ERDŐS-SZEKERES RESULTS FOR SET PARTITIONS

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Abstract

We prove a Ramsey-theoretic result on set partitions of finite sets and a refinement based on the number of blocks in the set partition. A well-known bijection shows that our results are equivalent to results on finite sequences in the spirit of the Erdős-Szekeres theorem.

1. Background and Definitions

In their early work on Ramsey theory, Erdős and Szekeres established the following result:

Theorem 1 (Erdős-Szekeres [2]). *Any sequence of $(n-1)^2+1$ distinct numbers contains a monotonic (either increasing or decreasing) subsequence of length n. This bound is tight; i.e., there are sequences of $(n-1)^2$ distinct numbers not containing a montonic subsequence of length n.*

Our goal here is to establish an analogous result for set partitions, that is, to determine the minimum weight for a partition which will guarantee that the partition contains a monotonizable subpartition of a given weight. We start by specifying the meanings of the terms "weight", "subpartition", and "monotonizable". Throughout this paper, let $S \subset \mathbb{N}$ be finite. A *(set) partition* π of S is a collection $\{A_1, \ldots, A_k\}$ of pairwise disjoint nonempty subsets of S such that $\cup A_i = S$. We refer to the A_i's as *blocks*. We say that π has *weight* $|S|$ and refer to π as an $|S|$*-partition*. Weight will serve as our measure of size and will serve the same purpose in this paper as length does in the Erdős-Szekeres result.

A partition μ is said to be a *subpartition* of π if there exists $T \subseteq S$ such that μ is the set of all nonempty sets of the form $A_i \cap T$. In this case, we say that T *induces* μ and write $\mu = \pi|_T$. Terminology for partitions also applies to subpartitions, so, e.g., we say that μ is a $|T|$-*subpartition*. For example, the blocks of $\pi = 1378/2/49/56$ are 1378, 2, 49, and 56, and the weight of π is 9. If $T = \{2,5,6,9\}$ then $\pi|_T = 2/9/56$ and the weight of $\pi|_T$ is 4.

Here and elsewhere, when writing particular partitions, we omit commas and set braces and simply list the elements of each block and separate blocks with slashes. The *flattening* of such a representation of a partition is obtained by dropping slashes. For example, the flattening of 1378/2/49/56 is 137824956. Observe that the elements of each block of π in this example are in their natural order and the blocks of π are ordered by smallest element. This way of representating a partition is called its *canonical form*.

We say that a partition is *monotonizable* if it has no subpartition of the form ac/b, where $a < b < c$. Monotonizable partitions are precisely those from which one can obtain a monotonic sequence (either increasing or decreasing) by flattening the partition after imposing a suitable ordering on the blocks and their elements; specifically, the canonical ordering of a monotonizable partition flattens to an increasing sequence. A partition $A_1/\cdots/A_k$ in canonical form is monotonizable if and only if it has the property that $i < j$ implies every element of A_i is less than every element of A_j. For example, π in our earlier example is not monotonizable since 3 and 8 are in the same block and 4 is in a different block and $3 < 4 < 8$. On the other hand, $\pi|_T$ is monotonizable, since the canonical form of $\pi|_T$ is $2/56/9$. In this paper, monotonizability is analogous to monotonicity in the Erdős-Szekeres result.

We will consider the function $f(n)$ which is defined to be the least integer w such that that every w-partition has a monotonizable n-subpartition. That such an integer exists follows since any partition with at least n blocks or a block of weight at least n trivially has a monotonizable n-subpartition (i.e., $f(n) \leq (n-1)^2+1$). In Section 2, we establish the value of $f(n)$. In Section 3, we refine these results based on the number of blocks of the set partitions in question. In Section 4, we remind the reader of a well-known bijection between set partitions and finite sequences and show how this bijection can be used to reinterpret our results in the language of sequences. In Section 5, we examine those partitions just smaller than $f(n)$ that have no orderly subpartition of size n.

The following results are for partitions of arbitrary sets of numbers, but by relabeling we can always assume that a given partition of weight w consists of the consecutive numbers $1, 2, ..., w$.

