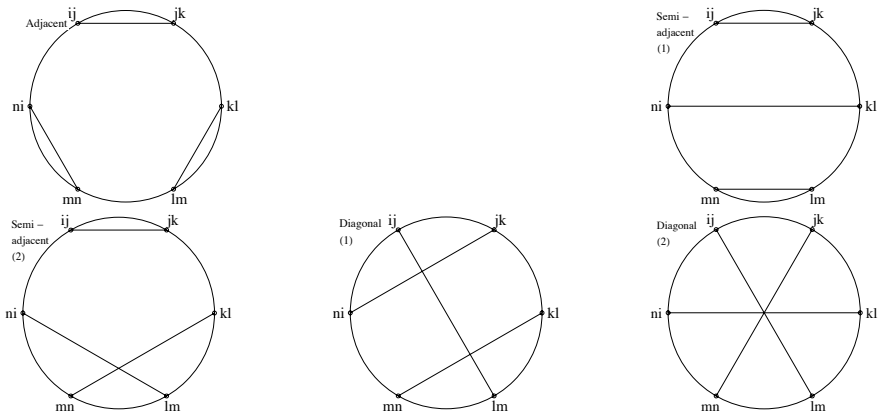
Figure 1:   The distinct configurations for the 6th moment. The multiplicity under rotation of the five patterns are 2, 3, 6, 3 and 1.
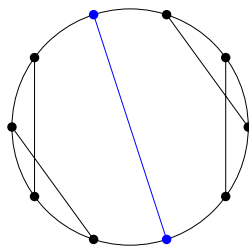

Figure 2: A pairing of 10 vertices with 8 crossing vertices (in two symmetric sets of 4 vertices), and 2 dividing vertices (connected by a main diagonal).

Definition 5. *A pair of vertices* (a, b), $a < b$, *is* crossing *if there exists a pair of vertices* (x, y) *such that the order of the four vertices, as we travel clockwise around the circle from a to b, is either a, x, b, y or a, y, b, x. A pair (a, b) is* non-crossing *if for every pair (x, y), x is between a and b (as we travel clockwise around the circle from a to b) if and only if y is.*

In other words, a pair is crossing if the line contained in the circle (i.e., the chord) connecting its two vertices crosses another line connecting two other vertices. In Figure 1 the first two configurations have no crossing vertices, the third has four, and all vertices are crossing for the fourth and fifth. The number of crossing vertices is always even and never two.

Definition 6. *A non-crossing pair of vertices (a, b) (with $a < b$) is* dividing *if the following two conditions hold (all other non-crossing pairs are* non-crossing non-dividing *pairs):*

1. *There exist two pairs of crossing vertices, (x, y) and (w, z), such that as we travel around the circle from a to b we have x, y, w and z are between a and b.*

2. *There exist two pairs of crossing vertices, (p, q) and (r, s), such that as we travel around the circle from b to a we have p, q, r and s are between b and a.*

In other words, a pair is dividing if it divides the circle into two regions of pairs, where each region has at least one crossing pair. No pair crosses a dividing edge since its pair must be non-crossing; see Figure 2 for an illustration. From the definition, every pairing with a dividing pair has at least 10 vertices. This creates additional complications arise when studying the higher moments. A similar situation arises in weighted d-regular graphs, where there is a marked change in behavior at the eighth moment; see [13] for details. Note that all pairings with a given configuration have the same number of crossing pairs and the same number of dividing pairs.

We show that the contribution of each pairing in the unsigned case is weighted by a factor depending on the number of crossing pairs in that pairing. We then prove some combinatorial formulas which give closed form expressions for the number of pairings with m vertices crossing for small k. As the combinatorics are prohibitively difficult for large k, we find the limiting behavior in §4.

3.1. Weighted Contributions

The following theorem is central to our determination of the moments. It reduces the calculation to two steps. First, we need to know the contribution of a pairing in the unsigned case (equivalently, when $p = 1$). For the singly palindromic Toeplitz ensemble each pairing contributes 1. For the Toeplitz ensemble, upper and lower bounds on the contribution of each pairing are known, but the exact value is not.

Second, we need to determine the number of vertices involved in crossing pairs, which we do in part in §3.2.

Theorem 3. *For each choice of a pairing c of the vertices $(1, \ldots, 2k)$, let $x(c)$ denote the total contribution of all tuples with paired terms matching this pairing in the unsigned case. (i.e., $x(c) = M'_{k,c}$ in the unsigned ensemble for the partition underlying the pairing.) For the Toeplitz and singly palindromic Toeplitz ensembles, the contribution in the signed case is $x(c)(2p-1)^{e(c)}$, where $e(c)$ represents the number of vertices in crossing pairs in the configuration corresponding to c.*

Recall that the contribution from a pairing c is

$$
\begin{aligned}
M'_{k,c} &= \sum_{(i_1, \ldots, i_k) \in S'_c} \mathbb{E}(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_{2k} i_1} b_{|i_{2k} - i_1|}) \\
&= \sum_{(i_1, \ldots, i_k) \in S'_c} \mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \mathbb{E}(b_{|i_1 - i_2|} \cdots b_{|i_k - i_1|}). \quad (3.13)
\end{aligned}
$$

We must show that $\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) = (2p-1)^{e(c)}$. We do this by showing that for each pair $(i_j, i_{j+1}), (i_k, i_{k+1})$ where $b_{|i_j - i_{j+1}|} = b_{|i_k - i_{k+1}|}$,

$$
\mathbb{E}\left(\epsilon_{i_j i_{j+1}} \epsilon_{i_k i_{k+1}}\right) = \begin{cases} (2p-1)^2 & \text{if } (i_j, i_{j+1}), (i_k, i_{k+1}) \text{ are a crossing pair} \\ 1 & \text{otherwise.} \end{cases} \quad (3.14)
$$

Notice that

$$
\mathbb{E}\left(\epsilon_\alpha\right) = 1 \cdot p + (-1) \cdot (1-p) = 2p-1, \quad \mathbb{E}\left(\epsilon_\alpha^2\right) = 1. \quad (3.15)
$$

Therefore, if m epsilons are chosen independently, the expected value of their product is $(2p-1)^m$. In particular, we see that $M_{k,c} = (2p-1)^{e(c)} x(c)$.

