CONVERGENCE OF DAVID'S 2-DIMENSIONAL CONTINUED FRACTION

Fritz Schweiger
FB Mathematik, Universität Salzburg, Hellbrunnerstr, Salzburg, Austria
fritz.schweiger@sbg.ac.at

Received: 10/28/14, Revised: 8/11/15, Accepted: 9/19/15, Published: 10/13/15

Abstract

In 1949 and 1950, two papers by M. David on modifications of the Jacobi algorithm appeared. In a recent paper of Schweiger some comments were made on the periodicity problem. Here a proof of convergence (with one notable exception) is given. The underlying map of the algorithm is ergodic and admits an absolutely continuous invariant measure.

1. Introduction

This paper is a continuation of [6]. We first describe briefly one of David's algorithms ([2], [3]) in the framework of Schweiger's book ([5]). Let $x_1, x_2 \in \mathbb{R}$ such that $0 < x_1 \leq 1$, $0 \leq x_2 \leq 1$. Then we define

$$b + 1 = \left\lceil \frac{1}{x_1} \right\rceil, \, a = \left\lfloor \frac{x_2}{x_1} \right\rfloor$$

and the map T by

$$T(x_1, x_2) = (y_1, y_2) = \left(\frac{x_2}{x_1} - a, -\frac{1}{x_1} + b + 1 \right)$$

which is equivalent to

$$\begin{pmatrix} 1 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -1 & b+1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}.$$

Then we find $0 \leq a \leq b$, $1 \leq b$, and the equation $a = b$ implies $y_1 + y_2 \leq 1$.
If $x_1 + x_2 = 1$ then $y_1 + y_2 = 1$. The restriction of T to this line is just continued fraction algorithm for $t = x_1$. Therefore we exclude from our considerations all

points on this line and all points with $T^n(x_1,x_2)_1 = 0$ for some $n \geq 1$.
The inverse branches of the algorithm are given by the matrices

$$\beta(a,b) = \begin{pmatrix} b+1 & 0 & -1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix} = \begin{pmatrix} B_0^{(1)} & -B_0^{(-1)} & -B_0^{(0)} \\ B_1^{(1)} & -B_1^{(-1)} & -B_1^{(0)} \\ B_2^{(1)} & -B_2^{(-1)} & -B_2^{(0)} \end{pmatrix}.$$

Matrix multiplication gives the recursion relations

$$B_j^{(n)} = (b_n+1)B_j^{(n-1)} - a_n B_j^{(n-2)} - B_j^{(n-3)},$$

$$a_n = a(T^{n-1}(x_1,x_2)), b_n = b(T^{n-1}(x_1,x_2)),\ j = 0,1,2,\ n \geq 2.$$

In [6] the problem of periodicity of this algorithm was discussed. Here we show that the algorithm is convergent with one obvious exception. The main problem is related to the fact that not all of the entries of the matrices $\beta(a,b)$ are non-negative. We further prove the basic ingredients for an ergodic theory of the map T.

2. Proof of Convergence

The characteristic polynomial of the periodic expansion

$$(x_1,x_2) = \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix}$$

is

$$\phi(\lambda) = \lambda^3 - 2\lambda^2 + 1$$

which shows $\phi(1) = 0$. The two points (γ,γ^2) and $(1,1)$ which correspond to the eigenvalues $\lambda = \Gamma$ and $\lambda' = 1$ are invariant. Here $\Gamma > 1$ is the number of the Golden Section; it satisfies $\Gamma^2 = \Gamma + 1$ and $\Gamma\gamma = 1$. Therefore the whole segment between (γ,γ^2) and $(1,1)$ is invariant. The algorithm cannot be convergent at all points.

We first observe the following facts. Let

$$B^- = \{(x_1,x_2) \in B : x_1 + x_2 \leq 1\}$$

and

$$B^+ = \{(x_1,x_2) \in B : x_1 + x_2 \geq 1\}.$$

- As already mentioned, the line $x_1 + x_2 = 1$ is invariant under the map T. The restriction of T to this line is equivalent to the continued fraction map. Note that $q_s = B_0^{(s)} - B_0^{(s-1)}$ obeys the usual relation $q_s = b_s q_{s-1} + q_{s-2}$. Therefore we will not consider these points further.

