REPRESENTATIONS BY QUATERNARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 7 OR 14

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Abstract

Using modular forms we determine explicit formulas for the number of representations of a positive integer n by quaternary quadratic forms with coefficients 1, 2, 7 or 14.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of positive integers, non-negative integers, integers, rational numbers and complex numbers, respectively. The sum of divisors function $\sigma(n)$ for $n \in \mathbb{N}$ is given by

$$\sigma(n) = \sum_{1 \leq m \mid n} m. \tag{1.1}$$

If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$. For $a_1, a_2, a_3, a_4 \in \mathbb{N}$, and $n \in \mathbb{N}_0$ we define

$$N(a_1, a_2, a_3, a_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

There are twenty-six quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ with $a_1, a_2, a_3, a_4 \in \{1, 2, 7, 14\}$, $\gcd(a_1, a_2, a_3, a_4) = 1$ and $a_1 \leq a_2 \leq a_3 \leq a_4$, namely

$$\begin{aligned}(a_1, a_2, a_3, a_4) =& (1,1,1,1), (1,1,1,2), (1,1,1,7), (1,1,1,14), (1,1,2,2), (1,1,2,7),\\ & (1,1,2,14), (1,1,7,7), (1,1,7,14), (1,1,14,14), (1,2,2,2), (1,2,2,7),\\ & (1,2,2,14), (1,2,7,7), (1,2,7,14), (1,2,14,14), (1,7,7,7),\\ & (1,7,7,14), (1,7,14,14), (1,14,14,14), (2,2,2,7), (2,2,7,7),\\ & (2,2,7,14), (2,7,7,7), (2,7,7,14), (2,7,14,14).\end{aligned} \tag{1.2}$$

Formulas for $N(1,1,1,1;n)$, $N(1,1,1,2;n)$, $N(1,1,2,2;n)$ and $N(1,2,2,2;n)$ are known. The formula

$$N(1,1,1,1;n) = 8\sigma(n) - 32\sigma(n/4) \tag{1.3}$$

is due to Jacobi, see [7], [2], [16, Theorem 9.5]. The formula

$$N(1,1,2,2;n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8) \tag{1.4}$$

was stated by Liouville, see [2], [11], [16, Theorem 18.1]. Formulas for $N(1,1,1,2;n)$ and $N(1,2,2,2;n)$ were also stated by Liouville, see [12], [16, Theorem 18.2].

In this paper we determine an explicit formula for $N(a_1,a_2,a_3,a_4;n)$ (Theorems 2.1–2.4) for each of the remaining twenty-two quaternary quadratic forms among which $(1,1,1,7)$, $(1,1,2,7)$, $(1,1,2,14)$, and $(1,2,2,7)$ are universal forms. To the best of our knowledge Theorems 2.1–2.4 are new.

For $q \in \mathbb{C}$ with $|q| < 1$ we have

$$\sum_{n=1}^{\infty} N(a_1,a_2,a_3,a_4;n)q^n = \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}), \tag{1.5}$$

where $\varphi(q)$ denotes Ramanujan's theta function defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

It is convenient to define $F(q)$ for $q \in \mathbb{C}$ with $|q| < 1$ by

$$F(q) = \prod_{n=1}^{\infty}(1-q^n). \tag{1.6}$$

The infinite product representation of $\varphi(q)$ is due to Jacobi (see, for example [4, Corollary 1.3.4]), namely,

$$\varphi(q) = \frac{F^5(q^2)}{F^2(q)F^2(q^4)}. \tag{1.7}$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\eta(z) = e^{\pi i z/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n z}). \tag{1.8}$$

Throughout the remainder of the paper we take $q = q(z) := e^{2\pi i z}$ with $z \in \mathbb{H}$ and so $|q| < 1$. Appealing to (1.6) we can express the Dedekind eta function (1.8) as

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = q^{1/24}F(q). \tag{1.9}$$

An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z), \tag{1.10}$$

where δ runs through a finite set of positive integers and the exponents r_δ are non-zero integers. Appealing to (1.7) and (1.9) we have

$$\varphi(q) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \tag{1.11}$$

Let χ and ψ be Dirichlet characters. For $n \in \mathbb{N}$ we define $\sigma_{\chi,\psi}(n)$ by

$$\sigma_{\chi,\psi}(n) := \sum_{1 \le m \mid n} \psi(m)\chi(n/m)m. \tag{1.12}$$

If $n \notin \mathbb{N}$ we set $\sigma_{\chi,\psi}(n) = 0$. Let χ_0 denote the trivial character, that is $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$. Then $\sigma_{\chi_0,\chi_0}(n)$ coincides with the sum of divisors function $\sigma(n)$ in (1.1). For $m \in \mathbb{Z}$ we define six Dirichlet characters by

$$\begin{cases} \chi_1(m) = \left(\frac{-7}{m}\right), \ \chi_2(m) = \left(\frac{-4}{m}\right), \ \chi_3(m) = \left(\frac{28}{m}\right), \\ \chi_4(m) = \left(\frac{-8}{m}\right), \ \chi_5(m) = \left(\frac{8}{m}\right), \ \chi_6(m) = \left(\frac{56}{m}\right). \end{cases} \tag{1.13}$$

2. Statements of Main Results

We define the following five eta quotients

$$A_1(q) = \eta(2z)\eta(4z)\eta(14z)\eta(28z), \tag{2.1}$$

$$A_2(q) = \frac{\eta^3(2z)\eta^3(28z)}{\eta(4z)\eta(14z)}, \tag{2.2}$$

$$A_3(q) = \frac{\eta^4(2z)\eta^4(14z)}{\eta(z)\eta(4z)\eta(7z)\eta(28z)}, \tag{2.3}$$

$$A_4(q) = \frac{\eta^4(4z)\eta^4(28z)}{\eta(2z)\eta(8z)\eta(14z)\eta(56z)}, \tag{2.4}$$

$$A_5(q) = \frac{\eta^3(2z)\eta(8z)\eta^3(14z)\eta(56z)}{\eta(z)\eta(4z)\eta(7z)\eta(28z)}, \tag{2.5}$$

and integers $a_r(n)$ $(n \in \mathbb{N})$ by

$$A_r(q) = \sum_{n=1}^{\infty} a_r(n)q^n, \quad r \in \{1, 2, 3, 4, 5\}. \tag{2.6}$$