For ease of exposition, we prove the following lemmas in the Toeplitz case, and comment on the proofs (or barriers to proof) in the singly palindromic and highly palindromic cases.[5]

Lemma 2. *For the Toeplitz and singly palindromic Toeplitz ensembles,*

$$
\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \geq (2p-1)^{e(c)}. \quad (3.16)
$$

Proof. To prove $\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \geq (2p-1)^{e(c)}$, we show that pairs not in a crossing contribute 1. Consider a non-crossing pair $(i_r, i_{r+1}), (i_p, i_{p+1})$ (corresponding to vertices r and p on the circle with $2k$ labeled vertices), with $r < p$. For each

---

[5]For the palindromic case, by (2.7) and (2.8) of [16] we must add $\mathcal{C}_1$ and $\mathcal{C}_2$ into equation (2.12), as well as parts of the proof for Lemma 1; however, some minor changes to our proofs show that these lemmas still hold in the palindromic Toeplitz case.

(i_q, i_{q+1}) paired with $(i_{q'}, i_{q'+1})$, we have $r < q < p$ if and only if $r < q' < p$. Recall from (2.12) and Lemma 1 that in the Toeplitz case, $i_q - i_{q+1} = -(i_{q'} - i_{q'+1})$, while in the singly palindromic Toeplitz case $i_q - i_{q+1} = -(i_{q'} - i_{q'+1}) + Q(q, q')$ where $Q(q, q') \in \{-(N-1), 0, N-1\}$. Thus $\sum_{k=r}^{p}(i_k - i_{k+1}) = t(N-1)$ for some integer t because each difference in the sum is paired with its additive inverse, which is also in the sum. As $\sum_{k=r}^{p}(i_k - i_{k+1}) = (i_r - i_{r+1}) + (i_{r+1} - i_{r+1}) + \cdots + (i_p - i_{p+1}) = i_r - i_{p+1}$, we must have $i_r = i_{p+1} \pm t(N-1)$. It is clearly impossible to have $|t| > 1$, and if $t = \pm 1$, this forces $\{i_r, i_{p+1}\} = \{1, N\}$; thus $t = 0$. Since this situation uses up a degree of freedom, this implies that $i_r = i_{p+1}$. By a similar argument applied to the sum $\sum_{k=p}^{r}(i_k - i_{k+1})$ (taking indices cyclically), $i_{r+1} = i_p$. Therefore $\epsilon_{i_r i_{r+1}} = \epsilon_{i_p i_{p+1}}$, and $\mathbb{E}(\epsilon_{i_r i_{r+1}} \epsilon_{i_p i_{p+1}}) = 1$. □

Lemma 3. *For the Toeplitz, singly palindromic Toeplitz, and highly palindromic Toeplitz ensembles,*

$$\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \leq (2p-1)^{e(c)}. \tag{3.17}$$

Proof. We show $\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \leq (2p-1)^{e(c)}$ by showing that if $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, $a < b$, then $(i_a, i_{a+1}), (i_b, i_{b+1})$ are non-crossing. This suffices to prove the result since the only dependency between the ϵ's arises from the requirement that the matrix is real symmetric. Thus there is a dependency between $\epsilon_{i_s i_{s+1}}$ and $\epsilon_{i_p i_{p+1}}$ if and only if we know they are equal. In showing that a dependency between ϵ's implies the corresponding vertex pair must be non-crossing, we show that crossing pairs imply independent ϵ's and thus contribute $(2p-1)^2$.

If $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$ then the unordered sets $\{i_a, i_{a+1}\}$ and $\{i_b, i_{b+1}\}$ are equal. This implies that $|i_a - i_{a+1}| = |i_b - i_{b+1}|$, so $(i_a, i_{a+1}), (i_b, i_{b+1})$ must be paired on the circle. Since the only contributing terms are when they are paired in opposite orientation, $i_a = i_{b+1}$, and $\sum_{k=a}^{b}(i_k - i_{k+1}) = i_a - i_{b+1} = \sum_k \pm \mathcal{C}_{r_k}$. We can rewrite this sum as

$\sum_{k=b}^{d} \delta_k |i_k - i_{k+1}| = \sum_k \pm \mathcal{C}_{r_k}$, where δ_k is ± 1 if the vertex k is paired with is less than a or greater than b, and 0 if and only if the vertex k is paired with is between a and b. However, since the number of possible values for $\sum_k \pm \mathcal{C}_{r_k}$ is independent of N, a linear dependence among the differences is impossible, as we need to have N^{k+1} degrees of freedom for each configuration (see the proof of Lemma 1). So each $\delta_k = 0$, and each vertex between vertices a and b is paired with something else between a and b. Thus, no edges cross the edge between vertices a and b. □

Proof of Theorem 3. For Toeplitz and singly palindromic Toeplitz matrices, we proved that an epsilon is unmatched if and only if its edge is in a crossing. Thus, an epsilon is not paired if and only if its edge is not in a crossing. Therefore the contribution is weighted by $\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1})$, which by Lemmas 2 and 3 is $(2p-1)^{e(c)}$, completing the proof. □

Remark 3. *In the doubly palindromic Toeplitz case, Lemma 2 does not hold for the sixth moment, as we shall see in Lemma 5. In particular, this means the determination of the limiting rescaled spectral measures for general signed ensembles and general p is harder.*

Lemma 4. *For the Toeplitz, singly palindromic Toeplitz, and highly palindromic Toeplitz ensembles, if the contribution from a non-crossing configuration was x before the weighting, it is at most $(2p-1)^4(x-1)+1$ after applying the weighting.*

Proof. In the Toeplitz and singly palindromic Toeplitz cases, $x = 1$ and the claim is trivial. In the highly palindromic case, we note that there is a contribution of 1 from the terms which also contribute in the real symmetric case. The remaining terms contain at least 2 pairs of vertices which are not matched in the real symmetric case, since one mismatched pair (relative to the real symmetric ensemble) implies a second mismatched pair as $\sum_{k=1}^{2m}(i_k - i_{k+1}) = 0$. Hence $\mathbb{E}(\epsilon_{i_1 i_2}\epsilon_{i_2 i_3}\cdots\epsilon_{i_{2k}i_1}) \leq (2p-1)^4$ for these terms, completing the proof. □

Remark 4. *A slightly modified version of this proof shows that for other real symmetric ensembles, if the contribution from a non-crossing configuration was x before the weighting, it is at most $(2p-1)^2(x-1)+1$ after applying the weighting. Similarly, for crossing configurations, if the contribution was x before the weighting, it is at most $(2p-1)^2 x$ after applying the weighting.*

Lemma 5. *For the sixth moment of signed doubly palindromic Toeplitz ensembles, the contribution from a configuration is not determined uniquely by the number of crossings.*

See §A.1 for a proof.

