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RAMSEY FUNCTIONS FOR GENERALIZED PROGRESSIONS

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Abstract

Given positive integers m and k, a k-term semi-progression of scope m is a sequence $x_1, x_2, \ldots, x_k$ such that $x_{j+1} - x_j \in \{d, 2d, \ldots, md\}$, for $1 \leq j \leq k-1$, for some positive integer d. Thus an arithmetic progression is a semi-progression of scope 1. Let $S_m(k)$ denote the least integer for which every 2-coloring of $\{1, 2, \ldots, S_m(k)\}$ yields a monochromatic k-term semi-progression of scope m. We obtain an exponential lower bound on $S_m(k)$ for all $m = O(1)$. Our approach also yields a marginal improvement on the best known lower bound for the analogous Ramsey function for quasi-progressions, which are sequences whose successive differences lie in a small interval.

1. Introduction

In 1927, B.L. van der Waerden [6] proved that given positive integers r and k, there exists an integer $W(r,k)$ such that any r-coloring of $\{1, 2, \ldots, W(r,k)\}$ yields a monochromatic k-term arithmetic progression. Even after nearly 90 years, the gap between the lower and upper bounds on $W(r,k)$ remains enormous, with the best known lower bound of the order of r^k, whereas the best known upper bound is a five-times iterated tower of exponents (see [1]). Analogues of the Van der Waerden threshold $W(r,k)$ have been studied for many variants of arithmetic progressions, including semi-progressions and quasi-progressions (see [4]).

Let m and k be positive integers. A k-term semi-progression of scope m is a sequence $x_1, x_2, \ldots, x_k$ such that for some positive integer d, we have $x_{j+1} - x_j \in \{d, 2d, \ldots, md\}$. The integer d is called the *low-difference* of the semi-progression.

We define $S_m(k)$ as the least integer for which any 2-coloring of $\{1, 2, \ldots, S_m(k)\}$ yields a monochromatic k-term semi-progression of scope m. Note that $S_m(k) \leq W(k)$ with equality if $m = 1$.

2. An Exponential Lower Bound for $S_m(k)$

Landman [3] showed that $S_m(k) \geq (2k^2/m)(1 + o(1))$. We improve this to an exponential lower bound for all $m = O(1)$.

Theorem 1 Let $k \geq 3, m = O(1)$ and $\alpha = \sqrt{2^m/(2^m - 1)}$. Then $S_m(k) > \alpha^k$.

Proof. Let $f(N, k, m)$ denote the number of 2-colorings of $[1, N]$ with a monochromatic k-term semi-progression of scope m. (In the remainder of the proof, we only consider k-term semi-progressions of scope m.) Note that $S_m(k)$ is the least integer N such that $f(N, k, m) = 2^N$. We derive an upper bound on $f(N, k, m)$ as follows.

Given a semi-progression $P = (a_1, a_2, \ldots, a_k)$ of low-difference d, we define the *conjugate vector* of P as $(u_1, u_2, \ldots, u_{k-1})$ where $u_i = (a_{i+1} - a_i - d)/d$. Likewise, the *frequency vector* of P is defined as $\mathbf{v} = (v_0, v_1, \ldots, v_{m-1})$ where v_j is the number of times j occurs in the conjugate vector of P. Note that $\Sigma_{j=0}^{m-1} v_j = k - 1$. Finally, the *weight* of a frequency vector $\mathbf{v}$, denoted $w(\mathbf{v})$, is defined as $\sum_{j=0}^{m-1} j v_j$.

Given a coloring χ, we define the (a, d)*-primary semi-progression* of χ as the semi-progression P whose conjugate vector is lexicographically least among the conjugate vectors of all semi-progressions (with first term a and low-difference d) that are monochromatic under χ. Let $P = (a_1, a_2, \ldots, a_k)$ be a semi-progression with first term $a_1 = a$ and low-difference d. We will give an upper bound for the number of colorings χ such that P is the (a, d)*-primary semi-progression* of χ.

Since P is monochromatic, all elements of P have the same color under χ. Furthermore, if $(v_0, v_1, \ldots, v_{m-1})$ is the frequency vector of P, it follows from the fact that P is the (a, d)-primary semi-progression of χ that $w(\mathbf{v})$ elements in the arithmetic progression $\{a, a + d, \ldots, a + m(k - 1)d\}$ must be of the color different from the color of the elements of P. For example, let $a = 17, d = 5, m = 3, k = 6$ and $P = \{17, 32, 42, 47, 62, 72\}$ with conjugate vector $(2, 1, 0, 2, 1)$. If the two colors are red and blue, and the elements of P are all red, then 22, 27, 37, 52, 57 and 67 must all be blue. Indeed, if 57 is red, then the semi-progression $P' = \{17, 32, 42, 47, 57, 62\}$ would have a lexicographically lower conjugate vector $(2, 1, 0, 1, 0)$. Thus there are at most 2^{N-11} colorings of $[1, N]$ whose (a, d)-primary semi-progression is P.