Theorem 2.1. *Let* $n \in \mathbb{N}$ *and* $a_r(n)$ $(r \in \{1,2,3,4,5\})$ *be as in* (2.6). *Then*

$$\begin{aligned}
\textbf{(a)}\ N(1,1,7,7;n) =& \frac{4}{3}\sigma(n) - \frac{8}{3}\sigma(n/2) + \frac{16}{3}\sigma(n/4) - \frac{28}{3}\sigma(n/7) + \frac{56}{3}\sigma(n/14) \\
& - \frac{112}{3}\sigma(n/28) + \frac{8}{3}a_3(n), \\
\textbf{(b)}\ N(1,1,14,14;n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) \\
& + \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) + \frac{2}{3}a_1(n) \\
& - 4a_2(n) + \frac{10}{3}a_3(n) - \frac{4}{3}a_4(n) + 8a_5(n), \\
\textbf{(c)}\ N(1,2,7,14;n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) \\
& + \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) + \frac{2}{3}a_1(n) \\
& + \frac{4}{3}a_3(n) - \frac{4}{3}a_4(n) + 4a_5(n), \\
\textbf{(d)}\ N(2,2,7,7;n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) \\
& + \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) - \frac{10}{3}a_1(n) \\
& + 4a_2(n) - \frac{2}{3}a_3(n) + \frac{20}{3}a_4(n) - 8a_5(n).
\end{aligned}$$

We define the following four eta quotients

$$B_1(q) = \frac{\eta^2(2z)\eta^3(7z)}{\eta(z)}, \tag{2.7}$$

$$B_2(q) = \frac{\eta^3(8z)\eta^2(28z)}{\eta(56z)}, \tag{2.8}$$

$$B_3(q) = \frac{\eta^2(4z)\eta^3(56z)}{\eta(8z)}, \tag{2.9}$$

$$B_4(q) = \frac{\eta^3(z)\eta^2(14z)}{\eta(7z)}, \tag{2.10}$$

and integers $b_r(n)$ $(n \in \mathbb{N})$ by

$$B_r(q) = \sum_{n=1}^{\infty} b_r(n)q^n, \quad r \in \{1,2,3,4\}. \tag{2.11}$$

Theorem 2.2. *Let* $n \in \mathbb{N}$. *Let* $\sigma_{\chi_i,\chi_j}(n)$ *be as in* (1.12) *for* $i,j \in \{0,1,2,3\}$, *and*

$b_r(n)$ $(r \in \{1,2,3,4\})$ *be as in* (2.11). *Then*

$$\textbf{(a)}\ N(1,1,1,7;n) = \frac{7}{2}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n) - \frac{1}{2}\sigma_{\chi_2,\chi_1}(n) + 3b_2(n) - 21b_3(n) - \frac{3}{2}b_4(n),$$

$$\textbf{(b)}\ N(1,1,2,14;n) = \frac{7}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n/2) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) + \frac{21}{8}b_1(n) - \frac{1}{2}b_2(n) + \frac{7}{2}b_3(n) + \frac{3}{8}b_4(n),$$

$$\textbf{(c)}\ N(1,2,2,7;n) = \frac{7}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n/2) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) - \frac{7}{8}b_1(n) + \frac{3}{2}b_2(n) - \frac{21}{2}b_3(n) - \frac{1}{8}b_4(n),$$

$$\textbf{(d)}\ N(1,7,7,7;n) = \frac{1}{2}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n) + \frac{1}{2}\sigma_{\chi_2,\chi_1}(n) - \frac{3}{2}b_1(n) + 3b_2(n) + 3b_3(n),$$

$$\textbf{(e)}\ N(1,7,14,14;n) = \frac{1}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n/2) + \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) - \frac{1}{8}b_1(n) + \frac{3}{2}b_2(n) + \frac{3}{2}b_3(n) + \frac{1}{8}b_4(n),$$

$$\textbf{(f)}\ N(2,7,7,14;n) = \frac{1}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n/2) + \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) + \frac{3}{8}b_1(n) - \frac{1}{2}b_2(n) - \frac{1}{2}b_3(n) - \frac{3}{8}b_4(n).$$

We define the following six eta quotients

$$C_1(q) = \frac{\eta^3(2z)\eta(7z)\eta^2(8z)\eta(28z)}{\eta(z)\eta^2(4z)}, \tag{2.12}$$

$$C_2(q) = \frac{\eta(2z)\eta^2(7z)\eta(8z)\eta^3(28z)}{\eta^2(14z)\eta(56z)}, \tag{2.13}$$

$$C_3(q) = \frac{\eta^2(z)\eta^3(4z)\eta(14z)\eta(56z)}{\eta^2(2z)\eta(8z)}, \tag{2.14}$$

$$C_4(q) = \frac{\eta^6(2z)\eta(8z)\eta^4(28z)}{\eta^2(z)\eta^3(4z)\eta(14z)\eta(56z)}, \tag{2.15}$$

$$C_5(q) = \frac{\eta^4(2z)\eta(7z)\eta^6(28z)}{\eta(z)\eta(4z)\eta^3(14z)\eta^2(56z)}, \tag{2.16}$$

$$C_6(q) = \frac{\eta^4(4z)\eta^6(14z)\eta(56z)}{\eta(2z)\eta^2(7z)\eta(8z)\eta^3(28z)}, \tag{2.17}$$

and integers $c_r(n)$ $(n \in \mathbb{N})$ by

$$C_r(q) = \sum_{n=1}^{\infty} c_r(n)q^n, \quad r \in \{1,2,3,4,5,6\}. \tag{2.18}$$

