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**NOTE ON LEECH-TYPE QUESTIONS OF TREES****Mustafa Ozen***Department of Computer Engineering, Bogazici University, Istanbul, Turkey*  
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demet.yalman@boun.edu.tr*Received: 2/22/15, Revised: 10/26/15, Accepted: 3/9/16, Published: 4/22/16***Abstract**

The Leech tree, introduced by John Leech in 1975, is a tree on  $n$  vertices with positive integer edge weights such that the weighted distances between pairs of vertices are exactly  $1, 2, \dots, \binom{n}{2}$ . Only five Leech trees are known and some non-existence results have been presented through the years. Variations of Leech trees such as the minimal distinct distance trees and modular Leech trees have been considered. In this note we examine such Leech-type questions on distances between leaves. We also introduce some other labeling questions related to the original motivation for Leech trees.

**1. Introduction**

In a tree  $T$  on  $n$  vertices, there are  $\binom{n}{2}$  distinct paths in  $T$ . Let  $W$  be a function that assigns a positive integer (weight) to each edge in  $T$  and let the weight of a path be the sum of the edge weights on this path, a tree  $T$  together with a weight function  $W$  is called a *Leech tree* if the  $\binom{n}{2}$  path weights are exactly  $1, 2, \dots, \binom{n}{2}$ . This concept was proposed by John Leech in 1975 [4], motivated from the question of finding an efficient design for electrical circuits. Throughout the years various properties of Leech trees have been presented [5, 6, 7]. The following are the five

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known Leech trees (Figure 1) and it is conjectured that they are the only Leech trees.

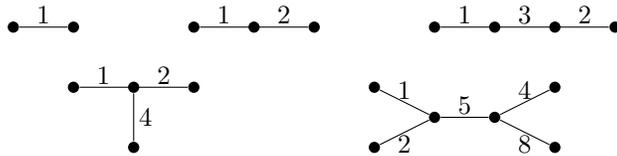


Figure 1: The known Leech trees

More recently, variations of Leech trees have been introduced and studied. Such concepts include the minimal distinct distance trees [1] and modular Leech trees [2, 3]. In this note we explore some related questions and provide simple but interesting observations.

In many areas of study, the distance between leaves are of interests in addition to the distances between all vertices. Motivated by the concept of Leech tree we define the *leaf-Leech tree* as follows.

**Definition 1.** A (not weighted) tree  $T$  with  $n$  leaves is a leaf-Leech tree if the distances between pairs of leaves are exactly  $3, 4, \dots, \binom{n}{2} + 2$ .

For example, in Figure 2, the tree on the left has three leaves that are at distances 3, 4, and 5 from each other; the tree on the right has four leaves with distances 3, 4, 5, 6, 7, 8 between them.

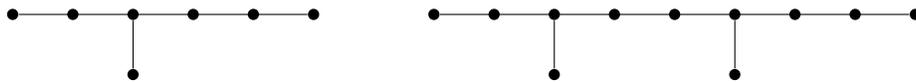


Figure 2: Examples of leaf-Leech trees

**Remark 1.** Note that the leaf-Leech trees are not weighted since any positive integer weighted edge can be replaced by a path of the corresponding length. Furthermore, distance 1 between two leaves forces the tree to have exactly two vertices; distance 2 between two leaves  $u$  and  $w$  would imply that  $u$  and  $w$  are at the same distance from any other vertex. Thus we do not require the presence of 1 and 2 among the distances between the leaves in a leaf-Leech tree.

## 2. Some Properties of Leaf-Leech Trees

For Leech trees, the beautiful Taylor’s condition [6] asserts that the number of vertices in a Leech tree must be a perfect square or a perfect square plus two. Below is an analogous statement following very similar argument as that in [6].

**Proposition 1.** *If there is a leaf-Leech tree with  $n$  leaves, we must have  $n = k^2$  or  $n = k^2 + 2$  for some  $k$ .*

*Proof.* Let  $T$  be a leaf-Leech tree on  $n$  leaves and  $v$  be one of the leaves. Define

- $O$  to be the set of leaves at odd distance from  $v$ ;
- $E$  to be the set of leaves at even distance from  $v$  (note that  $v \in E$ ).

First note that the distance between a pair of vertices in  $O$  or  $E$  is even and the distance between a vertex in  $O$  and a vertex in  $E$  is odd. Now consider two cases:

- If  $\binom{n}{2}$  is even, then the number of odd distances between leaves is the same as the number of even distances. Consequently

$$\binom{|O|}{2} + \binom{|E|}{2} = |O| \cdot |E|,$$

which can be rewritten as

$$n = |O| + |E| = (|O| - |E|)^2.$$

- If  $\binom{n}{2}$  is odd, then the number of odd distances between leaves is one more than the number of even distances. Consequently

$$\binom{|O|}{2} + \binom{|E|}{2} + 1 = |O| \cdot |E|,$$

which can be rewritten as

$$n = |O| + |E| = (|O| - |E|)^2 + 2.$$

□

It is shown in [5], among more general results, that other than the ones shown in Figure 1, no Leech tree can be a star. Similar arguments yield the analogous conclusion that there are only a few *starlike* (with exactly one vertex of degree at least 3) leaf-Leech trees.

