



ABEL'S LEMMA AND IDENTITIES ON HARMONIC NUMBERS

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Recently, Chen, Hou and Jin used both Abel's lemma on summation by parts and Zeilberger's algorithm to generate recurrence relations for definite summations. They also proposed the Abel-Gosper method to evaluate some indefinite sums involving harmonic numbers. In this paper, we use the Abel-Gosper method to prove an identity involving the generalized harmonic numbers. Special cases of this result reduce to many famous identities. In addition, we use both Abel's lemma and the WZ method to verify and to discover identities involving harmonic numbers. Many interesting examples are also presented.

1. Introduction

The objective of this paper is to employ Abel's lemma on summation by parts and hypergeometric summation algorithms to verify and to discover identities on the harmonic as well as generalized harmonic numbers.

Recall that for a positive integer n and an integer r , the generalized harmonic numbers of power r are given by

$$H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r}.$$

For convenience, we set $H_n^{(r)} = 0$ for $n \leq 0$. As usual, $H_n = H_n^{(1)}$ are the classical

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harmonic numbers. We also define (see [6])

$$H_n(x) = \sum_{k=1}^n \frac{1}{k+x}, \quad x \neq -1, -2, \dots$$

for $n \geq 1$ and $H_n(x) = 0$ when $n \leq 0$. Identities involving these numbers have been extensively studied and applied in the literature, see, for example, [5, 6, 12, 16, 20]. Also recall that Abel’s lemma [1] on summation by parts is stated as follows.

Lemma 1 (Abel’s lemma). *For two arbitrary sequences $\{a_k\}$ and $\{b_k\}$, we have*

$$\sum_{k=m}^{n-1} (a_{k+1} - a_k)b_k = \sum_{k=m}^{n-1} a_{k+1}(b_k - b_{k+1}) + a_n b_n - a_m b_m.$$

For a sequence $\{\tau_k\}$, define the forward difference operator Δ by $\Delta\tau_k = \tau_{k+1} - \tau_k$. Then Abel’s lemma can be written as

$$\sum_{k=m}^{n-1} b_k \Delta a_k = - \sum_{k=m}^{n-1} a_{k+1} \Delta b_k + a_n b_n - a_m b_m. \tag{1}$$

Graham, Knuth and Patashnik [12] reformulated Abel’s lemma in terms of finite calculus to evaluate several sums on harmonic numbers. Recently, Chen, Hou and Jin [4] proposed *the Abel-Gosper method* and derived some identities on harmonic numbers. The idea can be explained as follows. Let f_k be a hypergeometric term, i.e., f_{k+1}/f_k is a rational function of k . First, we use Gosper’s algorithm [17] to find a hypergeometric term a_k (if it exists) satisfying $\Delta a_k = f_k$. Then, by Abel’s lemma, we have

$$\sum_{k=m}^{n-1} f_k H_k = \sum_{k=m}^{n-1} H_k \Delta a_k = - \sum_{k=m}^{n-1} \frac{a_{k+1}}{k+1} + a_n H_n - a_m H_m. \tag{2}$$

Hence we can transform a summation involving harmonic numbers into a hypergeometric summation. For example, let $S(n) = \sum_{k=1}^n H_k$, we have

$$S(n) = \sum_{k=1}^n H_k \Delta k = - \sum_{k=1}^n (k+1) \Delta H_k + (n+1)H_{n+1} - H_1 = (n+1)H_n - n.$$

In this framework, they combine both Abel’s lemma and Zeilberger’s algorithm to find recurrence relations for definite summations involving non-hypergeometric terms. For example, they can prove the Paule-Schneider identity [16]

$$\sum_{k=0}^n (1 + 3(n-2k)H_k) \binom{n}{k}^3 = (-1)^n,$$

and Calkin's identity [3]

$$\sum_{k=0}^n \left(\sum_{j=0}^k \binom{n}{j} \right)^3 = n2^{3n-1} + 2^{3n} - 3n2^{n-2} \binom{2n}{n}.$$

