AN *N*-IN-A-ROW GAME

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Abstract

The ordinary n-in-a-row game is a positional game in which two players alternately claim points in $\mathbb{Z}^2$ with the winner being the first player to claim n consecutive points in a line. We consider a variant of the game, suggested by Croft, where the number of points claimed increases by 1 each turn, and so on turn t a player claims t points. Croft asked how long it takes to win this game. We show that, perhaps surprisingly, the time needed to win this game is $(1 - o(1))n$.

1. Introduction

A *positional game* is a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F} \subset \mathbb{P}(X)$. We call X the *board*, and the members $F \in \mathcal{F}$ are *winning sets*. The game is played by two players, Red and Blue, who alternately claim unclaimed points from the board. The first player to claim all points from a winning set wins. The *n-in-a-row game* is a positional game played on $\mathbb{Z}^2$ where the winning sets are any n consecutive points in a row, either horizontally, vertically or diagonally. A simple strategy stealing argument shows that any positional game is either a first player win or a draw. It is known [1] that for $n \leq 4$ the n-in-a-row game is a first player win, and a draw for $n \geq 8$. It is believed that for $n = 5$ the game is a first player win, and a draw for $n \geq 6$.

In this note we consider a related game. Given a function $f : \mathbb{N} \to \mathbb{N}$ we define the (n, f) *game* to be a positional game played on the same board with the same winning sets as n-in-a-row, however, now during the tth turn a player claims $f(t)$ points. By this we mean that in the first turn Red will claim $f(1)$ points, and in the second turn Blue will claim $f(2)$ points, and so on. We will think of the game as consisting of a series of timesteps, with the tth turn happening at time t.

The ordinary n-in-a-row game corresponds to the $(n, 1)$ game, where 1 is the constant function taking value 1. In this note we will consider the (n, ι) game, where ι is the identity function. Unlike the n-in-a-row game the (n, ι) game is not

(with perfect play) a draw, since at time n some player will claim n points and so can claim a winning set. A small case analysis shows that player 1 wins for $n = 1, 3, 4, 6, 7$ and player 2 wins for $n = 2, 5$, and so a strategy stealing argument cannot apply. Since the game is never a draw, for each n, either the first or second player must have a winning strategy, Croft [3] asked two questions about this game:

Question 1. For general n, is the (n, ι) game a first or second player win?

Question 2. How long does it take for that player to win?

Our aim is to provide a partial answer to Question 2 by showing that neither player can win in time less than $(1 - o(1))n$. Question 1 is still an open problem. In fact, we prove a stronger result by considering the Maker-Breaker version of the game.

A *Maker-Breaker game* is a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F} \subset \mathbb{P}(X)$, and as before we call X the board and the members $F \in \mathcal{F}$ winning sets. Two player, Maker and Breaker, alternately claim unclaimed points from the board, Maker colouring his points red and Breaker blue. If Maker is able to claim all points from a winning set he wins, otherwise Breaker wins. For more on Maker-Breaker games see the monograph of Beck [1].

The *Maker-Breaker (n, f) game* is played on the same board with the same winning sets as the (n, f) game. It is obvious that the Maker-Breaker (n, ι) game is a Maker win. We will consider the question of how long it takes for Maker to win. More formally given a strategy Φ for Breaker and a winning strategy Ψ for Maker, at some time $T(\Phi, \Psi)_n$ Maker will first fully occupy a winning set. We let $T_n = \max_\Phi \min_\Psi T(\Phi, \Psi)_n$, that is, T_n is the first time at which, with perfect play, Maker is guaranteed to have won. It is simple to see that

$$2\sqrt{n} - 1 \leq T_n \leq n + 1. \tag{1}$$

the lower bound follows since before $t = 2\sqrt{n} - 1$ Maker has not claimed n points in total. In Section 2 we show that, perhaps surprisingly, Breaker has a strategy that gives $T_n \geq (1 - o(1))n$.

Theorem 1. *We have $T_n \geq n - o(\sqrt{n} \log n)$.*

This strategy will give similar lower bounds for the ordinary (non Maker-Breaker) (n, ι) game when adopted by either player.

2. Proof of Theorem 1

When we refer to properties of the game "at time t", we will mean immediately prior to the turn where t points are claimed.