3.2. Counting Crossing Configurations

Theorem 3 reduces the determination of the moments to counting the number of pairings with a given contribution $x(c)$ and weighting by $(2p-1)^{e(c)}$, where $e(c)$ is the number of vertices in crossings in the configuration. As remarked above, in the singly palindromic Toeplitz case each $x(c) = 1$, while in the general Toeplitz case we only have bounds on the $x(c)$'s, and thus must leave these as parameters in the final answer.

We turn to computing the $e(c)$'s for various configurations. These and similar numbers have been studied in knot theory where these chord diagrams appear in the study of Vassiliev invariants (see [19, 22, 12, 31, 33]). While we cannot determine exact formulas in general, we are able to solve many special cases.

Definition 7. *Let $\mathrm{Cr}_{2k,2m}$ denote the number of pairings involving $2k$ vertices where exactly $2m$ vertices are involved in a crossing.*

Let $C_k = \frac{1}{k+1}\binom{2k}{k}$ denote the kth Catalan number. One of its many definitions is as the number of ways to match $2k$ objects on a circle in pairs without any crossings; this interpretation is the reason why Wigner's Semi-Circle Law holds. Thus, we immediately deduce the following.

Lemma 6. *We have* $\text{Cr}_{2k,0} = C_k$.

We use this result to prove the following theorem, which is instrumental in our counting.

Theorem 4. *Consider $2k$ vertices on a circle, with a partial pairing on a subset of $2v$ vertices. There are $\binom{2k}{k-v}$ ways to place the remaining $2k - 2v$ vertices in non-crossing, non-dividing pairs.*

Proof. Let $\mathcal{W}$ denote the desired quantity. Each of the remaining $2k - 2v$ vertices must be placed between two of the $2v$ already paired vertices on the circle. These $2v$ vertices have created $2v$ regions. A necessary and sufficient condition for these $2k - 2v$ vertices to be in non-crossing, non-dividing pairs is that the vertices in each of these $2v$ regions pair only with other vertices in that region in a non-crossing configuration. Thus, if there are $2s$ vertices in a region, by Lemma 6 the number of valid ways they can pair is C_{2s}. As the number of valid matchings in each region depends only on the number of vertices in that region and not on the matchings in the other regions, we obtain a factor of $C_{2s_1}C_{2s_2}\cdots C_{2s_{2v}}$, where $2s_1 + 2s_2 + \cdots + 2s_{2v} = 2k - 2v$.

We need only determine how many pairings this factor corresponds to. Notice that by specifying one index and $(s_1, s_2, \ldots, s_{2v})$, we completely specify a pairing of the $2k$ vertices. However, as we are pairing on a circle, this specification does not uniquely determine a pairing since the labeling of $(s_1, s_2, \ldots, s_{2v})$ is arbitrary. Each pairing can be written as any of the $2v$ circular permutations of some choice of $(s_1, s_2, \ldots, s_{2v})$ and one index. Thus we are interested in

$$\mathcal{W} = \frac{2k}{2v} \sum_{2s_1 + 2s_2 + \cdots + 2s_{2v} = 2k - 2v} C_{s_1}C_{s_2}\cdots C_{s_{2v}}. \tag{3.18}$$

We evaluate it with the k-fold self-convolution identity of Catalan numbers, which states

$$\sum_{i_1 + \cdots + i_r = n} C_{i_r - 1}\cdots C_{i_r - 1} = \frac{r}{2n - r}\binom{2n - r}{n}. \tag{3.19}$$

Setting $i_j = s_j + 1$, $r = 2v$ and $n = k + v$ and rewriting yields

$$\frac{2k}{2v} \sum_{2s_1 + 2s_2 + \cdots + 2s_{2v} = 2k - 2v} C_{s_1}C_{s_2}\cdots C_{s_{2v}} = \binom{2k}{k - v}, \tag{3.20}$$

which completes the proof as the left hand side is just (3.18). $\square$

Given Theorem 4, our ability to find formulas for $\text{Cr}_{2k,2m}$ rests on our ability to find the number of ways to pair $2v$ vertices where $2m$ vertices are crossing and $2v - 2m$ vertices are dividing. We can do this for small values of m, but for large m the combinatorics become very involved.

Definition 8. *Let* $\text{P}_{2k,2m,i}$ *represent the number of pairings of $2k$ vertices with $2m$ crossing vertices in i partitions. We define a* partition *to be a set of crossing vertices separated from all other sets of crossing vertices by at least one dividing edge.*

It takes a minimum of 4 vertices to form a partition, so the maximum number of partitions possible is $\lfloor 2m/4 \rfloor$. Our method of counting involves writing

$$\text{Cr}_{2k,2m} \;\; = \;\; \sum_{i=1}^{\lfloor 2m/4 \rfloor} \text{P}_{2k,2m,i}. \qquad (3.21)$$

Our first combinatorial result is the following.

Lemma 7. *We have* $\text{P}_{2k,2m,1} = \text{Cr}_{2m,2m} \binom{2k}{k-m}$.

Proof. The proof is immediate from Theorem 4. If there is only one partition, there are no dividing edges and we multiply the number of ways we can choose $2k - 2m$ non-crossing non-dividing pairs by the number of ways to then choose how the $2m$ crossing vertices are paired. ∎

Lemma 8. *We have*

$$\text{P}_{2k,2m,2} = \sum_{d=1}^{k-m} \binom{2k}{k-m-d} (m+d) \left(\sum_{0<a<m} \text{Cr}_{2a,2a} \text{Cr}_{2m-2a,2m-2a} \right). \qquad (3.22)$$

Proof. Let d be the number of dividing edges. To have two partitions we need at least one of the $k-m$ non-crossing edges to be dividing. We sum d from 1 to $k-m$. Given d, we can pair and place the non-crossing non-dividing edges in $\binom{2k}{k-m-d}$ ways from Theorem 4. We then choose a way to pair the $2m$ crossing vertices into 2 partitions, one with $2a$ vertices, the other with $2b$ vertices. If $a = b$, there are $m + d$ distinct spots where we may place the dividing edge. If $a \neq b$, there are $2m + 2d$ spots. As each choice of $a \neq b$ appears twice in the above sum, the result follows. ∎

Determining $\text{P}_{2k,2m,3}$ requires the analysis of several more cases, and we were unable to find a nice way to generalize the results of Lemmas 7 and 8. However, these two results do allow us to write down the following formulas.