Note that there are at most $N^2/(k - 1)$ choices for the pair (a, d). We say that two progressions P_1 and P_2 with the same a and d are equivalent if they have the same frequency vector. Note that for any a and d, there are at most

$$M(\mathbf{v}) = \frac{(v_0 + v_1 + \cdots + v_{m-1})!}{v_0! v_1! \cdots v_{m-1}!}$$

semi-progressions with frequency vector $(v_0, v_1, \ldots, v_{m-1})$. Adding over all the equivalence classes of semi-progressions, we obtain

$$f(N,k,m) \leq \frac{N^2 2^{N-k+1}}{k-1} \sum_{\substack{v_0,v_1,\ldots,v_{m-1} \geq 0 \\ v_0+v_1+,\ldots+v_{m-1}=k-1}} M(\mathbf{v}) 2^{-w(\mathbf{v})}$$

It follows from the multinomial theorem that

$$f(N,k,m) \leq \frac{N^2 2^N}{k-1} \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m} \right)^{k-1}$$

Thus $f(N,k,m) < 2^N$ for $N = \alpha_m^k$ where $\alpha_m = \sqrt{2^m/(2^m-1)}$, completing the proof. □

3. Exponential Lower Bounds for $Q_n(r,k)$

We now apply the same technique to quasi-progressions. A k-term quasi-progression of low difference d and diameter n is a sequence $(a_1, a_2, \ldots, a_k)$ such that $d \leq a_{j+1} - a_j \leq d + n, 1 \leq j \leq k-1$. Let $Q_n(r,k)$ denote the least positive integer such that any r-coloring of $\{1, 2, \ldots, Q_n(r,k)\}$ yields a monochromatic k-term quasi-progression of diameter n. It is known (see [5]) that $Q_1(2,k) > \beta^k$ where $\beta = 1.08226\ldots$. Indeed, β^4 can be expressed in terms of two algebraic numbers of degrees 2 and 3, respectively, and is the smallest positive real root of the equation $y^6 + 8y^5 - 112y^4 - 128y^3 + 1792y^2 + 1024y - 4096 = 0$. It is also known that $Q_n(2,k) = O(k^2)$ for $n > k/2$ (see [2]). We apply the techniques of the previous section to obtain lower bounds on $Q_n(r,k)$. Let $g(r,N,k,n)$ denote the number of r-colorings of $[1,N]$ with a monochromatic k-term semi-progression of diameter n. Note that $Q_n(r,k)$ is the least positive integer N such that $g(r,N,k,n) = r^N$. We first discuss the simplest non-trivial case, namely $r = 2$ and $n = 1$.

Theorem 2 Let $k \geq 3$. Then $Q_1(2,k) > \beta_{2,1}^k$ where $\beta_{2,1} = \sqrt{4 - 2\sqrt{2}} = 1.08239\ldots$.

Proof. We define the conjugate vector of a quasi-progression $Q = \{a_1, a_2, \ldots, a_k\}$ of low-difference d as $(u_1, u_2, \ldots, u_{k-1})$ where $u_i = a_{i+1} - a_i - d$. Given a coloring χ, we define the (a,d)-*primary quasi-progression* of χ as the quasi-progression Q whose conjugate vector is lexicographically least among the conjugate vectors of all quasi-progressions (with first term a and low-difference d) that are monochromatic under χ. Let $Q = \{a_1, a_2, \ldots, a_k\}$ be a quasi-progression with first term $a_1 = a$ and low-difference d. We give an upper bound for the number of colorings χ such that Q is the (a, d)-*primary quasi-progression* of χ.

Since Q is monochromatic, all elements of Q have the same color under χ, say red. Let $(u_1, u_2, \ldots, u_{k-1})$ be the conjugate vector of Q. Observe that if $u_j = 1$ and $u_{j+1} = 0$ for some j, so that $a_j, a_j + d + 1$ and $a_j + 2d + 1$ are elements of Q, and therefore red, it follows that the color of $a_j + d$ is different from red (say blue), as $(P \cup \{a_j + d\}) \setminus \{a_j + d + 1\}$ has a lexicographically lower conjugate vector. We define the weight of Q, denoted $w(Q)$, as the sum of the last element of the conjugate vector of Q, and the number of occurrences of the string "10" in the conjugate vector of Q. Note that in view of the above observation, the color of $w(Q)$ integers in the set $\{a, a+d, a+d+1, \ldots, a+(k-1)d, \ldots, a+(k-1)(d+1)\}$ can be inferred to be blue.