In this paper, we use the Abel-Gosper method to generalize the following well-known inversion formula (see, for example [11, (1.46)])

$$\sum_k (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n}. \tag{3}$$

To be specific, we have

Theorem 1. *Let $m, s, p, n \in \mathbb{N}$ and $n \geq p, m \geq 1$. Then*

$$\sum_{k=p}^n (-1)^{k-1} \binom{n}{k} \binom{k}{p} H_{mk+s}(x) = \begin{cases} \frac{(-1)^p m^{n-p-1} n!}{(n-p)p!} \sum_{i=1}^m \frac{1}{\prod_{u=p}^{n-1} (mu+s+x+i)}, & n > p, \\ (-1)^{p-1} H_{mp+s}(x), & n = p. \end{cases} \tag{4}$$

It is readily seen that identity (4) reduces to inversion formula (3) by setting $p = 0, m = 1, s = 0$ and $x = 0$. More interesting special cases of (4) can be found in Section 2.

In addition, by combining Abel's lemma with the WZ method, we establish the *Abel-WZ method* to construct identities on harmonic numbers from known hypergeometric identities. For example, we shall reestablish the following identity due to Prodinger [15].

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} H_k^{(2)} = 2 \sum_{k=1}^n \frac{(-1)^{k-1}}{k^2}.$$

The paper is organized as follows. In Section 2, we shall give a proof of Theorem 1 by the Abel-Gosper method. Special cases of Theorem 1 and more examples are also displayed. In Section 3, we introduce the Abel-WZ method and then construct many interesting identities on harmonic numbers from hypergeometric identities.

2. The Abel-Gosper Method

We first make use of the Abel-Gosper method to prove Theorem 1.

Proof of Theorem 1. Denote the left hand side of (4) by $S_{m,s,p}(n, x)$ and let

$$F(n, k) = (-1)^{k-1} \binom{n}{k} \binom{k}{p}.$$

By Gosper's algorithm, we have

$$F(n, k) = \Delta_k G(n, k),$$

where

$$G(n, k) = \frac{(-1)^k(k-p)}{n-p} \binom{n}{k} \binom{k}{p}.$$

Thus it follows that

$$S_{m,s,p}(n, x) = \sum_k \Delta_k G(n, k) H_{mk+s}(x).$$

Employing Abel's lemma and noticing the boundary values, we find

$$S_{m,s,p}(n, x) = -\frac{1}{n-p} \sum_k (-1)^{k-1} (k+1-p) \binom{n}{k+1} \binom{k+1}{p} \sum_{i=1}^m \frac{1}{mk+s+i+x}.$$

For $1 \leq i \leq m$, set

$$S_i(n) = \sum_k (-1)^{k-1} (k+1-p) \binom{n}{k+1} \binom{k+1}{p} \frac{1}{mk+s+i+x}.$$

Then Zeilberger's algorithm (see [17]) returns the recurrence equation

$$(mn+s+i+x)S_i(n+1) - m(n+1)S_i(n) = 0.$$

By the initial value

$$S_i(p+1) = \frac{(-1)^{p+1}(p+1)}{mp+s+i+x},$$

we obtain

$$S_i(n) = (-1)^{p+1} \frac{m^{n-p-1}n!}{p! \prod_{u=p}^{n-1} (mu+s+i+x)}.$$

Equation (1) is then established by noticing that

$$S_{m,s,p}(n, x) = -\frac{1}{n-p} \sum_{i=1}^m S_i(n), \quad n > p,$$

and $S_{m,s,p}(p) = (-1)^{p-1} H_{mp+s}(x)$. □

Now let us show some special cases of Theorem 1. By setting $m = 1$ and $x = 0$, (4) reduces to the following identity.

