AN INFINITE FAMILY OF CONGRUENCES FOR ℓ-REGULAR OVERPARTITIONS

Abdulaziz M. Alanazi
School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
Abdulaziz.Alanazi@students.wits.ac.za

Augustine O. Munagi
School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
Augustine.Munagi@wits.ac.za

James A. Sellers
Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania
sellersj@psu.edu

Received: 11/3/15, Accepted: 5/22/16, Published: 6/10/16

Abstract

We consider new properties of the combinatorial objects known as overpartitions (which are natural generalizations of integer partitions). In particular, we establish an infinite set of Ramanujan-type congruences for the restricted overpartitions known as ℓ-regular overpartitions. This significantly extends the recent work of Shen which focused solely on 3–regular overpartitions and 4–regular overpartitions.

1. Introduction

An overpartition of a positive integer n is a nonincreasing sequence of positive integers that sum to n, where the first occurrence of parts of each size may be overlined. An extensive study of overpartitions can be found in the work of Corteel and Lovejoy [4]. We denote the number of overpartitions of n by $\overline{p}(n)$, with $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ enumerates the following overpartitions:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$$

The three overpartitions with no overlined parts are the ordinary partitions of 3. Given a positive integer ℓ a partition λ is called ℓ-regular if no part of λ is a multiple of ℓ.

In 2003, Lovejoy [10] considered the functions $\overline{A}_\ell(n)$ which enumerate the overpartitions of n which are ℓ-regular; in other words, $\overline{A}_\ell(n)$ counts the number of overpartitions of n which have no parts being a multiple of ℓ. Andrews [1] extended this idea by considering the enumeration of *singular overpartitions* of n which correspond to ℓ-regular overpartitions of n in which the parts satisfy prescribed congruences. In particular, Andrews [1] noted that one of his functions is the same as $\overline{A}_3(n)$, and he proved that, for all $n \geq 0$,

$$\overline{A}_3(9n+3) \equiv \overline{A}_3(9n+6) \equiv 0 \pmod{3} \tag{1}$$

using elementary generating function manipulations. Motivated by this congruence result, Chen, Hirschhorn and Sellers [3] extensively studied the arithmetic properties of these singular overpartition functions.

In recent days, Shen [13] returned to the functions of Lovejoy and proved a finite set of congruences satisfied by $\overline{A}_3$ and $\overline{A}_4$. In particular, Shen proved the following eight congruence results:

Theorem 1. *For all $n \geq 0$,*

$$\begin{aligned}
\overline{A}_3(4n+1) &\equiv 0 \pmod{2},\\
\overline{A}_3(4n+3) &\equiv 0 \pmod{6},\\
\overline{A}_3(9n+3) &\equiv 0 \pmod{6}, \text{ and}\\
\overline{A}_3(9n+6) &\equiv 0 \pmod{24}.
\end{aligned}$$

Theorem 2. *For all $n \geq 0$,*

$$\begin{aligned}
\overline{A}_4(12n+4) &\equiv 0 \pmod{3},\\
\overline{A}_4(12n+8) &\equiv 0 \pmod{3},\\
\overline{A}_4(12n+7) &\equiv 0 \pmod{24}, \text{ and}\\
\overline{A}_4(12n+11) &\equiv 0 \pmod{24}.
\end{aligned}$$

Our primary goal in this paper is to prove families of congruences satisfied by the functions $\overline{A}_\ell$ for infinitely many values of ℓ. The proof techniques used are classical, involving elementary generating function manipulation techniques as well as Ramanujan's theta functions.

Throughout this work, we will utilize the following standard generating functions:

$$\sum_{n=0}^{\infty} \overline{p}(n) = \prod_{n\geq 1} \frac{1+q^n}{1-q^n}.$$

$$\sum_{n=0}^{\infty} \overline{A}_{\ell}(n) = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})}. \tag{2}$$

We will also make use of Ramanujan's theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1+q^{2n-1})^2(1-q^{2n}). \tag{3}$$

(See Berndt's book [2] for a detailed discussion of the function $\varphi(q)$ and its relatives.)

2. New Congruence Results

Motivated by Andrews' congruences (1), Chen, Hirschhorn, and Sellers [3] have already provided an infinite family of congruences satisfied by $\overline{A}_3(n)$ modulo 3 and small powers of 2. Our first goal in this paper is to show that $\overline{A}_\ell$ satisfies at least one congruence modulo 3 for an infinite set of values ℓ.