Theorem 2.3. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in (1.12) for $i,j \in \{0,5\}$ and $c_r(n)$ ($r \in \{1,2,3,4,5,6\}$) be as in (2.18). Then*

$$\begin{aligned}\textbf{(a) } N(1,1,7,14;n) =& \frac{4}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{28}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &+ 4c_1(n) + \frac{10}{3}c_2(n) + \frac{4}{3}c_3(n) - 3c_4(n) - c_5(n) + c_6(n),\end{aligned}$$

$$\begin{aligned}\textbf{(b) } N(1,2,7,7;n) =& \frac{4}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{28}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &- 4c_1(n) - \frac{2}{3}c_2(n) + \frac{4}{3}c_3(n) + 3c_4(n) + c_5(n) - c_6(n),\end{aligned}$$

$$\begin{aligned}\textbf{(c) } N(1,2,14,14;n) =& \frac{2}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{14}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &+ 2c_1(n) + 3c_2(n) + c_3(n) - \frac{1}{3}c_4(n) - 2c_5(n) + \frac{2}{3}c_6(n),\end{aligned}$$

$$\begin{aligned}\textbf{(d) } N(2,2,7,14;n) =& \frac{2}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{14}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &- 2c_1(n) - 3c_2(n) - c_3(n) + \frac{5}{3}c_4(n) + 2c_5(n) + \frac{2}{3}c_6(n).\end{aligned}$$

We define the following six eta quotients

$$D_1(q) = \frac{\eta^4(2z)\eta^2(4z)\eta(14z)\eta(56z)}{\eta^2(z)\eta(8z)\eta(28z)}, \tag{2.19}$$

$$D_2(q) = \frac{\eta^2(2z)\eta^4(4z)\eta(7z)\eta(28z)}{\eta(z)\eta^2(8z)\eta(14z)}, \tag{2.20}$$

$$D_3(q) = \frac{\eta^3(2z)\eta(7z)\eta^2(56z)}{\eta(z)\eta(4z)}, \tag{2.21}$$

$$D_4(q) = \frac{\eta^3(4z)\eta^2(7z)\eta(56z)}{\eta(2z)\eta(8z)}, \tag{2.22}$$

$$D_5(q) = \frac{\eta(2z)\eta(8z)\eta^4(14z)\eta^2(28z)}{\eta(4z)\eta^2(7z)\eta(56z)}, \tag{2.23}$$

$$D_6(q) = \frac{\eta(z)\eta^2(8z)\eta^3(14z)}{\eta(7z)\eta(28z)}, \tag{2.24}$$

and integers $d_r(n)$ ($n \in \mathbb{N}$) by

$$D_r(q) = \sum_{n=1}^{\infty} d_r(n)q^n, \quad r \in \{1,2,3,4,5,6\}. \tag{2.25}$$

Theorem 2.4. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in (1.12) for $i,j \in \{0,1,4,6\}$ and $d_r(n)$ ($r \in \{1,2,3,4,5,6\}$) be as in (2.25). Then*

$$\begin{aligned}\textbf{(a) } N(1,1,1,14;n) =& -\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{14}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n)\\ &+ \frac{9}{10}d_1(n) + \frac{6}{5}d_2(n) + \frac{63}{5}d_3(n) + \frac{63}{10}d_5(n) + \frac{9}{5}d_6(n),\end{aligned}$$

$$\textbf{(b)}\ N(1,1,2,7;n) = -\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{14}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{1}{10}d_1(n) + \frac{6}{5}d_2(n) - \frac{7}{5}d_3(n) - \frac{7}{10}d_5(n) - \frac{1}{5}d_6(n),$$

$$\textbf{(c)}\ N(1,2,2,14;n) = -\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{7}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) + \frac{21}{20}d_1(n) - \frac{1}{10}d_2(n) - \frac{7}{10}d_3(n) + \frac{7}{5}d_4(n) + \frac{7}{20}d_5(n) + \frac{3}{10}d_6(n),$$

$$\textbf{(d)}\ N(1,7,7,14;n) = \frac{2}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{2}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{1}{2}d_1(n) - \frac{7}{5}d_3(n) + \frac{12}{5}d_4(n) + \frac{1}{2}d_5(n) + \frac{7}{5}d_6(n),$$

$$\textbf{(e)}\ N(1,14,14,14;n) = \frac{1}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{1}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{9}{20}d_1(n) + \frac{9}{10}d_2(n) + \frac{27}{10}d_3(n) + \frac{9}{5}d_4(n) - \frac{3}{20}d_5(n) + \frac{9}{10}d_6(n),$$

$$\textbf{(f)}\ N(2,2,2,7;n) = -\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{7}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) + \frac{51}{20}d_1(n) + \frac{9}{10}d_2(n) + \frac{63}{10}d_3(n) - \frac{63}{5}d_4(n) - \frac{63}{20}d_5(n) - \frac{27}{10}d_6(n),$$

$$\textbf{(g)}\ N(2,7,7,7;n) = \frac{2}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{2}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{3}{2}d_1(n) + \frac{3}{5}d_3(n) + \frac{12}{5}d_4(n) + \frac{3}{2}d_5(n) - \frac{3}{5}d_6(n),$$

$$\textbf{(h)}\ N(2,7,14,14;n) = \frac{1}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{1}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) + \frac{1}{20}d_1(n) - \frac{1}{10}d_2(n) - \frac{3}{10}d_3(n) - \frac{1}{5}d_4(n) + \frac{27}{20}d_5(n) - \frac{1}{10}d_6(n).$$