**Proposition 2.** *There is no starlike leaf-Leech trees with more than 4 leaves.*

It is also known that no path can be a Leech tree except for the ones in Figure 1. In fact, it is shown in [5] that a Leech tree cannot contain a long path. With regard to leaf-Leech trees, one can show in a similar way, that there are only a few *caterpillars* (trees whose removal of leaves result in paths) that can be leaf-Leech trees.

**Proposition 3.** *There is no leaf-Leech caterpillars with more than 4 leaves.*

Although the above statements can be shown through similar arguments as those in [5]. We will present a more general observation in the next section, from which Propositions 2 and 3 follow as immediate corollaries.

### 3. Connection Between Leaf-Leech Trees and Leech Trees

Given the very similar properties of leaf-Leech trees and what is known for Leech trees, it is natural to ask if there is any obvious connection between the two. Intuitively, a leaf-Leech tree seems to be easier to find than a Leech tree. Indeed this will be confirmed by our next observation.

Given a tree  $T$  with positive integer weights on edges, the *expansion*  $T^e$  of  $T$  is defined as follows (Figure 3):

- For any edge  $u_i u_j \in E(T)$  with weight  $w$ , subdividing this edge into a path of length  $w$  from  $u_i$  to  $u_j$ ;
- To every vertex  $u_i \in V(T)$  append a pendant edge  $u_i v_i$ .

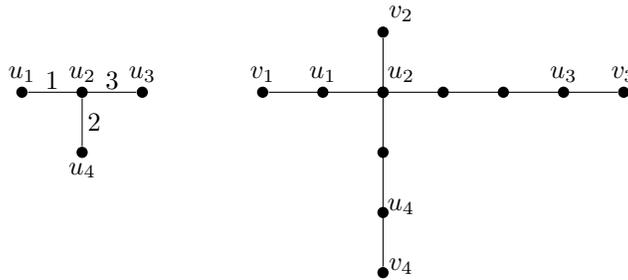


Figure 3: An edge-weighted tree  $T$  (on the left) and its expansion  $T^e$  (on the right)

It is easy to see that the expansion of a tree on  $n$  vertices is a tree with  $n$  leaves. We first show the following simple but interesting observation.

**Theorem 1.** *If there exists a Leech tree  $T$  on  $n$  vertices, there exists a leaf-Leech tree, namely  $T^e$ , with  $n$  leaves.*

*Proof.* Let  $T$  be a Leech tree on vertices  $u_1, u_2, \dots, u_n$  and  $T^e$  be generated as described above. By definition, the distance between  $u_i$  and  $u_j$  in  $T^e$ , denoted by  $d_{T^e}(u_i, u_j)$ , is exactly the same as the weighted distance between  $u_i$  and  $u_j$  (weight of the path connecting  $u_i$  and  $u_j$ ) in  $T$ , denoted by  $d_T(u_i, u_j)$ . Consequently

$$d_{T^e}(v_i, v_j) = 2 + d_{T^e}(u_i, u_j) = 2 + d_T(u_i, u_j).$$

Since  $T$  is a Leech tree,

$$\{d_T(u_i, u_j) : 1 \leq i < j \leq n\} = \left\{1, 2, \dots, \binom{n}{2}\right\}.$$

Hence

$$\{d_{T^e}(v_i, v_j) : 1 \leq i < j \leq n\} = \left\{3, 4, \dots, \binom{n}{2} + 2\right\},$$

implying that  $T^e$  is a leaf-Leech tree. □

On the other hand, it is not obvious whether the inverse is true. Figure 4 shows a tree with three leaves that is not the expansion of any tree. This is because of the vertex  $v$  with degree at least three and no leaf neighbors. We call such vertices *irreducible* vertices.

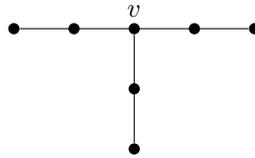


Figure 4: A tree with an irreducible vertex  $v$

Correspondingly, we show the following.

**Theorem 2.** *Every tree with no irreducible vertex and distance at least 3 between leaves is the expansion of some edge-weighted tree. In particular, a leaf-Leech tree consisting of  $n$  leaves with no irreducible vertex is the extension of a Leech tree on  $n$  vertices.*

*Proof.* Let  $T$  be a tree with no irreducible vertex and consisting of  $n$  leaves  $v_1, \dots, v_n$ . Since the distance between any pair of leaves is at least 3, no two leaves share a common neighbor. Label the unique neighbor of  $v_i$  as  $u_i$  for  $1 \leq i \leq n$ . Note that  $u_i$  is not a leaf.