Proof of Theorem 1. We first cover $\mathbb{Z}^2$ with a family of line segments of length $2n$. We let

$$F_{i,j} = \{(x,i) \,:\, jn \leq x \leq (j+2)n - 1\},$$
$$G_{i,j} = \{(i,y) \,:\, jn \leq y \leq (j+2)n - 1\},$$
$$H_{i,j} = \{(i+k,k) \,:\, jn \leq k \leq (j+2)n - 1\},$$
$$I_{i,j} = \{(i+k,-k) \,:\, jn \leq k \leq (j+2)n - 1\},$$

and take

$$\mathcal{F} = \{F_{i,j}, G_{i,j}, H_{i,j}, I_{i,j} : i,j \in \mathbb{Z}\}.$$

We call a segment *good* if Maker can no longer claim a winning line that is contained in that segment, otherwise the segment is *bad*. Breaker's strategy is as follows: At time $2t$ he identifies the t bad segments $F_1, F_2, \ldots, F_t \in \mathcal{F}$ with the most red points in them. Given such a segment F_i, without loss of generality let us suppose it is the segment $\{(x,0) \,:\, 0 \leq x \leq 2n - 1\}$, let x_i be the largest point (in terms of the first co-ordinate) between $(0,0)$ and $(n-1,0)$ which is either blue or unclaimed. Similarly, let y_i be the smallest point between $(n,0)$ and $(2n-1,0)$ which is either blue or unclaimed. For each i Breaker claims the points x_i and y_i, if either of these points are already claimed he takes an arbitrary move instead, note that these extra moves can only help Breaker.

We first note that Breaker can always perform this strategy. Indeed, if Maker has not yet won, then there must be at least one blue or unclaimed point between $(0,0)$ and $(n-1,0)$ and similarly between $(n,0)$ and $(2n-1,0)$. Secondly, we claim that after Breaker has claimed both x_i and y_i the set F_i is good. This will follow from the following lemma.

Lemma 1. *The segment $F_i = \{(x,0) \,:\, 0 \leq x \leq 2n - 1\}$ is good if and only if there is a blue point x_i between $(0,0)$ and $(n-1,0)$ and a blue point y_i between $(n,0)$ and $(2n-1,0)$ such that $||y_i - x_i|| < n$.*

Proof. Clearly, if no such x_i and y_i exist then there is a line of at least n points in a row which are either red or unclaimed, and hence Maker can still claim a winning line contained in F_i. Conversely, suppose such an x_i and y_i exist. Any winning line contained in the segment must start between $(0,0)$ and $(n,0)$. However, if it started between $(0,0)$ and x_i then it would contain the blue point x_i and if it started between x_i and $(n,0)$ then it would contain the blue point y_i, since $||y_i - x_i|| < n$. Therefore, every winning line contained in the segment contains either x_i or y_i and so if Breaker has claimed them both the segment is good. $\square$

Since Maker has not yet won we must have that, for the x_i and y_i defined by the strategy, $||y_i - x_i|| < n$. Therefore, by Lemma 1, after Breaker has claimed both x_i and y_i, each F_i is good. Note that, since each F_i was bad before Breaker's turn, on the $2t^{\text{th}}$ turn at least t segments become good.

We note that every point $v \in \mathbb{Z}^2$ is in 8 segments and also every winning line is contained in some segment. For a given play of the game, we define

$$\mathcal{A}_r^t = \{F \in \mathcal{F} : F \text{ is bad and the number of red points in } F \text{ at time } t \text{ is at least } r\}.$$

Suppose that Maker wins with the tth move such that $t < n - \delta\sqrt{n}\log n$, for some $\delta > 0$. Then we must have that $|\mathcal{A}_{n-t}^t| > 0$. Note that t is odd. We claim that for all integers $C \leq \min\{\frac{n-t}{\log n}, \frac{t}{8}\}$

$$\left|\mathcal{A}_{(n-t)-C\log n}^{t-2C}\right| > C\frac{t}{4} - C\frac{8t}{\log n}.$$

The claim clearly holds for $C = 0$. Suppose that it holds for a given value of $C < \min\{\frac{n-t}{\log n}, \frac{t}{8}\}$. Then we must have

$$\left|\mathcal{A}_{(n-t)-C\log n}^{t-2C-1}\right| > (C+1)\frac{t}{4} - C\frac{8t}{\log n}$$

since Breaker will make $\frac{t-2C-1}{2} \geq \frac{t}{4}$ of the $F \in \mathcal{F}$ good with his turn. This inequality holds since $C < \frac{t}{8}$. We claim that now

$$\left|\mathcal{A}_{(n-t)-(C+1)\log n}^{t-2C-2}\right| > (C+1)\frac{t}{4} - (C+1)\frac{8t}{\log n},$$

where, since $C < \frac{n-t}{\log n}$, the left hand side is defined. Indeed, since each point is in 8 of the $F \in \mathcal{F}$ then by claiming $t - 2C - 2$ points Maker can only claim $\log n$ points in at most $8\frac{t-2C-2}{\log n} \leq 8\frac{t}{\log n}$ segments. Therefore, the claim holds for all $C \leq \min\{\frac{n-t}{\log n}, \frac{t}{8}\}$.