3. Spaces $M_2(\Gamma_0(56), \chi_i)$ for $i = 0, 3, 5, 6$

Let $N \in \mathbb{N}$ and χ be a Dirichlet character of modulus dividing N, and $\Gamma_0(N)$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a,b,c,d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Let $k \in \mathbb{Z}$. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. It is known (see for example [14, p. 83] and [13]) that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{3.1}$$

Let χ_0 be the trivial character and χ_i ($i \in \{1, 2, 3, 4, 5, 6\}$) be as in (1.13). We define the Eisenstein series

$$L(q) := E_{\chi_0,\chi_0}(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{3.2}$$

$$E_{\chi_3,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_3,\chi_0}(n) q^n, \tag{3.3}$$

$$E_{\chi_0,\chi_3}(q) = -4 + \sum_{n=1}^{\infty} \sigma_{\chi_0,\chi_3}(n) q^n, \tag{3.4}$$

$$E_{\chi_1,\chi_2}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_2}(n) q^n, \tag{3.5}$$

$$E_{\chi_2,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_2,\chi_1}(n) q^n, \tag{3.6}$$

$$E_{\chi_5,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_5,\chi_0}(n) q^n, \tag{3.7}$$

$$E_{\chi_0,\chi_5}(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{\chi_0,\chi_5}(n) q^n, \tag{3.8}$$

$$E_{\chi_4,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_4,\chi_1}(n) q^n, \tag{3.9}$$

$$E_{\chi_1,\chi_4}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_4}(n) q^n, \tag{3.10}$$

$$E_{\chi_6,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_6,\chi_0}(n) q^n, \tag{3.11}$$

$$E_{\chi_0,\chi_6}(q) = -10 + \sum_{n=1}^{\infty} \sigma_{\chi_0,\chi_6}(n) q^n. \tag{3.12}$$

We use the following lemma to determine if certain eta quotients are modular forms. See [6, p. 174], [8, Corollary 2.3, p. 37], [9, Theorem 5.7, p. 99] and [10].

Lemma 3.1. (Ligozat's Criteria) *Let* $N \in \mathbb{N}$ *and* $f(z) = \prod_{1 \leq \delta | N} \eta^{r_\delta}(\delta z)$ *be an* *eta quotient. Let* $s = \prod_{1 \le \delta | N} \delta^{|r_\delta|}$ *and* $k = \frac{1}{2} \sum_{1 \le \delta | N} r_\delta$. *Suppose that the following conditions are satisfied:*

(L1) $\sum_{1 \le \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(L2) $\sum_{1 \le \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(L3) $\sum_{1 \le \delta | N} \frac{\gcd(d, \delta)^2 \cdot r_\delta}{\delta} \ge 0$ *for each positive divisor* d *of* N,

(L4) k *is an integer.*

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, *where the character* χ *is given by* $\chi(m) = \left(\frac{(-1)^k s}{m}\right)$.

(L3)$'$ *In addition to the above conditions, if the inequality in* (L3) *is strict for each positive divisor* d *of* N, *then* $f(z) \in S_k(\Gamma_0(N), \chi)$.

We note that the eta quotients given by (2.1)–(2.5), (2.7)–(2.10), (2.12)–(2.17) and (2.19)–(2.24) are constructed with MAPLE in a way that they satisfy the conditions of Lemma 3.1 for $N = 56$ and $k = 2$.

The quaternary quadratic forms (a_1, a_2, a_3, a_4) given in (1.2) are grouped in Table 3.1 according to the modular spaces to which $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ belong.

$M_2(\Gamma_0(56), \chi_0)$	$M_2(\Gamma_0(56), \chi_3)$	$M_2(\Gamma_0(56), \chi_5)$	$M_2(\Gamma_0(56), \chi_6)$
$(1, 1, 1, 1)$	$(1, 1, 1, 7)$	$(1, 1, 1, 2)$	$(1, 1, 1, 14)$
$(1, 1, 2, 2)$	$(1, 1, 2, 14)$	$(1, 1, 7, 14)$	$(1, 1, 2, 7)$
$(1, 1, 7, 7)$	$(1, 2, 2, 7)$	$(1, 2, 2, 2)$	$(1, 2, 2, 14)$
$(1, 1, 14, 14)$	$(1, 7, 7, 7)$	$(1, 1, 7, 14)$	$(1, 7, 7, 14)$
$(1, 2, 7, 14)$	$(1, 7, 14, 14)$	$(1, 2, 14, 14)$	$(1, 14, 14, 14)$
$(2, 2, 7, 7)$	$(2, 7, 7, 14)$	$(2, 2, 7, 14)$	$(2, 2, 2, 7)$
			$(2, 7, 7, 7)$
			$(2, 7, 14, 14)$

Table 3.1

Theorem 3.1. *Let* χ_0 *be the trivial character and* χ_3, χ_5, χ_6 *be as in* (1.13). *If* (a_1, a_2, a_3, a_4) *is in the first, second, third or fourth column of* Table 3.1, *then*

(a) $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_0)$,

(b) $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_3)$,

(c) $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_5)$,

(d) $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_6)$,

respectively.