2. Main Result

Let $\lfloor t \rfloor$ and $\lceil t \rceil$ denote the floor of t (i.e., the greatest integer less than or equal to t) and the ceiling of t (i.e., the least integer greater than or equal to t), respectively.

Let $g(n) = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$. In this section, we will prove that $f(n) = g(n)$. We begin by proving that $g(n)$ is an upper bound for $f(n)$. Let $[m] = \{1, \ldots, m\}$.

Lemma 1. *Every $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$-partition has a monotonizable n-subpartition.*

Proof. We will argue by induction that $f(n) \le g(n)$.

Partitions of weight $g(1) = 1$ and $g(2) = 2$ are monotonizable. Observe that $g(n) = g(n-2) + n$ for $n \ge 3$ and assume that every partition of $[g(n-2)]$ has a monotonizable $(n-2)$-subpartition. Let $\pi = B_1/\cdots/B_k$ be a partition of $[g(n)]$, not necessarily in canonical order, and suppose that $r = |B_1| \ge |B_j|$ when $2 \le j \le k$. We may assume $r < n$ or else B_1 itself provides the desired n-subpartition. Let $m = n - r + 2$. We consider two cases.

First, suppose that $|B_j \cap [m]| \ge 2$ for some j with $1 \le j \le k$, and let $C = [g(n)] \setminus (B_j \cup [m])$. Since $|B_j| \le r$, we have

$$|C| \ge g(n) - (r + m - 2) = g(n) - n = g(n-2).$$

By induction, $\pi|_C$ has a monotonizable $(n-2)$-subpartition, so $\pi|_{C\cup(B_j\cap[m])}$ is a monotonizable subpartition of π of weight at least n.

For the second case, suppose that $|B_j \cap [m]| \le 1$ for all j with $1 \le j \le k$. In particular, $|B_1 \cap [m]| \le 1$, so $|B_1 \setminus [m]| \ge r - 1$, so the partition

$$(B_1 \setminus [m])/(B_2 \cap [m])/\cdots/(B_k \cap [m])$$

(with all occurrences of the empty set deleted) is a monotonizable subpartition of π of weight at least $(r-1) + (m-1) = n$. □

Next, we will show that $g(n)$ is a lower bound for $f(n)$.

Lemma 2. *There exists a $\left(\left\lfloor \frac{(n+1)^2}{4} \right\rfloor - 1\right)$-partition that has no monotonizable n-subpartition.*

Proof. First, we prove that for any $k \in [n]$, there is a $(k(n-k+1)-1)$-partition with k blocks that has no monotonizable n-subpartition.

Let $\beta(n,k)$ be the partition of $[k(n-k+1)-1]$ into k blocks $\beta_m(n,k)$ for $1 \le m \le k$, where the elements of $\beta_m(n,k)$ are precisely those elements of $[k(n-k+1)-1]$ that are congruent to $m \mod k$. That is, $\beta_m(n,k)$ is the block

$$\beta_m(n,k) = \{m,\, m+k,\, m+2k,\, \ldots,\, m+(n-k)k\}$$

for $m = 1, \ldots, k-1$, and

$$\beta_k(n,k) = \{k,\, 2k,\, \ldots,\, (n-k)k\}.$$

For example, $\beta(5,3) = 147/258/36$ and $\beta(6,4) = 159/26\overline{10}/37\overline{11}/48$. Here and elsewhere, numbers having more than one decimal digit are overscored to avoid confusion. Note that $\beta(5,3)$ has no monotonizable 5-subpartition and that $\beta(6,4)$ has no monotonizable 6-subpartition.

Suppose μ is a monotonizable n-subpartition of $\beta(n,k)$. Consider the increasing sequence $a_1,\ldots,a_n$ obtained by flattening the canonical-form representation of μ. If a_i and a_{i+1} belong to the same block of μ, then $a_{i+1}-a_i \geq k$. If not, then $a_{i+1}-a_i \geq 1$.