- If $a_1(x) = b_1(x)$ then
$$y_1 + y_2 = \Big(\frac{x_2}{x_1} - b\Big) + \Big(b + 1 - \frac{1}{x_1}\Big) = \frac{x_1 + x_2 - 1}{x_1} < 1.$$
Therefore, $y = Tx \in B^-$.

- If $a_1(x) = b_1(x) - 1$ and $x \in B^-$ then
$$y_1 + y_2 = \frac{x_2 + 2x_1 - 1}{x_1} < 1.$$
Then again $y = Tx \in B^-$.

- If $a_1(x) = b_1(x) - 1$ then
$$|y_1 + y_2 - 1| = \frac{|x_1 + x_2 - 1|}{x_1}.$$
If $x_1 \leq \frac{1}{2}$ then $|y_1 + y_2 - 1| \geq 2|x_1 + x_2 - 1|$. If $a_g(x) = b_g(x) - 1$ for all $g \geq 1$ then $b_g \geq 2$ can appear only finitely often. Therefore, in this case, $a_g(x) = 0$ and $b_g(x) = 1$ for $g \geq g_0(x)$.

These considerations imply that the orbit of all points x which are not eventually excluded by the foregoing considerations will hit a cylinder with $2 \leq b$ and $a \leq b - 2$ infinitely often.

Theorem 1: *If $(a_n(x), b_n(x)) = (a, b)$ with $a \leq b - 2$, $2 \leq b$, infinitely often then the algorithm is convergent.*

Proof. : We first state an important property of this algorithm.
Let
$$\prod_{j=1}^{n} \beta(a_j, b_j) = \begin{pmatrix} B_0^{(n)} & -B_0^{(n-2)} & -B_0^{(n-1)} \\ B_1^{(n)} & -B_1^{(n-2)} & -B_1^{(n-1)} \\ B_2^{(n)} & -B_2^{(n-2)} & -B_2^{(n-1)} \end{pmatrix}.$$
Then the relations
$$B_i^{(n)} B_0^{(n-1)} - B_i^{(n-1)} B_0^{(n)} \geq 0, \; B_i^{(n)} B_0^{(n-2)} - B_i^{(n-2)} B_0^{(n)} \geq 0$$
hold for $i = 1, 2$ and $n \geq 1$.
Therefore the sequences $\frac{B_1^{(n)}}{B_0^{(n)}}$ and $\frac{B_2^{(n)}}{B_0^{(n)}}$ both are increasing. We denote their limit by $\overline{x} = (\overline{x}_1, \overline{x}_2)$. Now let $x = (x_1, x_2)$ be a point with an infinite expansion by this algorithm. Note that the points x and $\overline{x}$ have the same expansion.
If $T^g x = y$ then
$$x_1 = \frac{B_1^{(g)} - y_1 B_1^{(g-2)} - y_2 B_1^{(g-1)}}{B_0^{(g)} - y_1 B_0^{(g-2)} - y_2 B_0^{(g-1)}}$$

$$x_2 = \frac{B_2^{(g)} - y_1 B_2^{(g-2)} - y_2 B_2^{(g-1)}}{B_0^{(g)} - y_1 B_0^{(g-2)} - y_2 B_0^{(g-1)}}.$$

This implies

$$\frac{B_1^{(g)}}{B_0^{(g)}} < x_1$$

and

$$\frac{B_2^{(g)}}{B_0^{(g)}} < x_2.$$

Therefore, $\overline{x}_1 \leq x_1$ and $\overline{x}_2 \leq x_2$.