Proof. We prove only (a) as the remaining ones can be proven similarly. From (1.11) we have

$$\varphi^2(q)\varphi^2(q^7) = \frac{\eta^{10}(2z)\eta^{10}(14z)}{\eta^4(z)\eta^4(4z)\eta^4(7z)\eta^4(28z)}, \tag{3.13}$$

$$\varphi^2(q^2)\varphi^2(q^7) = \frac{\eta^{10}(4z)\eta^{10}(14z)}{\eta^4(2z)\eta^4(7z)\eta^4(8z)\eta^4(28z)}, \tag{3.14}$$

$$\varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14}) = \frac{\eta^3(2z)\eta^3(4z)\eta^3(14z)\eta^3(28z)}{\eta^2(z)\eta^2(7z)\eta^2(8z)\eta^2(56z)}, \tag{3.15}$$

$$\varphi^2(q)\varphi^2(q^{14}) = \frac{\eta^{10}(2z)\eta^{10}(28z)}{\eta^4(z)\eta^4(4z)\eta^4(14z)\eta^4(56z)}. \tag{3.16}$$

Then the assertions directly follow from (3.13)–(3.16) and Lemma 3.1. □

We deduce from [14, Sec. 6.1, p. 93] that

$$\dim(E_2(\Gamma_0(56), \chi_0)) = 7, \ \dim(S_2(\Gamma_0(56), \chi_0)) = 5. \tag{3.17}$$

Also we deduce from [14, Sec. 6.3, p. 98] that

$$\dim(E_2(\Gamma_0(56), \chi_3)) = 8, \ \dim(S_2(\Gamma_0(56), \chi_3)) = 4, \tag{3.18}$$

$$\dim(E_2(\Gamma_0(56), \chi_5)) = 4, \ \dim(S_2(\Gamma_0(56), \chi_5)) = 6, \tag{3.19}$$

$$\dim(E_2(\Gamma_0(56), \chi_6)) = 4, \ \dim(S_2(\Gamma_0(56), \chi_6)) = 6. \tag{3.20}$$

Theorem 3.2. *Let χ_0 be the trivial character and χ_3, χ_5, χ_6 be as in* (1.13).

(a) $A_r(q)$ $(r \in \{1, 2, 3, 4, 5\})$ *given by* (2.1)–(2.5) *form a basis for* $S_2(\Gamma_0(56), \chi_0)$.

(b) $B_r(q)$ $(r \in \{1, 2, 3, 4\})$ *given by* (2.7)–(2.10) *form a basis for* $S_2(\Gamma_0(56), \chi_3)$.

(c) $C_r(q)$ $(r \in \{1, 2, 3, 4, 5, 6\})$ *given by* (2.12)–(2.17) *form a basis for* $S_2(\Gamma_0(56), \chi_5)$.

(d) $D_r(q)$ $(r \in \{1, 2, 3, 4, 5, 6\})$ *given by* (2.19)–(2.24) *form a basis for* $S_2(\Gamma_0(56), \chi_6)$.

Proof. (a) The eta quotients $A_r(q)$ $(r \in \{1, 2, 3, 4, 5\})$ are linearly independent over $\mathbb{C}$. For each $r \in \{1, 2, 3, 4, 5\}$, $A_r(q) \in S_2(\Gamma_0(56), \chi_0)$ by Lemma 3.1. Then the assertion follows from (3.17). Similarly appealing to Lemma 3.1, the parts (b), (c) and (d) follow from (3.18), (3.19) and (3.20) respectively. □

Theorem 3.3. *Let χ_0 be the trivial character and χ_i $(1 \leq i \leq 6)$ be as in* (1.13).

(a) $\{L(q) - tL(q^t) \mid t = 2, 4, 7, 8, 14, 28, 56\} \cup \{A_r(q) \mid r = 1, 2, 3, 4, 5\}$ *is a basis for* $M_2(\Gamma_0(56), \chi_0)$.

(b) $\{E_{\chi_3,\chi_0}(q^t), E_{\chi_0,\chi_3}(q^t), E_{\chi_1,\chi_2}(q^t), E_{\chi_2,\chi_1}(q^t) \mid t = 1, 2\} \cup \{B_r(q) \mid r = 1, 2, 3, 4\}$ *is a basis for* $M_2(\Gamma_0(56), \chi_3)$.

(c) $\{E_{\chi_5,\chi_0}(q^t), E_{\chi_0,\chi_5}(q^t) \mid t = 1, 7\} \cup \{C_r(q) \mid r = 1, 2, 3, 4, 5, 6\}$ *is a basis for* $M_2(\Gamma_0(56), \chi_5)$.

(d) $\{E_{\chi_6,\chi_0}(q), E_{\chi_0,\chi_6}(q), E_{\chi_4,\chi_1}(q), E_{\chi_1,\chi_4}(q)\} \cup \{D_r(q) \mid r = 1, 2, 3, 4, 5, 6\}$ *is a basis for* $M_2(\Gamma_0(56), \chi_6)$.

Proof. (a) It follows from [14, Theorem 5.9, p. 88] with $\chi = \psi = \chi_0$ that $\{L(q) - tL(q^t) \mid t = 2, 4, 7, 8, 14, 28, 56\}$ is a basis for $E_2(\Gamma_0(56), \chi_0)$. Then the assertion follows from (3.1), (3.17) and Theorem 3.2(a).

(b) Appealing to [14, Theorem 5.9, p. 88] with $\epsilon = \chi_3$ and $\chi, \psi \in \{\chi_0, \chi_1, \chi_2, \chi_3\}$, we see that $\{E_{\chi_0,\chi_3}(q^t), E_{\chi_1,\chi_2}(q^t), E_{\chi_2,\chi_1}(q^t), E_{\chi_3,\chi_0}(q^t) \mid t = 1, 2\}$ is a basis for $E_2(\Gamma_0(56), \chi_3)$. Then the assertion follows from (3.1), (3.18) and Theorem 3.2(b).

(c) Appealing to [14, Theorem 5.9, p. 88] with $\epsilon = \chi_5$ and $\chi, \psi \in \{\chi_0, \chi_5\}$, we see that $\{E_{\chi_0,\chi_5}(q^t), E_{\chi_5,\chi_0}(q^t) \mid t = 1, 7\}$ is a basis for $E_2(\Gamma_0(56), \chi_5)$. Then the assertion follows from (3.1), (3.19) and Theorem 3.2(c).