Let q denote the number of indices i for which a_i and a_{i+1} are in the same block and let r denote the number of indices i for which a_i and a_{i+1} belong to different blocks, where $1 \leq i \leq n-1$. Then $q+r=n-1$ and $r \leq k-1$. Thus,

$$\begin{aligned}
k(n-k+1) > a_n &= (a_n-a_{n-1})+(a_{n-1}-a_{n-2})+\cdots+(a_2-a_1)+a_1\\
&\geq 1\cdot r + kq + 1\\
&= (n-1)k-(k-1)r+1\\
&\geq (n-1)k-(k-1)^2+1\\
&= k(n-k+1).
\end{aligned}$$

This contradiction shows that $\beta(n,k)$ has no monotonizable n-subpartition.

As a function of k, the expression $k(n-k+1)-1$ is maximized when $k=(n+1)/2$. Since k is an integer, the maximizing values are actually $\left\lfloor\frac{n+1}{2}\right\rfloor$ and $\left\lceil\frac{n+1}{2}\right\rceil$. Substituting either of these values for k in $k(n-k+1)-1$ gives $\left\lfloor\frac{(n+1)^2}{4}\right\rfloor-1$. Thus, $\beta\left(n,\left\lceil\frac{n+1}{2}\right\rceil\right)$ is a $\left(\left\lfloor\frac{(n+1)^2}{4}\right\rfloor-1\right)$-partition with no monotonizable n-subpartition. □

In Section 5, we will see that $\beta\left(n,\left\lceil\frac{n+1}{2}\right\rceil\right)$ is an instance of a wide class of $\left(\left\lfloor\frac{(n+1)^2}{4}\right\rfloor-1\right)$-partitions with no monotonizable n-subpartition.

Corollary 1. *Every $\left\lfloor\frac{(n+1)^2}{4}\right\rfloor$-partition has a monotonizable n-subpartition, and this is the least integer with this property.*

3. A Refinement Based on the Number of Blocks

In this section, we refine the previous section's results according to the number of blocks k of the partitions in question. Let $M(n,k)$ denote the least positive integer m such that an m-partition into exactly k blocks is guaranteed to have a monotonizable n-subpartition.

Theorem 2. *Let n and k be positive integers. Then*

$$M(n,k)=\begin{cases} k(n-k+1) & 1\leq k\leq (n+2)/2\\ \left\lfloor\frac{(n+1)^2}{4}\right\rfloor & (n+2)/2<k<n\\ k & k\geq n\end{cases}.$$

Proof. It is easy to see that $M(n,k) = k$ when $k \geq n$. Suppose $1 \leq k \leq (n+2)/2$. Lemma 2 shows that $M(n,k) \geq k(n-k+1)$. We will use induction on k to show that $M(n,k) \leq k(n-k+1)$.

If $k = 1$, then the unique partition of $[n]$ into a single block is a monotonizable n-subpartition of itself. For purposes of starting the induction, it is also useful to note that $\emptyset$ is a partition of $\emptyset$ into zero blocks.

Now assume that every partition of $[k(n-k+1)]$ into at most k blocks has a monotonizable n-subpartition whenever $n \geq 2(k-1)$. We show that if $n \geq 2((k+1)-1) = 2k$, then every partition of $[(k+1)(n-(k+1)+1)] = [(k+1)(n-k)]$ into at most $k+1$ blocks has a monotonizable n-subpartition.

Let $B_1/\cdots/B_p$ be such a partition, where $p \leq k+1$. Order the blocks so that B_1 is a block of maximum cardinality and let $r = |B_1|$. We may assume that $r < n$, since otherwise B_1 itself provides the desired subpartition. Let $m = n - r + 2$. We treat two cases.

First, suppose that $|B_j \cap [m]| \geq 2$ for some $j \leq p$. Let $C = [(k+1)(n-k)] \setminus (B_j \cup [m])$. Note that all elements of C are greater than m and C is disjoint from B_j. The original partition induces a partition on C into at most k blocks. Since $|B_j| \leq r$, we have

$$\begin{aligned} |C| &\geq (k+1)(n-k) - (r+m-2) \\ &= (k+1)(n-k) - n \\ &= k((n-2) - k + 1). \end{aligned}$$

Also, $n - 2 \geq 2(k-1)$. By the induction assumption the induced partition on C has a monotonizable subpartition μ of weight $n-2$. Then $\mu \cup \{B_j \cap [m]\}$ is a monotonizable subpartition of the original partition of weight at least n.