Corollary 1. For $n, p, s \in \mathbb{N}$ and $n > p$, we have

$$\sum_{k=p}^n (-1)^{k-1} \binom{n}{k} \binom{k}{p} H_{k+s} = \frac{(-1)^p \binom{p+s}{s}}{(n-p) \binom{n+s}{s}}. \tag{5}$$

The special cases $p = 0$ and $s = 0$ of (5) are given in [20, 21].

By setting $m = 2, s = 0$ and $x = 0$ in (4), we are led to the following identity .

Corollary 2. *For $n, p \in \mathbb{N}$ and $n > p$, we have*

$$\sum_{k=p}^n (-1)^{k-1} \binom{n}{k} \binom{k}{p} H_{2k} = \frac{(-1)^p}{(n-p)} \left(\frac{1}{2} + \frac{2^{2n-2p-2} \binom{2p}{p}}{\binom{2n-1}{n-1}} \right). \tag{6}$$

Using the relation $k^2 = 2\binom{k}{2} + \binom{k}{1}$ and the cases $p = 1, 2$ of (6), we arrive at an identity due to Sofo [19]:

$$\sum_k (-1)^{k-1} \binom{n}{k} k^2 H_{2k} = \frac{n}{2(n-1)(n-2)} + \frac{2^{2n-4}}{(n+2)\binom{2n-1}{n-3}}, \quad n > 2. \tag{7}$$

Note that we can also derive identities involving the generalized harmonic numbers $H_n^{(r)}$ from Theorem 1. To this end, we need the operators L and D which are defined by $L f(x) = f(0)$ and $D f(x) = f'(x)$. It is easy to see that

$$LD^m H_n(x) = (-1)^m m! H_n^{(m+1)}.$$

By setting $m = 1$ and $p = 0$ in (4), we get the following result (see [13]).

Corollary 3. *We have*

$$\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} H_{k+s}(x) = \frac{n!}{n(s+x+1)_n}. \tag{8}$$

Then applying the operator LD to both sides of (8), we obtain a formula given in [21]

$$\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} H_{k+s}^{(2)} = -\frac{1}{n} (H_s - H_{n+s}) \binom{n+s}{s}^{-1}. \tag{9}$$

Furthermore, applying the operator LD² to both its sides of (8) gives

$$\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} H_{k+s}^{(3)} = \frac{1}{2n} \left((H_{n+s} - H_s)^2 + H_{n+s}^{(2)} - H_s^{(2)} \right) \binom{n+s}{s}^{-1}.$$

More generally, (8) leads to the following inversion formula by applying the operator LD^m to both its sides.

Proposition 1. *For positive integers n and m , we have*

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} H_k^{(m+1)} = \frac{1}{n} \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_m \leq n} \frac{1}{j_1 j_2 \dots j_m}. \tag{10}$$

Proof. Setting $s = 0$ in (8) and applying the operator LD^m to its both sides, we have

$$(-1)^m m! \sum_k (-1)^{k-1} \binom{n}{k} H_k^{(m+1)} = \frac{n!}{n} LD^m \frac{1}{(x+1)_n}.$$

By the partial fraction decomposition

$$\frac{1}{(x+1)_n} = \sum_{k=1}^n \frac{1}{(x+k) \prod_{1 \leq j \neq k \leq n} (j-k)},$$

we find

$$\frac{n!}{n} LD^m \frac{1}{(x+1)_n} = (-1)^m m! \frac{1}{n} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1}}{k^m}.$$

Finally, using Dilcher's formula [10]

$$\sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1}}{k^m} = \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_m \leq n} \frac{1}{j_1 j_2 \dots j_m},$$

we arrive at (10). □

Similarly, we can use the Abel-Gosper method to find many other identities. Here are some examples.