Since $\overline{x} = (\overline{x}_1, \overline{x}_2)$ is the limit of the sequence $\left(\frac{B_1^{(n)}}{B_0^{(n)}}, \frac{B_2^{(n)}}{B_0^{(n)}}\right)$ for any $\varepsilon > 0$ there is a number g_0 such that for all $g \geq g_0$

$$0 < \overline{x}_1 - \frac{B_1^{(g)}}{B_0^{(g)}} < \varepsilon$$

and

$$0 < \overline{x}_2 - \frac{B_1^{(g)}}{B_0^{(g)}} < \varepsilon.$$

In the following we restrict to the first coordinate. Then

$$\overline{x}_1 - \frac{B_1^{(g)}}{B_0^{(g)}} = \frac{\overline{y}_1(B_1^{(g)}B_0^{(g-2)} - B_1^{(g-2)}B_0^{(g)}) + \overline{y}_2(B_1^{(g)}B_0^{(g-1)} - B_1^{(g-1)}B_0^{(g)})}{B_0^{(g)}(B_0^{(g)} - \overline{y}_1 B_0^{(g-2)} - \overline{y}_2 B_0^{(g-1)})}.$$

In a similar way we have

$$x_1 - \frac{B_1^{(g)}}{B_0^{(g)}} = \frac{y_1(B_1^{(g)}B_0^{(g-2)} - B_1^{(g-2)}B_0^{(g)}) + y_2(B_1^{(g)}B_0^{(g-1)} - B_1^{(g-1)}B_0^{(g)})}{B_0^{(g)}(B_0^{(g)} - y_1 B_0^{(g-2)} - y_2 B_0^{(g-1)})}.$$

Now we consider the quotient

$$\frac{x_1 - \frac{B_1^{(g)}}{B_0^{(g)}}}{\overline{x}_1 - \frac{B_1^{(g)}}{B_0^{(g)}}} = \frac{B_0^{(g)} - \overline{y}_1 B_0^{(g-2)} - \overline{y}_2 B_0^{(g-1)}}{B_0^{(g)} - y_1 B_0^{(g-2)} - y_2 B_0^{(g-1)}}$$

$$\times \frac{y_1(B_1^{(g)}B_0^{(g-2)} - B_1^{(g-2)}B_0^{(g)}) + y_2(B_1^{(g)}B_0^{(g-1)} - B_1^{(g-1)}B_0^{(g)})}{\overline{y}_1(B_1^{(g)}B_0^{(g-2)} - B_1^{(g-2)}B_0^{(g)}) + \overline{y}_2(B_1^{(g)}B_0^{(g-1)} - B_1^{(g-1)}B_0^{(g)})}.$$

Assume $z = Ty$ and $\overline{z} = T\overline{y}$ such that

$$y_1 = \frac{1}{b + 1 - z_2}, \; y_2 = \frac{a + z_1}{b + 1 - z_2}$$

and

$$\overline{y}_1 = \frac{1}{b+1-\overline{z}_2},\ \overline{y}_2 = \frac{a+\overline{z}_1}{b+1-\overline{z}_2}$$

then clearly

$$\frac{y_1}{\overline{y}_1} \leq \frac{b+1}{b} \leq 2.$$

If $a \geq 1$ then in a similar way we also get

$$\frac{y_2}{\overline{y}_2} \leq \frac{(a+1)(b+1)}{ab} \leq 4.$$

If $a = 0$ we have

$$\frac{y_2}{\overline{y}_2} \leq \frac{z_1(b+1)}{\overline{z}_1 b} \leq 4.$$

This shows that the second factor of the product is bounded by 4.

Now we consider the first factor. Let $b_{g+1} \geq 2$ and $a_{g+1} \leq b_{g+1} - 2$. Then we will show that

$$\frac{B_0^{(g)} - \overline{y}_1 B_0^{(g-2)} - \overline{y}_2 B_0^{(g-1)}}{B_0^{(g)} - y_1 B_0^{(g-2)} - y_2 B_0^{(g-1)}} \leq 4.$$

We write $z = Ty$ and $\overline{z} = T\overline{y}$. Then

$$\frac{B_0^{(g)} - \overline{y}_1 B_0^{(g-2)} - \overline{y}_2 B_0^{(g-1)}}{B_0^{(g)} - y_1 B_0^{(g-2)} - y_2 B_0^{(g-1)}}$$

$$= \frac{b_{g+1}+1-z_2}{b_{g+1}+1-\overline{z}_2} \times \frac{(b_{g+1}+1-\overline{z}_2)B_0^{(g)} - (a_{g+1}+\overline{z}_1)B_0^{(g-1)} - B_0^{(g-2)}}{(b_{g+1}+1-\overline{z}_2)B_0^{(g)} - (a_{g+1}+z_1)B_0^{(g-1)} - B_0^{(g-2)}}.$$

