A NON-CLASSICAL QUADRATIC FORM OF HESSIAN DISCRIMINANT 4 IS UNIVERSAL OVER $\mathbb{Q}(\sqrt{5})$

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Abstract

An adaptation of a quaternionic proof of the Sum of Four Squares Theorem over $\mathbb{Q}(\sqrt{5})$ is used to show that a particular non-classical quaternary quadratic form is universal.

1. Introduction and Notation

Universality of a quadratic form is a property of interest in number theory. It means that every totally positive integer of the number field can be represented by the quadratic form with arguments from the algebraic integers of that field. In previous papers the author was able to demonstrate the universality of certain quadratic forms over a number field of low discriminant. In one case the property was demonstrated for a non-classical form, that is a form whose cross product terms are not all even. See Deutsch [3, 4, 5, 6] for the specific examples. Here, another non-classical form of smaller "size" is shown to be universal over the ring of integers of $\mathbb{Q}(\sqrt{5})$.

The demonstration uses certain rings of quaternions. We use the notations of Deutsch [4, 6] in describing quaternions and related algebraic structures. An abbreviated review of notation is proper at this point. Bold characters are quaternions, while an overline ($\overline{}$) denotes a quaternion conjugate. N stands for the quaternion norm, also known as the reduced norm. It is the product of a quaternion with its conjugate. In this case, the order of multiplication is irrelevant.

Greek letters are used for elements of real quadratic fields, and a superscript star (*) is used for the quadratic field conjugate. We are particularly concerned with the algebraic integers in $\mathbb{Q}(\sqrt{5})$, and use the notation $O(\sqrt{5})$ for this ring. For R a ring with unity, the R–module over the quantities $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ is denoted $R[\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n]$. See Baake and Moody [1], Vignéras [8] and Deutsch [4, 6] for further notations and definitions. Also Lee [7] has determined all universal classical quaternary quadratic forms over $\mathbb{Q}(\sqrt{5})$.

2. Hessian Discriminant

The discriminant of a quadratic form $f(x_1, x_2, \ldots, x_n)$ is defined as the determinant of a particular matrix constructed out of the coefficients of f. Let

$$f(x_1, x_2, \ldots, x_n) \;=\; \sum_{\substack{i,\, j=1 \\ i \le j}}^{n} a_{i,j}\; x_i\; x_j\,. \tag{2.1}$$

Then the associated matrix $M = (m_{i,j})$ has coefficients

$$m_{i,i} \;=\; a_{i,i}\,, \text{for } \; i = 1, \ldots, n \text{ and } m_{i,j} \;=\; \frac{a_{i,j}}{2}\,, \text{for } \; i \ne j,\, 1 \le i, j \le n\,. \tag{2.2}$$

The discriminant of f is defined to be the determinant of M. Here we limit consideration to the case where all the coefficients are in $\mathbb{Z}$. The process produces rational integers for classical quadratic forms, that is, those that have even coefficients on the cross products. For non-classical quadratic forms the coefficients of the cross products are integral but not all even. In the general case we may end up with the discriminant equal to a fraction. As an alternative, we define the Hessian discriminant as the determinant of the Hessian of f.

$$\text{H-disc}\,(f) \;=\; \det\left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right)_{i,\, j\,=\,1,\, \ldots,\, n}\,. \tag{2.3}$$

The notation H-disc or H-discriminant can be used. The matrix has integral entries for non-classical quadratic forms. Its determinant is integral, and the value is equal to 2^n times the traditional discriminant of f. Under this definition the sum of four squares has the H-discriminant 16, while the G function of Deutsch [6] has H-discriminant 9. Recall

$$G(x,\, y,\, z,\, w) \;=\; x^2 \,+\, xy \,+\, y^2 \,+\, z^2 \,+\, zw \,+\, w^2\,. \tag{2.4}$$

Another nonstandard definition of discriminant has been made for ternary quadratic forms. See Berkovich [2] for details.

3. The Quadratic Form *K*

A computer search uncovered some quadratic forms of small Hessian discriminant. Investigation showed that at least one of these forms was likely to be universal. Set

$$K(x,\, y,\, z,\, w) \;=\; x^2 \,+\, y^2 \,+\, z^2 \,+\, w^2 \,+\, xy \,-\, xz \,-\, xw\,. \tag{3.1}$$

An easy computation shows that the Hessian discriminant of K is 4. It is helpful to note that K is a norm form. Recall $\mathbf{h} = (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})/2$. Computer algebra shows the following Lemma.