Example 1. For $n \in \mathbb{N}$ and $x \in \mathbb{C} \setminus \{-1, -2, \dots\}$, we have

$$\sum_{k=0}^n \frac{(x+1)_k}{k!} H_k = \frac{1}{x+1} \left(1 + \frac{(x+1)_{n+1}}{n!} \left(H_n - \frac{1}{x+1} \right) \right),$$

$$\sum_{k=0}^n \frac{k!}{(x+1)_k} H_k = \begin{cases} \frac{1}{(x-1)^2} \left(x - \frac{n!}{(x+1)_n} ((x-1)(n+1)H_n + n+x) \right), & \text{if } x \neq 1, \\ \frac{H_{n+1}^2 - H_{n+1}^{(2)}}{2}, & \text{if } x = 1. \end{cases}$$

We remark that the second identity also holds when x is a negative integer. In this case, it is equivalent to the following formula (see [12, Exercise 6.53]).

$$\sum_{k=0}^n \frac{(-1)^k}{\binom{m}{k}} H_k = \frac{(-1)^n}{\binom{m}{n}} \left[\frac{n+1}{m+2} H_n + \frac{m+1-n}{(m+2)^2} \right] - \frac{m+1}{(m+2)^2},$$

where $m, n \in \mathbb{N}$ and $n \leq m$.

Example 2. For $n, m, p \in \mathbb{N}$, we have the following three identities.

$$\begin{aligned} \sum_{k=0}^n (-1)^{k-1} \frac{\binom{n}{k}}{\binom{k+p}{p}} H_{k+p} &= \frac{n - p(n+p)H_p}{(n+p)^2}, \\ \sum_{k=0}^n (-1)^{k-1} \frac{k \binom{n}{k}}{\binom{k+p}{p}} H_k &= \frac{pn(1 + H_{p-1} - H_{n+p-2})}{(n+p)(n+p-1)}, \quad p \geq 2, \\ \sum_{k=0}^n (-1)^{k-1} \frac{k^2 \binom{n}{k}}{\binom{k+p}{p}} H_k &= \frac{pn((n-p)(H_{n+p-3} - H_{p-1}) - (2n-p))}{(n+p)(n+p-1)(n+p-2)}, \quad p \geq 3. \end{aligned}$$

We remark that the first formula is due to Sofo [18] and the remaining two were obtained by Chu [7].

Using the Abel-Gosper method iteratively, we can prove the following identity.

Example 3. For $n, p \in \mathbb{N}$ and $n > p$, we have

$$\sum_k (-1)^{k-1} \binom{n}{k} \binom{k}{p} H_k^2 = \frac{(-1)^p}{n-p} (H_n - 2H_{n-p-1} + H_p).$$

3. The Abel-WZ Method

In this section, we shall illustrate how to combine Abel’s lemma with the WZ method to derive identities on harmonic numbers.

Recall that a pair of hypergeometric functions $(F(n, k), G(n, k))$ is called a WZ pair if the following WZ equation holds

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).$$

For a given $F(n, k)$, the WZ method will give such $G(n, k)$ if it exists: see for example [17]. Now we are ready to describe the Abel-WZ method. In most cases, for a hypergeometric identity

$$\sum_k F(n, k) = f(n),$$

we can obtain a corresponding WZ pair

$$\left(\frac{F(n, k)}{f(n)}, G(n, k) \right).$$

Let $S(n) = \sum_{k \geq 0} F(n, k)b_k$, where b_k is a harmonic number. Then we have

$$\frac{S(n+1)}{f(n+1)} - \frac{S(n)}{f(n)} = \sum_k (G(n, k+1) - G(n, k))b_k.$$

Denote by $U(n) = \sum_k (G(n, k + 1) - G(n, k))b_k$. Then by Abel's lemma, we have (here we omit the boundary values)

$$U(n) = - \sum_k G(n, k + 1)\Delta_k b_k.$$

Again, if $\Delta_k b_k$ is hypergeometric, $U(n)$ can be treated by Zeilberger's algorithm. Moreover, if $U(n)$ can be expressed in closed form, we then establish an identity of the form

$$S(n) = f(n) \sum_{k \leq n-1} U(k).$$

We begin by an identity due to Prodinger [15].