The first factor is bounded by 2. If we assume that the second factor is also bounded by 2 then we have to estimate the expression

$$(b_{g+1}+1+\overline{z}_2-2z_2)B_0^{(g)} - (a_{g+1}+2z_1-\overline{z}_1)B_0^{(g-1)} - B_0^{(g-2)}$$

from below. Since the points x and $\overline{x}$ have the same expansion we use

$$2z_1 - \overline{z}_1 \leq 2 - \gamma$$

and

$$\overline{z}_2 - 2z_2 \geq -1 - \gamma.$$

We can assume $a_{g+1} = b_{g+1} - 2$ and the lower bound reduces to

$$(b_{g+1}-\gamma)B_0^{(g)} - (b_{g+1}-\gamma)B_0^{(g-1)} - B_0^{(g-2)}.$$

Since the points $(\overline{z}_1, \overline{z}_2)$ and (z_1, z_2) have the same expansion, it is easy to show that

$$2z_1 - \overline{z}_1 \leq 2 - \gamma$$

and

$$\overline{z}_2 - 2z_2 \geq -1 - \gamma.$$

Here we used the invariant segment between (γ, γ^2) and $(1,1)$ as the "worst case". If $a_g < b_g$ then for all $x \in B$ the inequality $B^{(g)} - x_2 B^{(g-1)} - x_1 B^{(g-2)} \geq 0$ holds and the proof would be complete.
If $a_g = b_g$ then we know that $a_{g-1} < b_{g-1} = d$. We insert

$$B_0^{(g)} = (d+1)B_0^{(g-1)} - (d-1)B_0^{(g-2)} - B_0^{(g-3)}$$

into the expression and obtain

$$(db_{g+1} - d\gamma + 2\gamma)B_0^{(g-1)} - (db_{g+1} - d\gamma - b_{g+1} + \gamma + 1)B_0^{(g-2)} - (b_{g+1} - \gamma)B_0^{(g-3)} \geq 0.$$

Note that $b_{g+1} \geq 2$.
This shows

$$0 < x_1 - \frac{B_1^{(g)}}{B_0^{(g)}} < 16\varepsilon.$$

□

3. Ergodic Theory of the Map

Theorem 2 *The underlying map is ergodic and admits an invariant density.*

Proof. Well-known techniques (here I will refer to [5] and to [4]) can be used if we show that the algorithm satisfies a *Rényi condition*. We first define a cylinder of rank g as the set

$$B((a_1, b_1), ..., (a_g, b_g)) = \{x : (a_1(x), b_1(x)) = (a_1, b_1), ..., (a_g(x), b_g(x)) = (a_g, b_g)\}.$$

The proof of convergence implies that for almost all points $x \in B$, the nested sequence of cylinders which contain the point x shrinks to this point. If we neglect a set of measure 0, we have to show the existence of a constant $C \geq 1$ such that for all points y and z with $y \in T^g B((a_1, b_1), ..., (a_g, b_g))$ and $z \in T^g B((a_1, b_1), ..., (a_g, b_g))$ the condition

$$\frac{B_0^{(g)} - y_1 B_0^{(g-1)} - y_2 B_0^{(g-2)}}{B_0^{(g)} - z_1 B_0^{(g-1)} - z_2 B_0^{(g-2)}} \leq C$$

is satisfied.
We first show that if $a_{g-1}(x) < b_{g-1}(x)$ and $a_g \le b_g - 2$, $b_g \ge 2$ then the estimate