Lemma 1 *We have*

$$K(x,\, y,\, z,\, w) \;=\; N(x + y\,\mathbf{h} + z\,\mathbf{jh} + w\,\mathbf{kh})\,. \tag{3.2}$$

For R any ring with unity contained in the reals $\mathbb{R}$, $R\,[1,\, \mathbf{h},\, \mathbf{jh},\, \mathbf{kh}]$ is closed under multiplication by Table I. Hence it is a subring of the quaternions.

	$\mathbf{1}$	$\mathbf{h}$	$\mathbf{jh}$	$\mathbf{kh}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{h}$	$\mathbf{jh}$	$\mathbf{kh}$
$\mathbf{h}$	$\mathbf{h}$	$-\mathbf{1}+\mathbf{h}$	$\mathbf{1}-\mathbf{h}+\mathbf{jh}+\mathbf{kh}$	$-\mathbf{1}-\mathbf{jh}$
$\mathbf{jh}$	$\mathbf{jh}$	$-\mathbf{1}-\mathbf{kh}$	$-\mathbf{1}-\mathbf{jh}$	$-\mathbf{1}+\mathbf{h}-\mathbf{jh}-\mathbf{kh}$
$\mathbf{kh}$	$\mathbf{kh}$	$\mathbf{1}-\mathbf{h}+\mathbf{jh}+\mathbf{kh}$	$\mathbf{1}-\mathbf{h}$	$-\mathbf{1}-\mathbf{kh}$

Table 1: Multiplication Table for $\mathbb{Z}[1,\, \mathbf{h},\, \mathbf{jh},\, \mathbf{kh}]$

Recall that the norm of the product of two quaternions is the product of the norms of each element. Computer algebra yields the following Lemma.

Lemma 2 *Let*

$$\mathbf{q}_1 \;=\; a_1 + b_1\,\mathbf{h} + c_1\,\mathbf{jh} + d_1\,\mathbf{kh} \quad \textit{and} \quad \mathbf{q}_2 \;=\; a_2 + b_2\,\mathbf{h} + c_2\,\mathbf{jh} + d_2\,\mathbf{kh} \tag{3.3}$$

be quaternions in the ring $\mathbb{R}\,[1,\, \mathbf{h},\, \mathbf{jh},\, \mathbf{kh}]$*. Then their product is*

$$\mathbf{q} \;=\; \mathbf{q}_1 \cdot \mathbf{q}_2 \;=\; A + B\,\mathbf{h} + C\,\mathbf{jh} + D\,\mathbf{kh} \tag{3.4}$$

with

$$\begin{aligned} A \;&=\; -\,d_1\,d_2 \,-\, c_1\,d_2 \,-\, b_1\,d_2 \,+\, c_2\,d_1 \,+\, b_2\,d_1 \,-\, c_1\,c_2 \,+\, b_1\,c_2 \\ &\quad\; -\,b_2\,c_1 \,-\, b_1\,b_2 \,+\, a_1\,a_2\,, \\ B \;&=\; c_1\,d_2 \,-\, c_2\,d_1 \,-\, b_2\,d_1 \,-\, b_1\,c_2 \,+\, b_1\,b_2 \,+\, a_1\,b_2 \,+\, a_2\,b_1\,, \\ C \;&=\; -\,c_1\,d_2 \,-\, b_1\,d_2 \,+\, b_2\,d_1 \,-\, c_1c_2 \,+\, b_1\,c_2 \,+\, a_1\,c_2 \,+\, a_2\,c_1\,, \\ D \;&=\; -\,d_1\,d_2 \,-\, c_1\,d_2 \,+\, a_1\,d_2 \,+\, b_2\,d_1 \,+\, a_2\,d_1 \,+\, b_1\,c_2 \,-\, b_2\,c_1\,. \end{aligned} \tag{3.5}$$

The equation $N(\mathbf{q}) = N(\mathbf{q}_1) \cdot N(\mathbf{q}_2)$ *becomes*

$$K\,(A,\, B,\, C,\, D) \;=\; K\,(a_1,\, b_1,\, c_1,\, d_1) \cdot K\,(a_2,\, b_2,\, c_2,\, d_2)\,. \tag{3.6}$$