Lemma 9. *We have*

$$
\begin{aligned}
\mathrm{Cr}_{2k,4} &= \binom{2k}{k-2} \\
\mathrm{Cr}_{2k,6} &= 4\binom{2k}{k-3} \\
\mathrm{Cr}_{2k,8} &= 31\binom{2k}{k-4} + \sum_{d=1}^{k-4}\binom{2k}{k-4-d}(4+d) \\
\mathrm{Cr}_{2k,10} &= 288\binom{2k}{k-5} + 8\sum_{d=1}^{k-5}\binom{2k}{k-5-d}(5+d). \quad\quad (3.23)
\end{aligned}
$$

The proof follows by deriving recursive formulas; see §A.2 for details.

By using Lemma 9 to calculate the number of terms with each of the possible contributions given in Theorem 3, we are able to calculate up to the 12th moment exactly (for the 12th moment we use the same recursive procedure as in the proof of Lemma 9 to calculate $\mathrm{Cr}_{12,12}$).

Remark 5. *The coefficients in front of the binomial coefficient of the leading term of* $\mathrm{Cr}_{2k,2m}$ *are sequence A081054 from the OEIS [20].*

4. Limiting Behavior of the Crossing Numbers

As we cannot find exact expressions for the number of pairings with exactly $2m$ crossing vertices for all m, we determine the expected value and variance of the number of vertices in a crossing. Such expressions and their limiting behavior are useful for obtaining bounds for the moments. We make frequent use of arguments on the probabilities of certain pairings, recognizing that since all configurations are equally likely the probability vertex i pairs with vertex j is just $\frac{1}{2k-1}$.

Theorem 5. *If all pairings of $2k$ vertices on a circle are equally likely, the expected number of vertices in crossings is*

$$
\frac{2k}{2k-1}\left(2k-2-\frac{{}_2F_1\left(1,\frac{3}{2},\frac{5}{2}-k;-1\right)}{2k-3}-(2k-1)\,{}_2F_1\left(1,\frac{1}{2}+k,\frac{3}{2};-1\right)\right)
$$
$$
= 2k-2-\frac{2}{k}+O\left(\frac{1}{k^2}\right). \quad (4.24)
$$

Proof. As we only need the asymptotic expression, we prove that below and give the proof of (4.24) in Appendix A.3. For a given pairing of $2k$ vertices, let $X_i = 1$

if vertex i is involved in a crossing and 0 otherwise. Then $Y_{2k} = \sum_{i=1}^{2k} X_i$ is the number of vertices involved in a crossing in this pairing. By linearity of expectation

$$\mathbb{E}(Y_{2k}) = \mathbb{E}\left(\sum_{i=1}^{2k} X_i\right) = 2k\mathbb{E}(X_i) = 2kp_{\text{cross}}, \qquad (4.25)$$

where p_{cross} is the probability that a given vertex is in a crossing (this is the same for all vertices). We may think of p_{cross} as the probability that vertex 1 is in a crossing. We notice that

1. Vertex 1 is matched with another odd indexed vertex with probability $\frac{k-1}{2k-1}$. In this case it is in a crossing as there are an odd number of vertices in the two regions created by the matching and the regions cannot only pair with themselves.

2. If vertex 1 is matched with an even indexed vertex, then it is in a crossing iff it doesn't partition the remaining vertices into two parts that pair exclusively with themselves. If it is matched with vertex $2m$ (happening with probability $\frac{1}{2k-1}$) then its edge divides the vertices into regions of $2m - 2$ and $2k - 2m$ vertices. As the number of ways to match 2ℓ objects in pairs with order immaterial is $(2\ell - 1)!! = (2\ell - 1)(2\ell - 3)\cdots 3 \cdot 1$, the probability that each region pairs only with itself is $(2m - 3)!!\,(2k - 2m - 1)!!/\,(2k - 3)!!$.

Thus the probability that vertex 1 is involved in a crossing is

$$
\begin{aligned}
p_{\text{cross}} &= \frac{k-1}{2k-1} + \sum_{m=2}^{k-1} \frac{1}{2k-1}\left(1 - \frac{(2m-3)!!\,(2k-2m-1)!!}{(2k-3)!!}\right) \\
&= \frac{2k-3}{2k-1} - \frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{\binom{k-2}{m-2}}{\binom{2k-3}{2m-3}}, \qquad (4.26)
\end{aligned}
$$

and therefore

$$\mathbb{E}(Y_{2k}) = 2kp_{\text{cross}} = (2k)\frac{2k-3}{2k-1} - (2k)\frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{\binom{k-2}{m-2}}{\binom{2k-3}{2m-3}}. \qquad (4.27)$$

The first and last terms above are both $\frac{1}{2k-3}$, as for $m = 2$ we have $\frac{\binom{k-2}{0}}{\binom{2k-3}{1}} = \frac{1}{2k-3}$, and for $m = k-1$ we have $\frac{\binom{k-2}{k-3}}{\binom{2k-3}{2k-5}} = \frac{\binom{k-2}{1}}{\binom{2k-3}{2}} = \frac{1}{2k-3}$. Looking at the ratio of subsequent terms, straightforward algebra shows