$$\frac{B_0^{(g)}}{B_0^{(g)} - y_2 B_0^{(g-1)} - y_1 B_0^{(g-2)}} \le 3$$

holds. This inequality amounts to prove

$$0 \le 2B_0^{(g)} - 3y_2 B_0^{(g-1)} - 3y_1 B_0^{(g-2)}.$$

We insert

$$B_0^{(g)} = (b+1)B_0^{(g-1)} - aB_0^{(g-2)} - B_0^{(g-3)}$$

and obtain

$$0 \le (2b + 2 - 3y_2)B_0^{(g-1)} - (2a + 3y_1)B_0^{(g-2)} - 2B_0^{(g-3)}.$$

We insert $(y_1, y_2) = (1, 1)$ and obtain

$$0 \le (2b - 1)B_0^{(g-1)} - (2a + 3)B_0^{(g-2)} - 2B_0^{(g)}.$$

Since $a = a_g \le b_g - 2 = b - 2$, the inequalities $2a + 3 \le 2b - 1$ and $2 \le 2b - 1$ are true and therefore the original inequality is correct.

We now have to show that this combination will appear infinitely often in the expansion of almost all points. We introduce the set

$$\Lambda_g = \{(x_1, x_2) : a_{2j-1}(x) \le b_{2j-1}(x) - 2; a_{2j}(x) = b_{2j}(x) \,\text{for}\, 1 \le j \le g\}.$$

We will show

$$\lambda(\bigcap_g \Lambda_g) = 0.$$

We first note

$$\begin{pmatrix} d+1 & 0 & -1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix} \begin{pmatrix} b+1 & 0 & -1 \\ 1 & 0 & 0 \\ b & 1 & 0 \end{pmatrix} = \begin{pmatrix} db+d+1 & -1 & -(d+1) \\ b+1 & 0 & -1 \\ ab+a+1 & 0 & -a \end{pmatrix}.$$

Letting $x = T^2 y$, the chain rule gives the estimate

$$\frac{1}{(B_0^{(2g)} - x_1 B_0^{(2g-2)} - x_2 B_0^{(2g-1)})^3}$$

$$= \frac{1}{(B_0^{(2g-2)} - y_1 B_0^{(2g-4)} - y_2 B_0^{(2g-3)})^3 (b_{2g-1} b_{2g} + b_{2g-1} + 1 - x_1 - (b_{2g-1} + 1)x_2)^3}$$

$$\leq \frac{1}{(B_0^{(2g-2)} - y_1 B_0^{(2g-4)} - y_2 B_0^{(2g-3)})^3 (b_{2g-1} b_{2g})^3}.$$

By induction we obtain

$$\frac{1}{(B_0^{(2g)} - x_1 B_0^{(2g-2)} - x_2 B_0^{(2g-1)})^3} \leq \frac{1}{(b_1 b_3 ... b_{2g-1})^3 (b_2 b_4 ... b_{2g})^3}.$$

Then we obtain

$$\lambda(\Lambda_g) \leq \frac{1}{2} (\sum_{b}^{\infty} \sum_{d=2}^{\infty} \sum_{a=0}^{d-2} \frac{1}{b^3 d^3})^g.$$

Since

$$\sum_{b=1}^{\infty} \sum_{d=2}^{\infty} \sum_{a=0}^{d-2} \frac{1}{b^3 d^3} \leq \sum_{b=1}^{\infty} \sum_{d=2}^{\infty} \frac{1}{b^3 d^2} < 1,$$

the proof is complete.

□

References

[1] Bernstein, L., *The Jacobi–Perron Algorithm - Its Theory and Application, LNM* 207 (1971), Berlin/Heidelberg/New York: Springer Verlag.

[2] David, M., Sur un algorithme voisin de celui de Jacobi. *C. R. Math. Acad. Sci. Paris* 229 (1949), 965-967.

[3] David, M., Sur trois algorithmes associés à l'algorithme de Jacobi. *C. R. Math. Acad. Sci. Paris* 230 (1950), 1445-1446.

[4] Ito, S. & M. Yuri, Number theoretical transformations with finite range structure and their ergodic properties. *Tokyo J. Math.* 10 (1987), 1-32.

[5] Schweiger, F., *Multidimensional Continued Fractions*. Oxford Univ. Press, 2000.

[6] Schweiger, F., On David's 2-dimensional continued fraction. *Integers* 14 (2014), #A58, 1-9.