4. The Subring $\mathbb{I}_K$ of the Icosians

We recall the ring of icosians $\mathbb{I}$ as defined in Vignéras [8]. Let $\tau = (1+\sqrt{5})/2$. Then τ is the fundamental unit of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Z}[1,\, \tau]$ are the algebraic integers of $\mathbb{Q}(\sqrt{5})$. The conjugate of τ is denoted τ^* and equals $(1-\sqrt{5})/2$. We have

$$\tau^{-1} \;=\; \frac{\sqrt{5}-1}{2} \;=\; -\,\tau^*\,. \tag{4.1}$$

The ring of icosians is the module $O(\sqrt{5})\,[\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4]$ where

$$\begin{aligned} \mathbf{e}_1 &= \tfrac{1}{2}(1+\tau^{-1}\mathbf{i}+\tau\mathbf{j})\,, \quad \mathbf{e}_2 = \tfrac{1}{2}(\tau^{-1}\mathbf{i}+\mathbf{j}+\tau\mathbf{k})\,, \\ \mathbf{e}_3 &= \tfrac{1}{2}(\tau\mathbf{i}+\tau^{-1}\mathbf{j}+\mathbf{k})\,, \quad \mathbf{e}_4 = \tfrac{1}{2}(\mathbf{i}+\tau\mathbf{j}+\tau^{-1}\mathbf{k})\,. \end{aligned} \tag{4.2}$$

We recall that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are elements of $\mathbb{I}$. See Vignéras [8] and Deutsch [6] for more details.

Definition 3 $\mathbb{I}_K$ is the module $O(\sqrt{5})\,[1,\, \mathbf{h},\, \mathbf{jh},\, \mathbf{kh}]$.

Clearly $\mathbb{I}_K$ is a subring of quaternions.

5. Proof of the Universality of K

At this point an analogue of Lemma 15 in Deutsch [4] is useful. We recall that result.

Lemma 4 *For all $\mathbf{q} \in \mathbb{I}$ there exist quaternion units $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{I}$ of norm 1 such that $\mathbf{u}_1\,\mathbf{q}\,\mathbf{u}_2 \in O(\sqrt{5})[1,\, \mathbf{i},\, \mathbf{j},\, \mathbf{k}]$.*

The analogous proposition is below.

Lemma 5 *For all $\mathbf{q} \in \mathbb{I}$ there exist quaternion units $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{I}$ of norm 1 such that $\mathbf{u}_1\,\mathbf{q}\,\mathbf{u}_2 \in \mathbb{I}_K$.*

Proof. It can be readily shown that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are in $\mathbb{I}_K$. Note that

$$\mathbf{i} \;=\; -1+\mathbf{h}-\mathbf{k}\,\mathbf{h}\,, \quad \mathbf{j} \;=\; 1+\mathbf{j}\,\mathbf{h}+\mathbf{k}\,\mathbf{h}\,, \quad \text{and} \quad \mathbf{k} \;=\; -1+\mathbf{h}-\mathbf{j}\,\mathbf{h}\,. \tag{5.1}$$

The lemma immediately follows. □

Lemma 16 of Deutsch [4] is the next step. It is repeated here.

Lemma 6 *Suppose ρ is a prime of the ring $O(\sqrt{5})$. Then there exists a unit $\lambda \in O(\sqrt{5})$ and a quaternion $\mathbf{q} \in \mathbb{I}$ such that $N(\mathbf{q}) = \lambda\rho$.*

The proof of universality continues.

Lemma 7 *Suppose ρ is a totally positive prime of the ring $O(\sqrt{5})$. Then there exist α, β, γ, and δ in $O(\sqrt{5})$ for which*

$$\rho \;=\; K\,(\alpha,\, \beta,\, \gamma,\, \delta)\,. \tag{5.2}$$

Proof. By the previous Lemma there exists a quaternion $\mathbf{q} \in \mathbb{I}$ and a unit $\lambda \in O(\sqrt{5})$ such that $N(\mathbf{q}) = \lambda\,\rho$. Since $\mathbf{q} \in \mathbb{I}$, by the definitions of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$ we may write $\mathbf{q}$ as a $\mathbb{Q}(\sqrt{5})$–linear combination of 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. From equations (5.1) we may then write $\mathbf{q}$ as a $\mathbb{Q}(\sqrt{5})$–linear combination of the basis elements of $\mathbb{I}_K$. Thus there exist α, β, γ, δ in $\mathbb{Q}(\sqrt{5})$ for which

$$\mathbf{q} \;=\; \alpha \,+\, \beta\,\mathbf{h} \,+\, \gamma\,\mathbf{jh} \,+\, \delta\,\mathbf{kh}\,. \tag{5.3}$$

So we have

$$\lambda\,\rho \;=\; N(\mathbf{q}) \;=\; N(\alpha \,+\, \beta\,\mathbf{h} \,+\, \gamma\,\mathbf{jh} \,+\, \delta\,\mathbf{kh})\,, \tag{5.4}$$

and by Equation (3.2)

$$\lambda\,\rho \;=\; K\,(\alpha,\, \beta,\, \gamma,\, \delta)\,. \tag{5.5}$$

Taking conjugates with respect to $\mathbb{Q}(\sqrt{5})$, since K has rational integer coefficients, we have

$$\lambda^*\,\rho^* \;=\; K\,(\alpha^*,\, \beta^*,\, \gamma^*,\, \delta^*) \;=\; N(\mathbf{q}^*)\,, \tag{5.6}$$

for some appropriate quaternion $\mathbf{q}^*$. Thus $\lambda\rho$ is totally positive, and consequently λ must be totally positive. Since λ is a unit in $O(\sqrt{5})$ it follows that $\lambda = \tau^{2m}$ for some nonnegative integer m.