Next, suppose that $|B_j \cap [m]| \leq 1$ for all $j \leq p$. In particular, $|B_1 \cap [m]| \leq 1$, so $|B_1 \setminus [m]| \geq r - 1$. Also, all elements of $B_1 \setminus [m]$ are greater than m. So, the partition $(B_1 \setminus [m])/(B_2 \cap [m])/\cdots/(B_p \cap [m])$ (with all occurrences of the empty set deleted) is a monotonizable subpartition of weight at least $(r-1)+(m-1) = n$. This concludes the proof of the case in which $1 \leq k \leq (n+2)/2$.

Now suppose that $(n+2)/2 < k < n$. By Lemma 1, we know that $M(n,k) \leq \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$. To see that $M(n,k) \geq \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$, we will modify the partition $\beta(n, \lceil \frac{n+1}{2} \rceil)$ of Lemma 2 to create a partition $\gamma(n,k) = \gamma_1(n,k)/\cdots/\gamma_k(n,k)$ of $\left[\left\lfloor \frac{(n+1)^2}{4} \right\rfloor - 1\right]$ with k blocks that has no monotonizable n-subpartition. Specifically, let

$$\gamma_m(n,k) = \begin{cases} \beta_m(n, \lceil \frac{n+1}{2} \rceil) & 1 \leq m \leq \lceil \frac{n+1}{2} \rceil - 1 \\ \{\lceil \frac{n+1}{2} \rceil, \ldots, \lceil \frac{n+1}{2} \rceil (n-k)\} & m = \lceil \frac{n+1}{2} \rceil \\ \{\lceil \frac{n+1}{2} \rceil \left(n - k + m - \lceil \frac{n+1}{2} \rceil\right)\} & \lceil \frac{n+1}{2} \rceil < m \leq k \end{cases}.$$

For example, none of

$$\gamma(8,5) = 1\,6\,\overline{11}\,\overline{16}/2\,7\,\overline{12}\,\overline{17}/3\,8\,\overline{13}\,\overline{18}/4\,9\,\overline{14}\,\overline{19}/5\,\overline{10}\,\overline{15}$$
$$\gamma(8,6) = 1\,6\,\overline{11}\,\overline{16}/2\,7\,\overline{12}\,\overline{17}/3\,8\,\overline{13}\,\overline{18}/4\,9\,\overline{14}\,\overline{19}/5\,\overline{10}/\overline{15}$$
$$\gamma(8,7) = 1\,6\,\overline{11}\,\overline{16}/2\,7\,\overline{12}\,\overline{17}/3\,8\,\overline{13}\,\overline{18}/4\,9\,\overline{14}\,\overline{19}/5/\overline{10}/\overline{15}$$

has a monotonizable subpartition of weight 8, which shows that $M(8,k) \geq 20$ for $k = 5,6,7$.

We now prove that $\gamma(n,k)$ has no monotonizable n-subpartition. As in Lemma 2, suppose μ is a monotonizable n-subpartition of $\gamma(n,k)$ and consider the increasing sequence $a_1, \ldots, a_n$ obtained from the canonical-form representation of μ. If a_{i+1} and a_i differ by a multiple of $\lceil \frac{n+1}{2} \rceil$, then $a_{i+1} - a_i \geq \lceil \frac{n+1}{2} \rceil$. If not, then $a_{i+1} - a_i \geq 1$.

For $1 \leq i \leq n-1$, let q denote the number of indices i for which a_i and a_{i+1} differ by a multiple of $\lceil \frac{n+1}{2} \rceil$. Let $r = n - q - 1$ be the number of indices where this is not true, so that $r \leq \lceil \frac{n+1}{2} \rceil - 1$. Thus,