$$\frac{\binom{k-2}{m-1}/\binom{2k-3}{2m-1}}{\binom{k-2}{m-2}/\binom{2k-3}{2m-3}} = \frac{2m-1}{2k-2m-1}. \qquad (4.28)$$
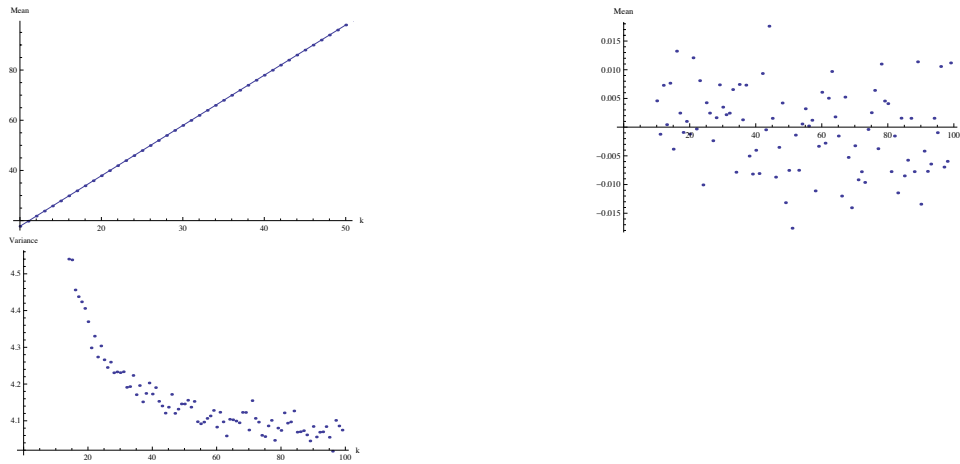
Figure 3: Numerical confirmation of formulas for the expected value and variance of vertices involved in crossing. The first plot is the expected value for $2k$ vertices (solid line is theory) versus k, the second plot is a plot of the deviations from theory, and the third plot is the observed variance; all plots are from 100,000 randomly chosen matchings of $2k$ vertices in pairs.

Thus for m up to the halfway point, each term in the sum is less than the previous. In particular, the $m = 3$ term is $5/(2k - 7)$ times the $m = 2$ term, and hence all of these terms are $O(1/k^2)$. Similarly, working from $m = k - 2$ to the middle we find all of these terms are also $O(1/k^2)$, and thus the sum in (4.27) can be rewritten, giving

$$
\begin{aligned}
\mathbb{E}\left(Y_{2k}\right) &= (2k)\frac{2k-3}{2k-1} - (2k)\frac{1}{2k-1}\left(\frac{2}{2k-3} + O\left(\frac{1}{k^2}\right)\right) \\
&= 2k - 2 - \frac{2}{k} + O\left(\frac{1}{k^2}\right).
\end{aligned}
\tag{4.29}
$$

$\square$

Theorem 6. *As $k \to \infty$, the variance of the number of vertices in a crossing approaches 4.*

As the proof is similar to the proof of Theorem 5, we leave the details to §A.4

Figure 3 provides a numerical verification of the formulas for the expected values and variances.

5. Limiting Spectral Measure

We complete the proof of Theorem 1 by showing convergence and determining the support.

Proof of Theorem 1. The proof of the claimed convergence is standard, and follows immediately from similar arguments as in [15, 27, 16, 24]. Those arguments rely only on degree of freedom counting arguments, and are thus applicable here as well. We are left with determining the limiting rescaled spectral measures.

- $p = 1/2$: If $p = 1/2$, we know from (3) that only those configurations with no crossings contribute. In particular, we may apply this to the ± 1 real symmetric weight matrix. Moreover, in the crossing configurations, it is simple to check that in $(N - n \cdot o(N))^n \approx N^n - o(N^n)$ of the N^n terms for the nth moment computation each random variable from the coefficients of the matrix ensemble occurs exactly twice. Since the moments of the original distribution are finite, the remaining $o(N^n)$ terms do not contribute. Thus, we may assume that each random variable occurs exactly twice in each term (and the variables are otherwise independent.) The claim follows directly from recalling that the number of non-crossing configurations are simply the Catalan numbers, which are also the moments of the semi-circle distribution.

- $p > 1/2$: We consider the case when $\widetilde{\mu}$ (the limiting rescaled spectral measure) has unbounded support; the case of bounded support is similar. To show that the limiting signed rescaled spectral measure has unbounded support it suffices to show that the moments of our distribution grow faster than any exponential bound, i.e., that for all B there exists some k such that $M_{2k} > B^{2k}$. Assume the moments of the unsigned ensemble grow faster than exponentially. We prove that our distribution similarly has unbounded support using this fact and by considering the "worst-case" scenario allowed for under Theorem 3. Namely, we suppose that each term contributes $x(c) (2p - 1)^{2k}$, which gives us the smallest moment possible. In this case, M_{2k} is decreased from the unsigned case by a factor of $(2p - 1)^{2k}$, and thus the growth is still faster than any exponential bound.

$\square$

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A. Appendix: Proofs of Some Claims

A.1. Proof of Lemma 5

We prove that the adjacent configuration and the non-adjacent non-crossing configuration (the upper-left and upper-middle configurations in Figure 1, respectively) have different contributions to the sixth moment. The main idea is that in the 'adjacent configuration', every contributing term has either all three pairings of the form $a_{ij}a_{ji}$, or exactly one pairing of this form. Since we know that the contribution

when all three pairings are of this form is 1, the contribution when there is exactly one pairing of this form is $(x-1)$. In this situation, the contribution to the moment is weighted by $(2p-1)^4$, giving a total of $(2p-1)^4(x-1)+1$.

Specifically, we have that

$$i_t - i_{t+1} \;=\; -(i_{t+1} - i_{t+2}) \pm \mathcal{C}_{r_{t,t+1}}, \tag{A.30}$$

where $\mathcal{C}_{r_{t,t+1}} = N/2$ or $N/2 - 1$ or 0 (N and $N-1$ are ruled out because we would lose a degree of freedom by forcing one value to be 1 and the other to be N). Moreover, $\mathcal{C}_{r_{t,t+1}} = 0$ if and only if $\epsilon_{i_t i_{t+1}} = \epsilon_{i_{t+1} i_{t+2}}$. If we choose three values from $\{0, \pm N/2, \pm N/2 - 1\}$ that add to 0, either one or three of the values must be 0. The cases where all three are 0 contribute fully while the case where two are non-zero is depressed by $(2p-1)^4$, so that contribution to the moment in the signed ensemble is exactly $(2p-1)^4(x-1)+1$.