Now if $\min\{\frac{n-t}{\log n}, \frac{t}{8}\} = \frac{t}{8}$, then $t \leq \frac{8n}{\log n}$, so we conclude that, with $C = \frac{t}{16}$, which for ease of presentation we assume to be an integer,

$$\left|\mathcal{A}_{\frac{n}{4}}^{\frac{7t}{8}}\right| \geq \left|\mathcal{A}_{(n-t)-\frac{t}{16}\log n}^{t-\frac{t}{8}}\right| > \frac{t^2}{64} - \frac{t^2}{2\log n} > \frac{t^2}{128},$$

for large enough n. But in order to claim at least $\frac{n}{4}$ points in at least $\Omega(t^2)$ of the $F \in \mathcal{F}$, Maker must have claimed at least $\frac{1}{8}\Omega(t^2)\frac{n}{4} = \Omega(nt^2)$ points. However, by time $\frac{7t}{8}$ Maker has claimed at most $O(t^2)$ points, a contradiction.

Similarly, in the case where $\min\{\frac{n-t}{\log n}, \frac{t}{8}\} = \frac{n-t}{\log n}$, we conclude that, with $C = \frac{n-t}{2\log n}$, which again we will assume to be an integer,

$$\left|\mathcal{A}_{\frac{(n-t)}{2}}^{t-2C}\right| \geq \left|\mathcal{A}_{(n-t)-\frac{n-t}{2\log n}\log n}^{t-2C}\right| > \left(\frac{n-t}{2\log n}\right)\left(\frac{t}{4}\right) - \left(\frac{n-t}{2\log n}\right)\left(\frac{8t}{\log n}\right) \geq \frac{(n-t)t}{16\log n},$$

for large enough n. But in order to claim at least $\frac{(n-t)}{2}$ points in at least $\Omega\left(\frac{(n-t)t}{\log n}\right)$ of the $F \in \mathcal{F}$, Maker must have claimed at least $\frac{1}{8}\Omega\left(\frac{(n-t)t}{\log n}\right)\frac{(n-t)}{2} = \Omega\left(\frac{(n-t)^2 t}{\log n}\right)$

points. The minimum of $\frac{(n-t)^2 t}{\log n}$ for $t \in [2\sqrt{n} - 1, n - \delta\sqrt{n}\log n]$, the lower bound here is from (1), is at $t = n - \delta\sqrt{n}\log n$, for large enough n, and so at time $t - 2C$ Maker must have claimed at least $\Omega(n^2 \log n)$ points. However, in the entire game Maker will claim at most $O(n^2)$ points, a contradiction. $\qquad\square$

We mention that the same strategy would show that Maker cannot win significantly before the time at which he is playing n points in a turn, even if at time $2t$ Maker claimed $100t$ points and at time $2t + 1$ Breaker claimed $0.01t$ points. Indeed, this strategy only uses the fact that both $f(2t)$ and $f(2t+1)$ are linear in t. Hence the same strategy will show that the (n, f) game has $T_n \geq n/a - o(\sqrt{n}\log n)$ as long as $f(2t) = at + b$ and $f(2t+1) = ct + d$ for $a, b, c, d \in \mathbb{R}^+$.

Kane noted [4] that a more careful analysis of the above strategy of Breaker's gives a bound of $T_n \geq n - O(\log n)$. In a paper in preparation with Mark Walters we analyse this strategy more closely in order to apply it to a more general class of games. Essentially the strategy can be viewed as comparing the game to an auxiliary "ball and bin" game, as in the classical paper of Chvátal and Erdős [2], where Maker gets to place balls spread out over some bins on each turn and Breaker gets to pick a bin and remove it from the game, with the aim being to minimise the number of balls in the bin with the most balls in at the end of the game. One can find optimal strategies for both Maker and Breaker in this game, and in fact Breaker's is the greedy strategy described in this paper, and from that deduce a simple bound for the number of balls in the largest bin at the end of the game.

From this work it follows that $T_n \geq n - O(\log n)$ is the best bound achievable with this strategy. It would be interesting to know if the bound could be improved, or if a strategy for Maker can be found to prove a corresponding upper bound on T_n.

As mentioned in the introduction, since the analysis of this strategies would be unchanged, up to a small constant, if Breaker were to play first, both of these strategies can be used by the losing player in the (n, ι) game and so the lower bounds on T_n are also applicable to the (n, ι) game.

References

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