(d) By [14, Theorem 5.9, p. 88] with $\epsilon = \chi_6$ and $\chi, \psi \in \{\chi_0, \chi_1, \chi_4, \chi_6\}$, $\{E_{\chi_0,\chi_6}(q), E_{\chi_6,\chi_0}(q), E_{\chi_1,\chi_4}(q), E_{\chi_4,\chi_1}(q)\}$ is a basis for $E_2(\Gamma_0(56), \chi_6)$. Then the assertion follows from (3.1), (3.20) and Theorem 3.2(d). □

4. Theta Function Identities

In this section we state some theta function identities (Theorems 4.1–4.4) from which we deduce Theorems 2.1–2.4.

Theorem 4.1.

$$\begin{aligned}
\textbf{(a)}\ \varphi^2(q)\varphi^2(q^7) =& \frac{4}{3}\ L(q) - \frac{8}{3}\ L(q^2) + \frac{16}{3}\ L(q^4) - \frac{28}{3}\ L(q^7) + \frac{56}{3}\ L(q^{14}) \\
& - \frac{112}{3}\ L(q^{28}) + \frac{8}{3} A_3(q), \\
\textbf{(b)}\ \varphi^2(q)\varphi^2(q^{14}) =& \frac{2}{3}\ L(q) - \frac{2}{3}\ L(q^2) - \frac{4}{3}\ L(q^4) - \frac{14}{3}\ L(q^7) + \frac{16}{3}\ L(q^8) \\
& + \frac{14}{3}\ L(q^{14}) + \frac{28}{3}\ L(q^{28}) - \frac{112}{3}\ L(q^{56}) + \frac{2}{3} A_1(q) - 4A_2(q) \\
& + \frac{10}{3} A_3(q) - \frac{4}{3} A_4(q) + 8A_5(q),
\end{aligned}$$

$$\textbf{(c)}\ \varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14}) = \frac{2}{3}\ L(q) - \frac{2}{3}\ L(q^2) - \frac{4}{3}\ L(q^4) - \frac{14}{3}\ L(q^7) + \frac{16}{3}\ L(q^8) + \frac{14}{3}\ L(q^{14}) + \frac{28}{3}\ L(q^{28}) - \frac{112}{3}\ L(q^{56}) + \frac{2}{3}A_1(q) + \frac{4}{3}A_3(q) - \frac{4}{3}A_4(q) + 4A_5(q),$$

$$\textbf{(d)}\ \varphi^2(q^2)\varphi^2(q^7) = \frac{2}{3}\ L(q) - \frac{2}{3}\ L(q^2) - \frac{4}{3}\ L(q^4) - \frac{14}{3}\ L(q^7) + \frac{16}{3}\ L(q^8) + \frac{14}{3}\ L(q^{14}) + \frac{28}{3}\ L(q^{28}) - \frac{112}{3}\ L(q^{56}) - \frac{10}{3}A_1(q) + 4A_2(q) - \frac{2}{3}A_3(q) + \frac{20}{3}A_4(q) - 8A_5(q).$$

Proof. Let (a_1, a_2, a_3, a_4) be any of the quaternary quadratic forms listed in the first column of Table 3.1. By Theorem 3.1(a), we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_0)$. Then, by Theorem 3.3(a), $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $L(q)-tL(q^t)$ $(t = 2, 4, 7, 8, 14, 28, 56)$ and $A_r(q)$ $(r = 1, 2, 3, 4, 5)$, namely there exist coefficients $x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_1, y_2, y_3, y_4, y_5 \in \mathbb{C}$ such that

$$\begin{aligned} \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \quad = \quad & x_1\ (L(q) - 2L(q^2)) + x_2\ (L(q) - 4L(q^4)) \\ & +x_3\ (L(q) - 5L(q^5)) + x_4\ (L(q) - 8L(q^8)) \\ & +x_5\ (L(q) - 10L(q^{10})) + x_6\ (L(q) - 20L(q^{20})) \\ & +x_7\ (L(q) - 40L(q^{40})) + y_1A_1(q) + y_2A_2(q) \\ & +y_3A_3(q) + y_4A_4(q) + y_5A_5(q). \end{aligned} \tag{4.1}$$

We now prove part (a) of the theorem as the remaining parts can be proven similarly. Let $(a_1, a_2, a_3, a_4) = (1, 1, 7, 7)$. Appealing to [9, Theorem 3.13], we find that the Sturm bound for the modular space $M_2(\Gamma_0(56))$ is 16. So, equating the coefficients of q^n for $0 \le n \le 16$ on both sides of (4.1), we find a system of linear equations with the unknowns x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_1, y_2, y_3, y_4 and y_5. Then, using MAPLE we solve this system, and find that

$$x_1 = x_3 = x_6 = \frac{4}{3},\ x_2 = x_5 = -\frac{4}{3},\ x_7 = 0,\ y_1 = y_2 = y_4 = y_5 = 0,\ y_3 = \frac{8}{3}.$$

Substituting these values into (4.1) and doing some simplifications, we obtain the equation in part (a). □

Theorems 4.2–4.4 below can be proven similarly.