In the other non-crossing configuration, the moment is at most $(2p-1)^4(x-1)+1$ by the proof of Lemma 4. Hence, to show the moment is smaller than this, it suffices to find a contributing group of terms whose moment is reduced by more than $(2p-1)^4$. We can take the vertices to be $a_{i,j}, a_{j,i+N/2}, a_{i+N/2,k+N/2}, a_{k+N/2,l}, a_{l,k}, a_{k,i}$, where $i, k < N/2$. While there is an additional inequality between i and j and between k and l, this does not remove a degree of freedom since there are still order N possible values. Hence some portion of the $(x-1)$ contribution is reduced by a factor of $(2p-1)^6 < (2p-1)^4$. Since the remaining portion of the contribution is reduced to at most $(2p-1)^4$ times its original value, the contribution to the 6th moment of the non-adjacent non-crossing configuration in the signed doubly palindromic case is strictly less than $(2p-1)^4(x-1)+1$, and thus not equal to the contribution from the adjacent non-crossing configuration. $\square$

A.2. Proof of Lemma 9

We recall that $\mathrm{Cr}_{2k,0} = C_k$ and $\mathrm{Cr}_{2k,2} = 0$, where the second equation follows from the fact that at least 4 vertices are needed for a crossing. From (3.21) and (7) we find $\mathrm{Cr}_{2k,4} = \mathrm{P}_{2k,4,1} = \mathrm{Cr}_{4,4}\binom{2k}{k-2}$. We can calculate $\mathrm{Cr}_{4,4}$ by using the above and the fact that

$$\sum_{m=0}^{k} \mathrm{Cr}_{2k,2m} \;=\; (2k-1)!!. \tag{A.31}$$

This follows as the number of ways to match $2k$ objects in pairs of 2 with order not mattering is $(2k-1)!!$, and thus the sum of all our matchings in pairs must equal this. Note this number is also the $2k^{\text{th}}$ moment of the standard normal; this is the reason the singly palindromic Toeplitz have a limiting rescaled spectral measure that is normal, as each contribution contributes fully. We find

$$\text{Cr}_{4,4} \ = \ (2 \cdot 2 - 1)!! - \text{Cr}_{4,2} - \text{Cr}_{4,0} \ = \ 3 - 2 \ = \ 1. \tag{A.32}$$

This completes the proof of the first formula: $\text{Cr}_{2k,4} = \binom{2k}{k-2}$.

The other coefficients are calculated in a similar recursive fashion – essentially, once we have values for $\text{Cr}_{2k,2l}$ for $l = 0, 1, 2, \ldots, m-1$, we can find $\text{Cr}_{2m,2m}$ by using (A.31), which allows us to write the general formulas above for $\text{Cr}_{2k,2m}$. We show the calculations below. We have

$$\text{Cr}_{6,6} \ = \ (6-1)!! - \text{Cr}_{6,4} - \text{Cr}_{6,2} - \text{Cr}_{6,0} \ = \ 4, \tag{A.33}$$

so $\text{Cr}_{2k,6} = 4\binom{2k}{k-3}$,
and thus

$$\text{Cr}_{8,8} \ = \ (8-1)!! - \text{Cr}_{8,6} - \text{Cr}_{8,4} - \text{Cr}_{8,2} - \text{Cr}_{8,0} \ = \ 31. \tag{A.34}$$

To finish the calculation for $\text{Cr}_{2k,8}$ we compute

$$\sum_{0<a<4} \text{Cr}_{2a,2a}\text{Cr}_{8-2a,8-2a}$$
$$= \ \text{Cr}_{2,2}\text{Cr}_{6,6} + \text{Cr}_{4,4}\text{Cr}_{4,4} + \text{Cr}_{6,6}\text{Cr}_{2,2} \ = \ 0+1+0 \ = \ 1, \tag{A.35}$$

so that we get $\text{Cr}_{2k,8} \ = \ 31\binom{2k}{k-4} + \sum_{d=1}^{k-4} \binom{2k}{k-4-d}(4+d)$.
For the formula for $\text{Cr}_{2k,10}$,

$$\text{Cr}_{10,10} \ = \ (10-1)!! - \text{Cr}_{10,8} - \text{Cr}_{10,6} - \text{Cr}_{10,4} - \text{Cr}_{10,2} - \text{Cr}_{10,0} \ = \ 288, \tag{A.36}$$

and finally

$$\sum_{0<a<5} \text{Cr}_{2a,2a}\text{Cr}_{10-2a,10-2a}$$
$$= \text{Cr}_{2,2}\text{Cr}_{8,8} + \text{Cr}_{4,4}\text{Cr}_{6,6} + \text{Cr}_{6,6}\text{Cr}_{4,4} + \text{Cr}_{8,8}\text{Cr}_{2,2} \ = \ 8, \tag{A.37}$$

so $\text{Cr}_{2k,10} \ = \ 288\binom{2k}{k-5} + 8\sum_{d=1}^{k-5} \binom{2k}{k-5-d}(5+d)$. $\square$

A.3. Proof of Theorem 5 (Mean)

To prove (4.24), it suffices to simplify the sum in the expansion of p_{cross} in (4.26). We first extend the m-sum to include $m = k$; this adds 1 to the sum which must be

subtracted from the term outside. For notational convenience, set $n = k - 2$. We re-index and let m run from 0 to n, reducing to

$$S(n) = \sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{2n+1}{2m+1}}. \tag{A.38}$$

The following notation and properties are standard (see for example [14]). The Pochhammer symbol $(x)_m$ is defined for $m \geq 0$ by $(x)_m = \Gamma(x+m)/\Gamma(x) = x(x+1)\cdots(x+m-1)$, and the hypergeometric function $_2F_1$ by

$$_2F_1(a, b, c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \tag{A.39}$$

which converges for all $|z| < 1$ so long as c is not a negative integer.