Theorem 3. *For all $n \geq 0$ and all $j \geq 3$, $\overline{A}_{3^j}(27n+18) \equiv 0 \pmod 3$.*

Proof. From recent work of Munagi and Sellers [12], we define the function $R_\ell(n)$ to be the number of overpartitions of n in which only parts not divisible by ℓ may be overlined. (In [12], the function $R_\ell(n)$ is denoted $A_\ell(n)$.) We find from [12] that, for all $n \geq 0$ and all $j \geq 3$, $R_{3^j}(27n+18) \equiv 0 \pmod 3$ where

$$\sum_{n\geq 0} R_{3^j}(n)q^n = \prod_{n\geq 1} \frac{(1-q^{3^j n})}{(1-q^{2\cdot 3^j n})} \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^n)^2}. \tag{4}$$

Next, note that

$$\sum_{n\geq 0} \overline{A}_{3^j}(n)q^n = \prod_{n\geq 1} \frac{(1-q^{3^j n})}{(1+q^{3^j n})} \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^n)^2}. \tag{5}$$

Via elementary manipulations, it is then clear from (4) and (5) that

$$\sum_{n\geq 0} \overline{A}_{3^j}(n)q^n = \left(\prod_{n\geq 1}(1-q^{3^j n})\right) \sum_{n\geq 0} R_{3^j}(n)q^n.$$

Moreover,

$$\prod_{n\geq 1}(1-q^{3^j n})$$

is a function of q^{27}, which means that $\overline{A}_{3^j}(27n+18)$ is simply a linear combination of values of $R_{3^j}(27n+18)$ (no other terms can enter this sum). Therefore, thanks to the corresponding congruence result for R_{3^j} from [12], the theorem follows. □

Interestingly enough, it is also the case that $\overline{A}_9(n)$ satisfies congruences modulo 3. However, they appear to be of a different nature than those satisfied by $\overline{A}_3$ (as stated in [3]) and $\overline{A}_{3^j}$ for $j \geq 3$ (as given in Theorem 3). Thus we need to discuss $\overline{A}_9(n)$ separately.

In order to consider $\overline{A}_9(n)$ modulo 3, we will utilize a number of results of Hirschhorn and Sellers [7]. In particular, we will consider the two functions

$$D(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q)_\infty^2}{(q^2)_\infty}$$

and

$$Y(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = \frac{(q)_\infty (q^6)_\infty^2}{(q^2)_\infty (q^3)_\infty},$$

where the q-Pochhammer symbol is defined by $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$ with the shortened notation $(q;q)_\infty = (q)_\infty$.

It is worth noting that $D(q) = \varphi(-q)$ where $\varphi(q)$ is defined in (3).

In [7, Lemma 3.1] the following three identities are proved (where $\omega = e^{2\pi i/3}$):

$$D(q) = D(q^9) - 2qY(q^3),\ D(q)D(\omega q)D(\omega^2 q) = \frac{D(q^3)^4}{D(q^9)},\ \text{and } D(q^3)^3 - 8qY(q)^3 = \frac{D(q)^4}{D(q^3)}.$$

Now note the following:

$$\begin{aligned}
\sum_{n\geq 0} \overline{A}_9(n) q^n &= \frac{D(q^9)}{D(q)} \\
&= \frac{D(q^9)}{D(q)} \frac{D(\omega q)}{D(\omega q)} \frac{D(\omega^2 q)}{D(\omega^2 q)} \\
&= \frac{D(q^9)}{D(q^3)^4} (D(q^{81}) - 2q^9 Y(q^{27}))(D(q^9) - 2\omega q Y(q^3))(D(q^9) - 2\omega^2 q Y(q^3)) \\
&= \frac{D(q^9)}{D(q^3)^4} (D(q^{81}) - 2q^9 Y(q^{27}))(D(q^9)^2 + 2qD(q^9)Y(q^3) + 4q^2 Y(q^3)^2).
\end{aligned}$$

Thus, we can 3–dissect the generating function for $\overline{A}_9$ to obtain

$$\sum_{n\geq 0} \overline{A}_9(3n+2) q^n = \frac{D(q^3)}{D(q)^4} (D(q^{27}) - 2q^3 Y(q^9))(4Y(q)^2).$$

Next, we simplify this generating function modulo 3, utilizing the three identities mentioned above:

$$\begin{aligned}
\sum_{n\geq 0}\overline{A}_9(3n+2)q^n &= \frac{D(q^3)}{D(q)^4}(D(q^{27})-2q^3Y(q^9))(4Y(q)^2)\\
&\equiv \frac{D(q^3)}{D(q)^4}(D(q^9)^3-2q^3Y(q^3)^3)(4Y(q)^2) \pmod 3\\
&\equiv \frac{D(q^3)}{D(q)^4}(D(q)^3)(Y(q)^2) \pmod 3\\
&= \frac{D(q^3)}{D(q)}Y(q)^2\\
&= \frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\frac{(q;q)_\infty^2(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2}\\
&= \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty}
\end{aligned}$$

Therefore, we know that

$$\sum_{n\geq 0}\overline{A}_9(3n+2)q^n \equiv \sum_{n\geq 0}a_3(n)q^{2n} \pmod 3$$

where $a_3(n)$ is the number of 3–cores of n. This leads to two congruence results for $\overline{A}_9$.