Theorem 4.2. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in* (1.13). *Then*

$$\textbf{(a)}\ \varphi^3(q)\varphi(q^7) = \frac{7}{2}E_{\chi_3,\chi_0}(q) - \frac{1}{4}E_{\chi_0,\chi_3}(q) + \frac{7}{4}E_{\chi_1,\chi_2}(q) - \frac{1}{2}E_{\chi_2,\chi_1}(q) + 3B_2(q) - 21B_3(q) - \frac{3}{2}B_4(q),$$

(b) $$\varphi^2(q)\varphi(q^2)\varphi(q^{14}) = \frac{7}{4}E_{\chi_3,\chi_0}(q) - \frac{1}{4}E_{\chi_0,\chi_3}(q^2) + \frac{7}{4}E_{\chi_1,\chi_2}(q^2) - \frac{1}{4}E_{\chi_2,\chi_1}(q) + \frac{21}{8}B_1(q) - \frac{1}{2}B_2(q) + \frac{7}{2}B_3(q) + \frac{3}{8}B_4(q),$$

(c) $$\varphi(q)\varphi^2(q^2)\varphi(q^7) = \frac{7}{4}E_{\chi_3,\chi_0}(q) - \frac{1}{4}E_{\chi_0,\chi_3}(q^2) + \frac{7}{4}E_{\chi_1,\chi_2}(q^2) - \frac{1}{4}E_{\chi_2,\chi_1}(q) - \frac{7}{8}B_1(q) + \frac{3}{2}B_2(q) - \frac{21}{2}B_3(q) - \frac{1}{8}B_4(q),$$

(d) $$\varphi(q)\varphi^3(q^7) = \frac{1}{2}E_{\chi_3,\chi_0}(q) - \frac{1}{4}E_{\chi_0,\chi_3}(q) - \frac{1}{4}E_{\chi_1,\chi_2}(q) + \frac{1}{2}E_{\chi_2,\chi_1}(q) - \frac{3}{2}B_1(q) + 3B_2(q) + 3B_3(q),$$

(e) $$\varphi(q)\varphi(q^7)\varphi^2(q^{14}) = \frac{1}{4}E_{\chi_3,\chi_0}(q) - \frac{1}{4}E_{\chi_0,\chi_3}(q^2) - \frac{1}{4}E_{\chi_1,\chi_2}(q^2) + \frac{1}{4}E_{\chi_2,\chi_1}(q) - \frac{1}{8}B_1(q) + \frac{3}{2}B_2(q) + \frac{3}{2}B_3(q) + \frac{1}{8}B_4(q),$$

(f) $$\varphi(q^2)\varphi^2(q^7)\varphi(q^{14}) = \frac{1}{4}E_{\chi_3,\chi_0}(q) - \frac{1}{4}E_{\chi_0,\chi_3}(q^2) - \frac{1}{4}E_{\chi_1,\chi_2}(q^2) + \frac{1}{4}E_{\chi_2,\chi_1}(q) + \frac{3}{8}B_1(q) - \frac{1}{2}B_2(q) - \frac{1}{2}B_3(q) - \frac{3}{8}B_4(q).$$

Theorem 4.3. *Let χ_0 be the trivial character and χ_5 be as in* (1.13). *Then*

(a) $$\varphi^2(q)\varphi(q^7)\varphi(q^{14}) = \frac{4}{3}E_{\chi_5,\chi_0}(q) - \frac{28}{3}E_{\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{\chi_0,\chi_5}(q) - \frac{7}{3}E_{\chi_0,\chi_5}(q^7) + 4C_1(q) + \frac{10}{3}C_2(q) + \frac{4}{3}C_3(q) - 3C_4(q) - C_5(q) + C_6(q),$$

(b) $$\varphi(q)\varphi(q^2)\varphi^2(q^7) = \frac{4}{3}E_{\chi_5,\chi_0}(q) - \frac{28}{3}E_{\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{\chi_0,\chi_5}(q) - \frac{7}{3}E_{\chi_0,\chi_5}(q^7) - 4C_1(q) - \frac{2}{3}C_2(q) + \frac{4}{3}C_3(q) + 3C_4(q) + C_5(q) - C_6(q),$$

(c) $$\varphi(q)\varphi(q^2)\varphi^2(q^{14}) = \frac{2}{3}E_{\chi_5,\chi_0}(q) - \frac{14}{3}E_{\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{\chi_0,\chi_5}(q) - \frac{7}{3}E_{\chi_0,\chi_5}(q^7) + 2C_1(q) + 3C_2(q) + C_3(q) - \frac{1}{3}C_4(q) - 2C_5(q) + \frac{2}{3}C_6(q),$$

(d) $$\varphi^2(q^2)\varphi(q^7)\varphi(q^{14}) = \frac{2}{3}E_{\chi_5,\chi_0}(q) - \frac{14}{3}E_{\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{\chi_0,\chi_5}(q) - \frac{7}{3}E_{\chi_0,\chi_5}(q^7) - 2C_1(q) - 3C_2(q) - C_3(q) + \frac{5}{3}C_4(q) + 2C_5(q) + \frac{2}{3}C_6(q).$$

Theorem 4.4. *Let χ_0 be the trivial character and χ_1, χ_4, χ_6 be as in* (1.13). *Then*

(a) $$\varphi^3(q)\varphi(q^{14}) = -\frac{2}{5}E_{\chi_4,\chi_1}(q) + \frac{7}{10}E_{\chi_1,\chi_4}(q) + \frac{14}{5}E_{\chi_6,\chi_0}(q) - \frac{1}{10}E_{\chi_0,\chi_6}(q) + \frac{9}{10}D_1(q) + \frac{6}{5}D_2(q) + \frac{63}{5}D_3(q) + \frac{63}{10}D_5(q) + \frac{9}{5}D_6(q),$$