Example 4. For $n \in \mathbb{N}$, we have

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} H_k^{(2)} = 2 \sum_{k=1}^n \frac{(-1)^{k-1}}{k^2}. \tag{11}$$

Proof. Denote the left side of (11) by $S(n)$. For $F(n, k) = (-1)^{n-k} \binom{n}{k} \binom{n+k}{k}$, the WZ method gives

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k),$$

where

$$G(n, k) = \frac{2(-1)^{n-k} k^2 \binom{n}{k} \binom{n+k}{k}}{(n - k + 1)(n + 1)}.$$

Multiplying both sides of the WZ equation by $H_k^{(2)}$ and summing over k gives

$$S(n + 1) - S(n) = \sum_k (G(n, k + 1) - G(n, k))H_k^{(2)}.$$

Then applying Abel's lemma to the right hand side of the above identity and noting the boundary values, we have

$$\begin{aligned} S(n + 1) - S(n) &= \sum_k \frac{-G(n, k + 1)}{(k + 1)^2} \\ &= \sum_{k \geq 0} (T(k + 1) - T(k)) \\ &= 2 \frac{(-1)^n}{(n + 1)^2}, \end{aligned}$$

where

$$T(k) = \frac{2(-1)^{n-k-1} (k + 1)^2 \binom{n}{k+1} \binom{n+k+1}{k+1}}{(n - k)(n + 1)^3}.$$

Thus we have

$$S(n) = S(0) + 2 \sum_{k=1}^n \frac{(-1)^{k-1}}{k^2}.$$

By the initial value $S(0) = 0$, we complete the proof. □

The underlying hypergeometric identity of the above theorem is the special case $p = 0$ of

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k}{p} = (-1)^n \binom{n+p}{p} \binom{n}{p},$$

which enables us to establish the following identities.

Example 5. For $n, p \in \mathbb{N}$ and $n \geq p$, we have

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} H_{2k} &= 3H_n - H_{\lfloor \frac{n}{2} \rfloor}, \\ \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k}{p} H_k &= (-1)^n \binom{n+p}{p} \binom{n}{p} (2H_n - H_p), \\ \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k}{p} H_{n+k} &= (-1)^n \binom{n+p}{p} \binom{n}{p} (H_{n+p} + H_n - H_p). \end{aligned}$$

The cases $p = 0, 1$ of the last two formulas can be found in [15] and [14] respectively.

We conclude this paper by giving the following examples.

Example 6. From the binomial theorem $\sum_k \binom{n}{k} \lambda^{n-k} \mu^k = (\lambda + \mu)^n$, we can derive the following formula due to Boyadzhiev [2].

$$\sum_{k=1}^n \binom{n}{k} H_k \lambda^{n-k} \mu^k = (\lambda + \mu)^n H_n - \left(\lambda(\lambda + \mu)^{n-1} + \frac{\lambda^2}{2}(\lambda + \mu)^{n-2} + \dots + \frac{\lambda^n}{n} \right).$$

Example 7. From identity

$$\sum_{k=p}^n \binom{n}{k}^2 \binom{k}{p} = \binom{2n-p}{n} \binom{n}{p},$$

we can derive

$$\sum_{k=p}^n \binom{n}{k}^2 \binom{k}{p} H_k = \binom{2n-p}{n} \binom{n}{p} (2H_n - H_{2n-p}).$$

The special cases $p = 0$ and $p = 1$ are due to Paule and Schneider [16].

Example 8. From the identities

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$$

and

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3},$$

we have

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 H_k &= (-1)^n \binom{2n}{n} \frac{H_n + H_{2n}}{2}, \\ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 H_k &= (-1)^n \frac{(3n)!}{n!^3} \frac{H_n + 2H_{2n} - H_{3n}}{2}, \\ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 H_k^{(2)} &= (-1)^n \frac{(3n)!}{n!^3} \frac{H_n^{(2)} + H_{2n}^{(2)}}{2}. \end{aligned}$$

The last two formulas can be found in [9] and [8] respectively.

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