Theorem 4. *For all $n\geq 0$, we have $\overline{A}_9(6n+5)\equiv 0 \pmod 3$.*

Proof. This result follows immediately from the fact that

$$\sum_{n\geq 0}\overline{A}_9(3n+2)q^n \equiv \sum_{n\geq 0}a_3(n)q^{2n} \pmod 3$$

and the fact that the series on the right–hand side is an even function of q. Therefore, for all $n\geq 0$, we have $\overline{A}_9(3(2n+1)+2)=\overline{A}_9(6n+5)\equiv 0 \pmod 3$. □

In a similar vein, the generating function work above also proves that, for all $n\geq 0$,

$$\overline{A}_9(6n+2)\equiv a_3(n) \pmod 3. \tag{6}$$

This is truly significant as it provides infinitely many Ramanujan–like congruences modulo 3 satisfied by $\overline{A}_9$. One way to see this is to note that $a_3(n)$ is infinitely often identical to zero (see the work of Hirschhorn and Sellers [8] for elementary proofs of some of the arithmetic properties of $a_3(n)$). Indeed, we can easily prove the following result.

Theorem 5. *Let $p \equiv 2 \pmod 3$ be prime. For each $1 \leq k \leq p-1$, let r be the least nonnegative integer such that*

$$r \equiv \frac{p^2-1}{3} + kp \pmod{p^2}.$$

Then, for all $n \geq 0$, we have $\overline{A}_9(6(p^2n+r)+2) \equiv 0 \pmod 3$.

Proof. The proof relies on a result found in Hirschhorn and Sellers [8]. Namely, under the hypothesis of this theorem, it is the case that $a_3(p^2n+r) = 0$. Thanks to this fact and (6), the proof is complete. □

We now turn our attention to congruences satisfied by $\overline{A}_\ell$ modulo small powers of 2. As with numerous other overpartition functions, it is clear that, for each ℓ, $\overline{A}_\ell(n)$ satisfies many congruences modulo small powers of 2. (See, for example, [5, 6, 7, 9, 11] where this phenomenon is also noted.)

With the goal of proving such congruences modulo small powers of 2, we develop an extremely beneficial way to rewrite the generating function for $\overline{A}_\ell(n)$ in terms of Ramanujan's theta function $\varphi(q)$.

We state the following lemmas, the proofs of which may be found in [12]:

Lemma 1. *We have*

$$\varphi(-q^2)^2 = \varphi(q)\varphi(-q)$$

Lemma 2. *We have*

$$\frac{1}{\varphi(-q)} = \varphi(q)\varphi(q^2)^2\varphi(q^4)^4\ldots$$

Combining (2) with Lemma 2, we have

$$\sum_{n\geq 0} \overline{A}_\ell(n)q^n = \frac{\varphi(q)\varphi(q^2)^2\varphi(q^4)^4\ldots}{\varphi(q^\ell)\varphi(q^{2\ell})^2\varphi(q^{4\ell})^4\ldots}. \tag{7}$$

Corollary 1. *For all $n \geq 1$, we have $\overline{A}_\ell(n) \equiv 0 \pmod 2$.*

Proof. Since $\varphi(q) = 1 + 2\sum_{n\geq 1} q^{n^2}$, we know that

$$\varphi(q) \equiv 1 \pmod 2.$$

So (7) gives

$$\sum_{n\geq 0} \overline{A}_\ell(n)q^n \equiv \frac{1\cdot 1\cdot 1\ldots}{1\cdot 1\cdot 1\ldots} \equiv 1 \pmod 2.$$

□

Corollary 2. *For all $n \geq 1$ and an integer $k > 0$,*

$$\overline{A}_\ell(n) \equiv \begin{cases} 2 \pmod 4 & \textit{if } n = k^2 \textit{ or } n = \ell k^2, \textit{ where } \ell \textit{ is not a square;} \\ 0 \pmod 4 & \textit{otherwise,} \end{cases}$$