$$\begin{aligned}
\textbf{(b)}\ \varphi^2(q)\varphi(q^2)\varphi(q^7) = & -\frac{2}{5}E_{\chi_4,\chi_1}(q)+\frac{7}{10}E_{\chi_1,\chi_4}(q)+\frac{14}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& -\frac{1}{10}D_1(q)+\frac{6}{5}D_2(q)-\frac{7}{5}D_3(q)-\frac{7}{10}D_5(q)-\frac{1}{5}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(c)}\ \varphi(q)\varphi^2(q^2)\varphi(q^{14}) = & -\frac{1}{5}E_{\chi_4,\chi_1}(q)+\frac{7}{10}E_{\chi_1,\chi_4}(q)+\frac{7}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& +\frac{21}{20}D_1(q)-\frac{1}{10}D_2(q)-\frac{7}{10}D_3(q)\\
& +\frac{7}{5}D_4(q)+\frac{7}{20}D_5(q)+\frac{3}{10}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(d)}\ \varphi(q)\varphi^2(q^7)\varphi(q^{14}) = & \frac{2}{5}E_{\chi_4,\chi_1}(q)-\frac{1}{10}E_{\chi_1,\chi_4}(q)+\frac{2}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& -\frac{1}{2}D_1(q)-\frac{7}{5}D_3(q)+\frac{12}{5}D_4(q)+\frac{1}{2}D_5(q)+\frac{7}{5}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(e)}\ \varphi(q)\varphi^3(q^{14}) = & \frac{1}{5}E_{\chi_4,\chi_1}(q)-\frac{1}{10}E_{\chi_1,\chi_4}(q)+\frac{1}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& -\frac{9}{20}D_1(q)+\frac{9}{10}D_2(q)+\frac{27}{10}D_3(q)+\frac{9}{5}D_4(q)-\frac{3}{20}D_5(q)\\
& +\frac{9}{10}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(f)}\ \varphi^3(q^2)\varphi(q^7) = & -\frac{1}{5}E_{\chi_4,\chi_1}(q)+\frac{7}{10}E_{\chi_1,\chi_4}(q)+\frac{7}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& +\frac{51}{20}D_1(q)+\frac{9}{10}D_2(q)+\frac{63}{10}D_3(q)-\frac{63}{5}D_4(q)-\frac{63}{20}D_5(q)\\
& -\frac{27}{10}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(g)}\ \varphi(q^2)\varphi^3(q^7) = & \frac{2}{5}E_{\chi_4,\chi_1}(q)-\frac{1}{10}E_{\chi_1,\chi_4}(q)+\frac{2}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& -\frac{3}{2}D_1(q)+\frac{3}{5}D_3(q)+\frac{12}{5}D_4(q)+\frac{3}{2}D_5(q)-\frac{3}{5}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(h)}\ \varphi(q^2)\varphi(q^7)\varphi^2(q^{14}) = & \frac{1}{5}E_{\chi_4,\chi_1}(q)-\frac{1}{10}E_{\chi_1,\chi_4}(q)+\frac{1}{5}E_{\chi_6,\chi_0}(q)-\frac{1}{10}E_{\chi_0,\chi_6}(q)\\
& +\frac{1}{20}D_1(q)-\frac{1}{10}D_2(q)-\frac{3}{10}D_3(q)\\
& -\frac{1}{5}D_4(q)+\frac{27}{20}D_5(q)-\frac{1}{10}D_6(q).
\end{aligned}$$

5. Proofs of Main Results

Theorem 2.1 follows from (1.5), (2.6), (3.2) and Theorem 4.1. Theorem 2.2 follows from (1.5), (2.11), (3.3)–(3.6) and Theorem 4.2. Theorem 2.3 follows from (1.5), (2.18), (3.7), (3.8) and Theorem 4.3. Theorem 2.4 follows from (1.5), (2.25), (3.9)–(3.12) and Theorem 4.4.

6. Remarks

By Theorem 3.1(a), $\varphi^4(q)$ and $\varphi^2(q)\varphi^2(q^2)$ are in $M_2(\Gamma_0(56),\chi_0)$. Appealing to Theorem 3.3(a), we have

$$\begin{aligned}\varphi^4(q) =&8L(q) - 32L(q^4),\\ \varphi^2(q)\varphi^2(q^2) =&4L(q) - 4L(q^2) + 8L(q^4) - 32L(q^8),\end{aligned}$$

from which we deduce the well-known results (1.3) and (1.4) respectively.

By Theorem 3.1(c), $\varphi^3(q)\varphi(q^2)$ and $\varphi(q)\varphi^3(q^2)$ are in $M_2(\Gamma_0(56),\chi_5)$. Appealing to Theorem 3.3(c), we obtain

$$\varphi^3(q)\varphi(q^2) = -2E_{\chi_0,\chi_5}(q) + 8E_{\chi_5,\chi_0}(q), \tag{6.1}$$

$$\varphi(q)\varphi^3(q^2) = -2E_{\chi_0,\chi_5}(q) + 4E_{\chi_5,\chi_0}(q). \tag{6.2}$$

Then appealing to (1.5), (3.7), (3.8), (6.1) and (6.2) we deduce that for $n \in \mathbb{N}$

$$N(1,1,1,2;n) = -2\sigma_{\chi_0,\chi_5}(n) + 8\sigma_{\chi_5,\chi_0}(n) = -2\sum_{d|n}\left(\frac{8}{d}\right)d + 8\sum_{d|n}\left(\frac{8}{d}\right)\frac{n}{d},$$

$$N(1,2,2,2;n) = -2\sigma_{\chi_0,\chi_5}(n) + 4\sigma_{\chi_5,\chi_0}(n) = -2\sum_{d|n}\left(\frac{8}{d}\right)d + 4\sum_{d|n}\left(\frac{8}{d}\right)\frac{n}{d},$$

which agree with the results in [15, (4.3), (5.3)] and [1, Theorem 5.1, Theorem 5.2].

One can show that $A_3(-q) = -\eta(z)\eta(2z)\eta(7z)\eta(14z)$, where $A_3(q)$ is given by (2.3). We note that $-A_3(-q)$ is used in [5, (3.16)] to give an expression for $\varphi^2(-q)\varphi^2(-q^7)$ as a linear combination of Eisenstein series and a cusp form, from which we deduce the formula for $N(1,1,7,7;n)$ given in Theorem 2.1(a).

It would be interesting to determine general formulas for the number of representations of a positive integer n by the quaternary quadratic forms with coefficients in $\{1, p_1, p_2, p_1p_2\}$, where p_1 and p_2 are distinct prime numbers. The case when $p_1 = 2$ and $p_2 = 5$ is treated in [3].

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