$$\begin{aligned}
\left\lfloor \frac{(n+1)^2}{4} \right\rfloor &> a_n \\
&= (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1) + a_1 \\
&\geq 1 \cdot r + \left\lceil \frac{n+1}{2} \right\rceil q + 1 \\
&= 1 \cdot r + \left\lceil \frac{n+1}{2} \right\rceil (n - r - 1) + 1 \\
&= \left\lceil \frac{n+1}{2} \right\rceil (n-1) - \left(\left\lceil \frac{n+1}{2} \right\rceil - 1 \right) r + 1 \\
&\geq \left\lceil \frac{n+1}{2} \right\rceil (n-1) - \left(\left\lceil \frac{n+1}{2} \right\rceil - 1 \right)^2 + 1 \\
&= \left\lceil \frac{n+1}{2} \right\rceil \left(n + 1 - \left\lceil \frac{n+1}{2} \right\rceil \right) \\
&\geq \left\lceil \frac{n+1}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor \\
&= \left\lfloor \frac{(n+1)^2}{4} \right\rfloor .
\end{aligned}$$

Thus $\gamma(n,k)$ has no monotonizable n-subpartition. □

4. Parallel Results for Sequences

A *restricted growth function* is a sequence $a_1 a_2 \ldots$ of natural numbers that satisfies $a_j \leq 1 + \max_{i<j} a_i$. There is a well-known bijection mapping r-partitions with k

blocks onto surjective restricted growth functions on $[k]$ of length r. Specifically, suppose that the partition $A_1/\cdots/A_k$ is in canonical form. Then $A_1/\cdots/A_k$ is mapped to the sequence $a_1\dots a_r$ where $a_j = m$ when $a_j \in A_m$. For example, $1378/2/49/56 \mapsto 121344113$. Under this map, each of the results from the previous section can be interpreted in terms of restricted growth functions. In fact, the results apply to all sequences, so we state them in this generality.

We say that a sequence $a_1\dots a_r$ is *separated* if $i \le m \le j$ and $a_i = a_j$ imply $a_m = a_i$. Equivalently, a separated sequence is one in which all like terms appear consecutively. A partition is monotonizable precisely when the corresponding sequence is separated. In this paper, subsequences need not be consecutive.

Theorem 3. *Every sequence of length* $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ *has a separated subsequence of length* n*, and this is the least integer with this property.*

Let $S(n,k)$ denote the least positive integer s such that every sequence of length s on exactly k letters has a separated subsequence of length n.

Theorem 4. *Let* n *and* k *be positive integers. Then*

$$S(n,k) = \begin{cases} k(n-k+1) & 1 \le k \le 1 + n/2 \\ \left\lfloor \frac{(n+1)^2}{4} \right\rfloor & 1 + n/2 < k < n \\ k & k \ge n \end{cases}.$$

5. n-Extremal Partitions and Sequences

In the case of the Erdős-Szekeres result, $(n^2 - 2n + 2) - 1 = (n-1)^2$ is the greatest length that a permutation can have without having a monotonic n-subsequence. Say that an $(n-1)^2$-permutation is *n-extremal* if it has no monotonic n-subsequence. How many n-extremal permutations are there?

The Robinson-Schensted-Knuth correspondence [4, 5, 6] provides a bijection from the set of n-extremal permutations to the set of pairs of $(n-1) \times (n-1)$ standard Young tableaux. This shows that the number of n-extremal permutations is a square. The hook length formula [3] can then be used to calculate the number of such tableaux.

By analogy, we say that a $\left(\left\lfloor \frac{(n+1)^2}{4} \right\rfloor - 1\right)$-partition is *$n$-extremal* if it has no monotonizable n-subpartition. It is natural to inquire about $t(n)$, the number of n-extremal partitions. A sequence is *n-extremal* if it is of length $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor - 1$ and has no separated n-subsequence. The function $t(n)$ also counts the number of n-extremal restricted growth functions. We computed the first six values of $t(n)$ by computer. The seventh and eighth values were computed by Butler and Graham[1];

they appear in the following table.

n	1	2	3	4	5	6	7	8
$t(n)$	1	1	1	4	9	121	1080	88788

Suggestively, $t(n)$ is square if $1 \le n \le 6$, which initially led us to conjecture that $t(n)$ is always a square. Butler and Graham's determination that $t(7) = 33^2 - 1$ and $t(8) = 88788 = 298^2 - 16$ resolved that conjecture in the negative. We consider the determination of the values of $t(n)$, exact or asymptotic, and the classification of n-extremal partitions to be interesting questions worthy of further investigation.

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