Proof. Thanks to (7), we know

$$\sum_{n\geq 0}\overline{A}_\ell(n)q^n \equiv \frac{\varphi(q)}{\varphi(q^\ell)} \pmod 4$$

since $\varphi(q^i)^j \equiv 1 \pmod 4$ for any $j \geq 2$. Next, we know, in view of Lemma 1, that

$$\varphi(q) = \frac{\varphi(-q^2)^2}{\varphi(-q)}.$$

Thus,

$$\begin{aligned}\sum_{n\geq 0}\overline{A}_\ell(n)q^n &\equiv \frac{\varphi(q)}{\varphi(q^\ell)} \pmod 4\\ &\equiv \frac{\varphi(q)\varphi(-q^\ell)}{\varphi(-q^{2\ell})^2} \pmod 4\\ &\equiv \varphi(q)\varphi(-q^\ell) \pmod 4\end{aligned}$$

since $\varphi(-q^{2\ell})^2 \equiv 1 \pmod 4$.

Therefore,

$$\begin{aligned}\sum_{n\geq 0}\overline{A}_\ell(n)q^n &\equiv \varphi(q)\varphi(-q^\ell) \pmod 4\\ &= (1+2\sum_{n\geq 1}q^{n^2})(1+2\sum_{n\geq 1}(-q^\ell)^{n^2})\\ &\equiv 1+2\sum_{n\geq 1}q^{n^2}+2\sum_{n\geq 1}(-q^\ell)^{n^2} \pmod 4\\ &\equiv 1+2\sum_{n\geq 1}q^{n^2}+2\sum_{n\geq 1}q^{\ell n^2} \pmod 4.\end{aligned}$$

The result follows. □

It is clear that Corollary 2 provides a framework from which we can write down infinitely many congruences modulo 4 satisfied by $\overline{A}_\ell$ for certain values of ℓ. We provide such an infinite family of results here.

Corollary 3. *Let ℓ be a square, p be a prime, and let r be a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $\overline{A}_\ell(pn+r) \equiv 0 \pmod 4$.*

Proof. Assume ℓ is a square. Thanks to Corollary 2, we know that $\overline{A}_\ell(n)$ is divisible by 4 unless n is a square. Thus, in order for this result to be false, we must have $pn+r = k^2$ for some k. But this implies that $r \equiv k^2 \pmod p$, and this cannot occur because r is assumed to be a quadratic nonresidue modulo p. The result follows. □

Clearly, Corollary 3 provides $\frac{p-1}{2}$ congruences modulo 4 for each prime p and for each square value of ℓ. Thus, we have demonstrated infinitely many congruences modulo 4 which are satisfied by $\overline{A}_\ell$ (for a specific set of values of ℓ).

It is worth noting that the proof technique used in Corollary 2 could be extended to write down similar results for moduli which are higher powers of 2. However, the results will undoubtedly be less elegant as those above, so we refrain from doing so here.

We close by acknowledging that the results above do not provide an exhaustive list of congruences of this form for $\overline{A}_\ell$. Even so, it is the case that we have greatly extended the set of congruences proven by Shen [13].

References

[1] G. E. Andrews, Singular overpartitions, *Int. J. Number Theory* **11** (2015), 1523–1533.

[2] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, 2006.

[3] S.–C. Chen, M. D. Hirschhorn, and J. A. Sellers, Arithmetic properties of Andrews' singular overpartitions, *Int. J. Number Theory* **11** (2015), 1463–1476.

[4] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623–1635.

[5] M. D. Hirschhorn and J. A. Sellers, An infinite family of overpartition congruences modulo 12, *INTEGERS* **5** (2005), Article A20.

[6] M. D. Hirschhorn and J. A. Sellers, Arithmetic relations for overpartitions, *J. Combin. Math. Combin. Comput.* **53** (2005), 65–73.

[7] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of overpartitions into odd parts, *Ann. Comb.* **10** (2006), 353–367.

[8] M. D. Hirschhorn and J. A. Sellers, Elementary proofs of various facts about 3–cores, *Bull. Aust. Math. Soc.* 79 (2009), 507–512.

[9] B. Kim, A short note on the overpartition function, *Discrete Math.* **309** (2009), 2528–2532.

[10] J. Lovejoy, Gordon's theorem for overpartitions, *J. Combin. Theory Ser. A* **103** (2003), 393–401.

[11] K. Mahlburg, The overpatition function modulo small powers of 2, *Discrete Math.* **286** (2004), 263–267.

[12] A. O. Munagi and J. A. Sellers, Refining overlined parts in overpartitions via residue classes: bijections, generating functions, and congruences, *Util. Math.* **95** (2014), 33–49.

[13] E. Y. Y. Shen, Arithmetic properties of ℓ-regular overpartitions, *Int. J. Number Theory* **12** (2016), 841–852.