For ease of exposition, we work backwards from the answer.[6] Using $\Gamma(1+z) = z\Gamma(z)$ and $\Gamma(1+\ell) = \ell!$ (for integral ℓ), we find

$$\begin{aligned}
_2F_1(1, \frac{3}{2}, \frac{1}{2} - n, -1) &= \sum_{m=0}^{\infty} \frac{(1)_m (3/2)_m}{(1/2-n)_m} \frac{(-1)^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{\Gamma(1+m)}{\Gamma(1)} \frac{\Gamma(3/2+m)}{\Gamma(3/2)} \frac{\Gamma(1/2-n)}{\Gamma(1/2-n+m)} \frac{(-1)^m}{m!} \\
&=: T_1(n) + T_2(n), \tag{A.40}
\end{aligned}$$

where $T_1(n)$ is the sum over $m \leq n$ and $T_2(n)$ is the sum over $m > n$. From the functional equation of the Gamma function and using $\ell!! = \ell(\ell-2)(\ell-4)\cdots$ down to 2 or 1, we find

$$\begin{aligned}
\Gamma(3/2+m) &= 2^m (2m+1)!! \Gamma(3/2) \\
\Gamma(1/2-n+m) &= (-1)^m 2^m (2n-1)(2n-3)\cdots(2n-2m+1)\Gamma(1/2-n).
\end{aligned} \tag{A.41}$$

Substituting, we find

$$T_1(n) = \sum_{m=0}^{n} \frac{(2m+1)!!(2n-2m-1)!!}{(2n-1)!!} = (2n+1)\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{2n+1}{2m+1}}; \tag{A.42}$$

note this is our desired sum. Thus

$$\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{2n+1}{2m+1}} = \frac{_2F_1(1, 3/2, 1/2-n, -1) - T_2(n)}{2n+1}, \tag{A.43}$$

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[6]Mathematica can evaluate such sums and suggest the correct hypergeometric combinations. One has to be a little careful, though, as it incorrectly evaluated $S(n)$, erroneously stating that there was zero contribution if we extend the sum to all m.

and the proof is completed by analyzing $T_2(n)$. To determine this term's contribution, we re-index. Writing $m = n + 1 + u$, we find

$$T_2(n)$$

$$= \sum_{u=0}^{\infty} \frac{\Gamma(1 + n + 1 + u)}{\Gamma(1)} \frac{\Gamma(\frac{3}{2} + n + 1 + u)}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{1}{2} - n)}{\Gamma(\frac{1}{2} - n + n + 1 + u)} \frac{(-1)^{n+1+u}}{(n+1+u)!} \frac{u!}{u!}$$

$$= \sum_{u=0}^{\infty} \frac{\Gamma(1 + u)}{\Gamma(1)} \frac{\Gamma(\frac{5}{2} + n + u)}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{1}{2} - n)}{\Gamma(\frac{3}{2} + u)} \frac{(-1)^{n+1}(-1)^u}{u!}$$

$$= \frac{(-1)^{n+1}\Gamma(\frac{1}{2} - n)\Gamma(\frac{5}{2} + n)}{\Gamma(\frac{3}{2})^2} \sum_{u=0}^{\infty} \frac{\Gamma(1 + u)}{\Gamma(1)} \frac{\Gamma(\frac{5}{2} + n + u)}{\Gamma(\frac{5}{2} + n)} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} + u)} \frac{(-1)^u}{u!}$$

$$= -(2n + 3)(2n + 1) \, _2F_1(1, 1/2 + k, 3/2, -1), \tag{A.44}$$

where we used $\Gamma(1 - z)\Gamma(z) = \pi / \sin(\pi z)$ with $z = n + \frac{1}{2}$ to simplify the Gamma factors depending only on n. Combining the above proves (4.24).

A.4. Proof of Theorem 5 (Variance)

We need to calculate $\mathrm{Var}\left(Y_{2k}\right) = \mathbb{E}\left(Y_{2k}^2\right) - \mathbb{E}\left(Y_{2k}\right)^2$. As we know the second term by Theorem 5, we concentrate on the first term:

$$\mathbb{E}\left(Y_{2k}^2\right) = \sum_{i,j \in \{1,\dots,2k\}} \mathbb{E}\left(X_i X_j\right). \tag{A.45}$$

The above sum has $4k^2$ terms. For $2k$ of those terms, $i = j$ so $\mathbb{E}\left(X_i X_j\right) = \mathbb{E}\left(X_i^2\right) = \mathbb{E}\left(X_i\right) = p_{\mathrm{cross}}$ as the X_ℓ's are binary indicator variables with probability of success p_{cross}. For another $2k$ terms, we have i and j are paired on the same edge, so $\mathbb{E}\left(X_i X_j\right) = \mathbb{E}\left(X_i\right) = p_{\mathrm{cross}}$ as before.

For the remaining $4k^2 - 4k$ terms, i and j are on different edges and we must find the probability that both those edges are in crossings. We separate this into two disjoint probabilities, the probability p_a that they cross each other, and the probability that they don't cross each other but are each crossed by at least one other pairing. We denote this second probability by $(1 - p_a)\, p_b$, where p_b is the conditional probability they are each crossing given that they don't cross each other. We find these probabilities by summing over the placements of k, m, p, q above as appropriate and calculating for each the probability of observing one of our desired configurations. We have shown

$$\mathbb{E}\left(Y_{2k}^2\right) = 4k p_{\mathrm{cross}} + \left(4k^2 - 4k\right)\left(p_a + (1 - p_a)\, p_b\right), \tag{A.46}$$

thus reducing the problem to the determination of p_a and p_b.

We label our edges as $\{1, m\}$ and $\{p, q\}$. They cross iff one of $\{p, q\}$ is one of the $m - 2$ vertices between 1 and m, and the other is one of the $2k - m$ vertices between m and $2k$. Thus

$$p_a = \sum_{m=2}^{2k} \frac{1}{2k-1} \cdot 2 \cdot \frac{m-2}{2k-2} \cdot \frac{2k-m}{2k-3}$$

$$= \frac{2}{(2k-1)(2k-2)(2k-3)} \left[\sum_{m=2}^{2k} -4k - \sum_{m=2}^{2k} m^2 + (2k+2) \sum_{m=2}^{2k} m \right].$$
(A.47)

From formulas for the sum of the first n integers and squares, we simplify the second factor and find

$$p_a = \frac{2}{(2k-1)(2k-2)(2k-3)} \frac{(2k-1)(2k-2)(2k-3)}{6} = \frac{1}{3}. \text{(A.48)}$$

We now calculate p_b, the probability that $\{1, m\}$ and $\{p, q\}$ are both involved in crossings given they don't cross each other. We must place $\{1, m\}, \{p, q\}$. Relabeling if necessary, we may assume $1 < m < p < q$; such a labeling is possible if and only if $\{1, m\}$ and $\{p, q\}$ do not cross each other. We compute the complement of our desired probability by finding the number of configurations where at most one of $\{1, m\}$ and $\{p, q\}$ is in a crossing. We denote the number of such configurations by $N_{k,m,p,q}$ and can thus write

$$p_b = 1 - \sum_{m=2}^{2k-2} \sum_{p=m+1}^{2k-1} \sum_{q=p+1}^{2k} \frac{N_{k,m,p,q}}{(2k-5)!!}. \tag{A.49}$$