Consider the path connecting  $u_i$  and  $u_j$  that does not contain any other labeled vertices. We claim that every vertex on this path (except possibly for  $u_i$  and  $u_j$ ) are of degree 2. This is because, for a vertex  $w$  on this path, it has no leaf-neighbors in  $T$  or it would have been labeled. Consequently  $w$  would be an irreducible vertex in  $T$  if its degree is at least 3.

Now we construct  $T'$  by replacing any path connecting  $u_i$  and  $u_j$  that does not contain any other labeled vertices by an edge with weight equal to the length of this path, and removing the pendant edge  $u_i v_i$  for  $1 \leq i \leq n$ .

It is easy to see that  $T$  is the expansion of  $T'$ . If  $T$  is a leaf-Leech tree, then

$$\{d_T(v_i, v_j) : 1 \leq i < j \leq n\} = \left\{ 3, 4, \dots, \binom{n}{2} + 2 \right\}$$

and

$$\{d_T(u_i, u_j) : 1 \leq i < j \leq n\} = \left\{ 1, 2, \dots, \binom{n}{2} \right\}.$$

Consequently

$$\{d_{T'}(u_i, u_j) : 1 \leq i < j \leq n\} = \left\{ 1, 2, \dots, \binom{n}{2} \right\},$$

implying that  $T'$  is a Leech tree.  $\square$

As immediate consequences, we provide the proofs of Propositions 2 and 3 as follows.

*Proof of Proposition 2.* Supposing (for contradiction) that  $T$  is a starlike leaf-Leech tree on at least 5 leaves. Then the distance between any pair of leaves in  $T$  is at least three.

In order to have distance 3 between leaves, we must have a pendant edge and a pendant path of length 2. Note that now the only vertex of degree larger than 2 is incident with this pendant edge and hence has a leaf neighbor.

Consequently  $T$  satisfies the conditions of Theorem 2 and is the expansion of some Leech star on at least 5 vertices. But as shown in [5], no such Leech stars exist, a contradiction.  $\square$

*Proof of Proposition 3.* Supposing (for contradiction) that  $T$  is a leaf-Leech caterpillar on at least 5 vertices. From the definition of a caterpillar it is obvious that every vertex with degree at least three is adjacent to some leaf. Hence by Theorem 2,  $T$  is the expansion of some Leech path on at least 5 vertices. But no such Leech paths exist as shown in [5], a contradiction.  $\square$

A natural question follows:

**Question 3.1.** Does there exist a leaf-Leech tree that contains an irreducible vertex?

To further consider the relation between these two concepts one may allow rational weights in both Leech trees and leaf-Leech trees. Note that even in this case the definition of Leech trees requires all edge weights to be positive integers. It is natural to guess that many (or all) trees are leaf-Leech with rational edge weights (as opposed to the original definition of non-weighted trees). Unfortunately, similar arguments show that starlike trees (or stars) are not leaf-Leech (even with rational edge weights) in general. In fact, it is not easy to find a non-trivial leaf-Leech tree with rational edge weights.

#### 4. Concluding Remarks and Other Related Questions

We studied the characteristics of leaf-Leech trees and explored their connection with Leech trees. A negative answer to Question 3.1 would imply the equivalence of the Leech trees and leaf-Leech trees. In the other direction, we have not been able to find a leaf-Leech tree that is not the expansion of a Leech tree.

Given Leech's original motivation for finding Leech trees [4], some other variations may also be interesting. Knowing that no Leech tree exists on  $n$  vertices, the

minimal distinct distance tree [1] is one way to generalize the concept and to find the next best weighted tree in this aspect. It is also natural to pack, instead of distinct values, as many as possible values of  $\{1, \dots, \binom{n}{2}\}$  into the set of distances between vertices. Figure 5 is such an “almost Leech tree” on 5 vertices, where 6 is the only distance missing from the set  $\{1, \dots, 10\}$ . Of course, it is easy to see that the expansion of an “almost Leech tree” yields an “almost leaf-Leech tree”.

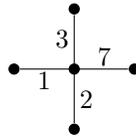


Figure 5: An “almost” Leech tree on 5 vertices

Similarly, it is known that there is no *modular Leech tree* (a tree with positive integer edge weights such that the weighted distances between vertices yields  $1, 2, \dots, \binom{n}{2}$  when taking modulo  $\binom{n}{2} + 1$  [2, 3]) on 5 vertices. Figure 6 shows such an “almost modulo Leech tree”.

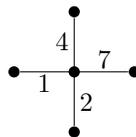


Figure 6: An “almost” modulo Leech tree on 5 vertices

Another question of obvious interest is to “pack” as many copies of each distance from a given set of values as possible, in a tree with as few vertices as possible. As an example, Figure 7 is a weighted binary tree where every sub-star on four vertices is an exact copy of the Leech star on four vertices. As the structure extend indefinitely, it is easy to see that each of the distances  $1, 2, \dots$  appear at least three times.

A simple improvement on the above construction is to replace the “1” with 2 and “2” with 4 in Figure 7. This will result in the inclusion of larger distances with fewer vertices. The general optimization does not seem to be easy and is not explored in this note.

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