Thus $N(\mathbf{q}) = \tau^{2m}\rho$. By Lemma 5 there are quaternion units $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{I}$ of norm 1 such that $\mathbf{u}_1\,\mathbf{q}\,\mathbf{u}_2 \in \mathbb{I}_K$. We may write

$$\mathbf{u}_1\,\mathbf{q}\,\mathbf{u}_2 \;=\; \alpha_1 \,+\, \beta_1\,\mathbf{h} \,+\, \gamma_1\,\mathbf{jh} \,+\, \delta_1\,\mathbf{kh} \tag{5.7}$$

with α_1, β_1, γ_1, $\delta_1 \in O(\sqrt{5})$. Thus

$$\tau^{2m}\rho \;=\; N(\mathbf{q}) \;=\; N(\mathbf{u}_1)\,N(\mathbf{q})\,N(\mathbf{u}_2) \;=\; N(\mathbf{u}_1\,\mathbf{q}\,\mathbf{u}_2) \;=\; K(\alpha_1,\, \beta_1,\, \gamma_1,\, \delta_1)\,. \tag{5.8}$$

Since K is homogeneous, we find

$$\rho \;=\; K\,(\tau^{-m}\alpha_1,\, \tau^{-m}\beta_1,\, \tau^{-m}\gamma_1,\, \tau^{-m}\delta_1)\,. \qquad \square \tag{5.9}$$

□

Theorem 8 *K is a universal form for the totally positive integers of $O(\sqrt{5})$.*

Proof. The demonstration almost exactly follows the proof of Götzky's Theorem in Deutsch [4]. Let $\eta \in O(\sqrt{5})$ be totally positive. Since $O(\sqrt{5})$ is a principal ideal domain and a unique factorization domain, we may write η as a unit of $O(\sqrt{5})$ times a product of primes. With no loss of generality, we may choose the primes to be totally positive. Thus we may write

$$\eta \;=\; \mu\,\rho_1\,\rho_2\; \ldots\; \rho_r \tag{5.10}$$

with μ a unit and ρ_1 through ρ_r totally positive primes. Then μ is totally positive, so $\mu = \tau^{2m}$ for some m in $\mathbb{Z}$. From equation (3.1) we have $\tau^{2m} = K(\tau^m, 0, 0, 0)$. By the previous Lemma, each of ρ_1 through ρ_r can be represented by K with arguments in $O(\sqrt{5})$. From the product formula equation (3.6), it follows that η can be represented by K with arguments in $O(\sqrt{5})$. □

6. The Computation

The formula for the determinant of the 4×4 Hessian matrix (2.3) was produced by the computer algebra system MAXIMA. A scan was made using the MinGW 32 bit C compiler on a Windows partition of a late 2000's era laptop. The scan ran over approximately 450 million cases, and took about 8.4 seconds with the GNU C compiler version 3.4.5. Using the TDM-GCC 4.8.1 compiler, the same scan took only 6.3 seconds. The optimizations used were "-O2 -mmmx -msse2" in the first case, and "-O2" in the second.

Other computer algebra efforts also used MAXIMA.

References

[1] M. Baake and R. Moody, Similarity submodules and root systems in four dimensions, Canad. J. Math. **51** (1999), 1258–1276.

[2] A. Berkovich, On the Gauss $E\Upsilon PHKA$ theorem and some allied inequalities, J. Number Theory **148** (2015), 1–18.

[3] J. I. Deutsch, Geometry of numbers proof of Götzky's four squares theorem, J. Number Theory **96** (2002), 417—431.

[4] J. I. Deutsch, An alternate proof of Cohn's four squares theorem. J. Number Theory **104** (2004), 263—278.

[5] J. I. Deutsch, Short proofs of the universality of certain diagonal quadratic forms, Arch. Math. (Basel) **91** (2008), 44—48.

[6] J. I. Deutsch, Universality of a non-classical integral quadratic form over $\mathbb{Q}(\sqrt{5})$, Acta Arith. **136** (2009), 229—242.

[7] Y. M. Lee, Universal forms over $\mathbb{Q}(\sqrt{5})$, Ramanujan J. **16** (2008), 97–104.

[8] M.-F. Vignéras, Arithmétique des Algèbres de Quaternions, Springer-Verlag, Berlin, 1980.