As there are $\binom{2k-1}{3}$ terms in the above sum (corresponding to the $\binom{2k-1}{3}$ possible choices of m, p, q as we have specified the location of vertex 1 and the order of m, p, q), we can rewrite (A.49) as

$$p_b = 1 - \frac{\sum_{m=2}^{2k-2} \sum_{p=m+1}^{2k-1} \sum_{q=p+1}^{2k} N_{k,m,p,q}}{\binom{2k-1}{3} (2k-5)!!}. \tag{A.50}$$

All that remains is to evaluate the sum above. To do so, we first define the following function $P(k)$, which counts the number of ways k vertices can be paired with each other:

$$P(x) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k = 0 \\ (k-1)!! & \text{otherwise.} \end{cases} \tag{A.51}$$

These two edges divide the remaining vertices into three regions: those between $\{1, m\}$ and $\{p, q\}$, of which there are $M = p - m - 1 + 2k - q$, those on the side of

$\{1, m\}$, of which there are $L = m - 2$, and those on the side of $\{p, q\}$, of which there are $R = q - p - 1$. We know that $\{1, m\}$ will not be crossed if the L vertices between 1 and m pair exclusively with each other. Likewise, $\{p, q\}$ will not be crossed if the vertices between p and q pair exclusively with each other. Our desired quantity is thus the union of these two events less their intersection:

$$P(L + M) P(R) + P(R + M) P(L) - P(L) P(M) P(R). \qquad (A.52)$$

If L or R is 0, one of $\{1, m\}, \{p, q\}$ is an adjacent edge, and therefore is not crossing. Thus

$$
N_{k,m,p,q}
$$
$$
= \begin{cases} (2k - 5)!! & \text{if } L \text{ or } R \text{ is } 0 \\ P(L + M) P(R) + P(R + M) P(L) - P(L) P(M) P(R) & \text{otherwise.} \end{cases}
$$
$$(A.53)$$

We now investigate the limiting behavior of p_b (given in (A.49)) by using the cases in (A.53).

- For the first case, we have L or R is zero, and thus $N_{k,m,p,q} = (2k - 5)!!$. We are reduced to counting the number of terms with L or R zero. Note that $L = 0$ when $m = 2$, and $R = 0$ when $q = p + 1$. Each of these events happens in $\binom{2k-2}{2}$ pairings (we have fixed either m or q, and the other 2 vertices are chosen from the remaining $2k - 2$ vertices), and their intersection is $\binom{2k-3}{1}$ (p is the only free index) pairings. In the limit, this case contributes

$$\frac{\left(2\binom{2k-2}{2} - \binom{2k-3}{1}\right)(2k - 5)!!}{\binom{2k-1}{3}(2k - 5)!!} = \frac{3}{k} + O\left(\frac{1}{k^3}\right). \qquad (A.54)$$

- For the second case, L and R are non-zero. We first evaluate the contribution of the first two terms (they contribute the same as we can relabel $\{1, m\}$ and $\{p, q\}$) and then the third, recalling that we only have to look for terms that are at least $O\left(\frac{1}{k^2}\right)$ since by (A.46) any other terms do not contribute as $k \to \infty$.

  - For $P(L + M) P(R)$, the largest terms are from when either $L + M = 2$, or when $R = 2$. In these cases, $N_{k,m,p,q} = (2k - 7)!!$. If $R = 2$ then $q = p + 3$ and m, p are free so there are $\binom{2k-4}{2}$ such terms corresponding to the $\binom{2k-4}{2}$ choices of m and p. If $L + M = 2$ and $L \neq 0$ then there are only two possible terms: either $L = 1, M = 1, R = 2k - 6$ or $L = 2, M =$

$0, R = 2k - 6$. Including the symmetric terms for $P(R+M) P(L)$, these terms thus have a combined contribution of

$$\frac{2 \left(\binom{2k-4}{2} + 2 \right) (2k-7)!!}{\binom{2k-1}{3} (2k-5)!!} \;=\; \frac{3}{2k^2} + O \left(\frac{1}{k^3} \right). \tag{A.55}$$

– For the third term, $-P(L) P(M) P(R)$, the largest contributions are when two regions combine for exactly 2 vertices, contributing $(2k - 7)!!$. If we disregard the requirement that L and R are nonzero in order to obtain an upper bound on the magnitude of this contribution, there are 3 possible terms. The next largest contribution is when two regions combine for exactly 4 vertices, contributing $(2k - 9)!!$. Proceeding with these diagonal terms, we know that the third term contributes at most in magnitude

$$3 \frac{(2k-7)!!}{\binom{2k-1}{3}(2k-5)!!} \;+\; 6 \frac{(2k-9)!!}{\binom{2k-1}{3}(2k-5)!!} \;+\; 9 \frac{(2k-11)!!}{\binom{2k-1}{3}(2k-5)!!} \;+\; \cdots$$

$$= \; O \left(\frac{1}{k^3} \right), \quad \text{(A.56)}$$

so they do not contribute to the main term in the limit.

Thus we have that, as $k \to \infty$,

$$p_b \;=\; 1 - \frac{3}{k} - \frac{3}{2k^2} + O \left(\frac{1}{k^3} \right). \tag{A.57}$$

Therefore if we substitute for p_a and p_b in (A.46) we find

$$\mathbb{E} \left(Y_{2k}^2 \right) \;=\; 4k - 4 + \left(4k^2 - 4k \right) \left(\frac{1}{3} + \frac{2}{3} \left(1 - \frac{3}{k} - \frac{3}{2k^2} \right) \right)$$

$$= \; 4k^2 - 8k + O \left(\frac{1}{k} \right). \tag{A.58}$$

Using (4.29), we also have that

$$\mathbb{E} \left(Y_{2k} \right)^2 \;=\; \left(2k - 2 - \frac{2}{k} + O \left(\frac{1}{k^2} \right) \right)^2$$

$$= \; 4k^2 - 8k - 4 + O \left(\frac{1}{k} \right). \tag{A.59}$$

The variance is $\mathbb{E} \left(Y_{2k}^2 \right) - \mathbb{E} \left(Y_{2k} \right)^2$, which is $4 + O(1/k)$ as $k \to \infty$. $\qquad \square$