We now derive an upper bound on $g(2, N, k, 1)$. There are $N^2/(k-1)$ choices for the pair (a, d). Of the 2^{k-1} possible conjugate vectors for a quasi-progression with first term a and common difference d, let w_ℓ be the number of conjugate vectors of weight ℓ. Let

$$S_t = \sum_{\ell=0}^{\lceil t/2 \rceil} w_\ell 2^{-\ell}$$

denote the weighted sum of all such vectors of length t. Clearly, $S_t = S_{t,0} + S_{t,1}$ where $S_{t,0}$ and $S_{t,1}$ denote the weighted sum of conjugate vectors that begin with 0 and 1 respectively, with $S_{1,0} = 1$ and $S_{1,1} = 1/2$. It is easy to see that $A[S_{t-1,0}\ S_{t-1,1}]^T = [S_{t,0}\ S_{t,1}]^T$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1/2 & 1 \end{bmatrix}$$

Since $\lambda_{max}(A) = 1 + \frac{1}{\sqrt{2}}$, we get

$$g(2, N, k, 1) < \frac{N^2 2^{N-k+1}\left[\left(1+\frac{1}{\sqrt{2}}\right)^k + \left(1-\frac{1}{\sqrt{2}}\right)^k\right]}{2(k-1)}$$

Thus $g(2, N, k, 1) < 2^N$ for $N = \beta_{2,1}^k$ where $\beta_{2,1} = \sqrt{4 - 2\sqrt{2}} = 1.08239\ldots$ is the smallest positive real root of the equation $y^4 - 8y^2 + 8 = 0$. It follows that $Q_1(2, k) > \beta_{2,1}^k$ yielding a tiny improvement over the lower bound in [5]. $\square$

In general, since there are r^N r-colorings of $[1, N]$ and at most $N^2(n+1)^{k-1}$ k-term quasi-progressions of diameter n, a lower bound of the form $Q_n(r, k) \geq \left(\sqrt{r/(n+1)}\right)^k$ follows immediately from the linearity of expectation. However, this bound is only useful when $n \leq r - 2$. Generalizing the approach outlined earlier, we represent the conjugate vector of Q as an r-ary string, and define the weight $w(Q)$ as the sum of the last element of the conjugate vector of Q, and the number of occurrences of strings of length two of the form "xy", counted with multiplicity $m(x, y) = \min(x, n - y)$. (Note that $m(x, y)$ denotes the number of conjugate vectors that are lexicographically lower than the given vector and correspond to quasi-progressions that differ from Q in exactly one element.)

As before, let $S_{t,j}$ denote the weighted sum of conjugate vectors of length t beginning with $j, 0 \le j \le n$, with $S_{1,j} = \alpha^j$ for all j where $\alpha = 1 - \frac{1}{r}$. Then $A[S_{t,0} \ \cdots \ S_{t,n}]^T = [S_{t+1,0} \ \cdots \ S_{t+1,n}]^T$ where

$$A_{r,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha & \alpha & \cdots & \alpha & 1 \\ \alpha^2 & \alpha^2 & \cdots & \alpha & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^n & \alpha^{n-1} & \cdots & \alpha & 1 \end{bmatrix}$$

Note that the $(i, j)^{th}$ entry of the matrix $A_{r,n}$ is $\alpha^{m(i-1,j-1)} = \alpha^{\min(i-1,n+1-j)}$. Then $Q_n(r, k) > \beta^k$ where $\beta = \beta_{r,n} = \sqrt{r/\lambda_{max}(A_{r,n})}$. Note that for each r, there are only finitely many values for which $\beta_{r,n} > 1$. The first few such values are shown in the following table.

n	1	2	3	4	5	6
$\beta_{2,n}$	1.08239	< 1	< 1	< 1	< 1	< 1
$\beta_{3,n}$	1.28511	1.11226	1.02236	< 1	< 1	< 1
$\beta_{4,n}$	1.46410	1.24686	1.12770	1.05338	1.00384	< 1

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References

[1] W. T. Gowers, *A new proof of Szemerédi's theorem*. Geom. Funct. Anal. 11, 2001.

[2] A. Jobson, A. Kezdy, H. Snevily and S. C. White, *Ramsey functions for quasi-progressions with large diameter*. J. Comb. 2 (2011), 557-573.

[3] B. M. Landman, *Monochromatic sequences whose gaps belong to* $\{d, 2d, \ldots, md\}$. Bull. Aust. Math. Soc. 58 (1998), 93-101.

[4] B. M. Landman and A. Robertson, *Ramsey Theory on the Integers*. American Mathematical Society, Providence, 2004.

[5] S. Vijay, *On a variant of Van der Waerden's Theorem*. Integers 10 (2010), A17, 5pp. (electronic).

[6] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*. Nieuw Arch. Wiskd. 15 (1927), 212-216.