VARIANTS OF BULGARIAN SOLITAIRE

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Abstract

A variant of the object redistribution game Bulgarian Solitaire may be defined using rules for pile reduction which vary among piles of different sizes, but remain consistent for piles of a particular size throughout a single game. Such a variant may produce behavior that deviates significantly from that established for the original version. Bounds on maximum cycle length found in such variants are provided, as are conditions associated with cycles of positions which are "isolated," in the sense that no position outside the cycle leads to any position within it.

1. Introduction

The game known as Bulgarian Solitaire came to the attention of mathematicians in the early 1980s. The papers about it from that time include [2, 8, 1]. In Bulgarian Solitaire, a set of cards (or any other type of stackable objects) is separated into any number of piles, each of any desired size. One "turn" of the game involves removing the top card from each pile, and using the collected cards to make a new pile, which is then reduced on subsequent turns along with whatever piles remain. Turns are taken until a repeating cycle of identical positions is reached. The original interest in the game came from the desire to explain the empirical observation that when a triangular number $T_k = 1 + 2 + \cdots + k$ of cards is used, one inevitably reaches the "triangular position" consisting of piles of size $1, 2, \ldots, k-1$, and k, regardless one's choice of starting position. If the number of cards is not triangular, then different starting positions may lead to different cycles of positions which are "nearly" triangular. Characterizing these cycles is the main contribution of Akin and Davis [1]; some of their results are summarized in Theorem 3 below. Because players only follow rules mechanically, and do not choose between different possible courses of action, Bulgarian Solitaire is not really "solitaire" in the sense most people mean the word. (Nor is it particularly Bulgarian! See [5] for information on the game's origins.)

Many modifications of this basic game have been suggested in the literature. Both Carolina Solitaire [4] and Montreal Solitaire [3] distinguish not just the size of piles but also the order of their creation; the latter also keeps track of some vacated positions. Austrian Solitaire [1] collects cards in a "bank," from which new piles are created only of a certain fixed size. In [9], a stochastic version is analyzed in which piles are reduced only with a certain probability. In another version [7], the player creates several new stacks via some fixed number of rounds of pile reduction before adding them all simultaneously to the playing field. Still another [5] requires the player to reduce only those piles which have at least a certain fixed height. (We are grateful to Brian Hopkins for sharing his knowledge about these variants.)

Common to all variants mentioned is that piles are reduced by either one or no cards. We wish to put forward what we believe is a novel variant of Bulgarian Solitaire — actually, a whole family of variants — based on rules for pile reduction which allow one to remove any number of cards from a given pile. The number removed varies among piles of different sizes, but remains fixed for all piles of a particular size throughout the game. As in the original, the collected cards are used to create a single new pile, which is reduced on subsequent turns, according to the rule for piles of its size. To give one simple example, a player could remove two cards from every pile, except for piles with a single card, which are eliminated. In a more interesting example, a player could remove three cards from any pile divisible by four, and otherwise remove one card. In this game, a player beginning with piles of sizes 11, 9, 8, 4, and 2, would, after a single move, have reduced piles of sizes 10, 8, 5, 1, and 1, and a new pile of size 9. The next position would have piles of size 9, 8, 8, 5, and 4.

Many of the questions asked and answered for Bulgarian Solitaire thirty years ago can be looked at anew with each such variant: How many turns until a position is repeated? How long can cycles of positions be? How many cycles are possible? To discuss some of these questions, we introduce notation and terminology for the formal representation of these variants.

Unless noted otherwise, n is a fixed but arbitrary positive integer. A *partition* of n is a way of writing n as a sum of positive integers, in which the order of the addends is irrelevant. We think of partitions as multisets of positive integers, written $p = (p_1, p_2, \ldots, p_m)$. By convention, p_i are placed in non-increasing order. Each p_i is called a *part* of p. The set of all partitions of n is denoted by P_n. A *rule* (*for* n) is a function $\sigma : \{1, 2, \ldots, n\} \longrightarrow \{0, 1, \ldots, n-1\}$ for which $\sigma(i) < i$. For a rule σ and $i \leq n$, let $\Delta i = i - \sigma(i)$. Also, for $p \in P_n$, let $\Delta p = \sum \Delta p_i$. (We should write something like $\Delta^\sigma i$ and so on, but σ will always be apparent from context.) A rule σ may be used to define a function $\bar\sigma$ with domain P_n as follows:

$$\bar\sigma\, p = (\ \sigma(p_1),\ \sigma(p_2),\ \ldots,\ \sigma(p_m),\ \Delta p\ ).$$

So $\bar\sigma\, p$ is a partition, and as such its parts may need reordering to conform with convention. It follows that $\bar{\sigma}\, p \in P_n$, because

$$\Delta p = \sum \Delta p_i = \sum (p_i - \sigma(p_i)) = \sum p_i - \sum \sigma(p_i) = n - \sum \sigma(p_i)$$

and therefore

$$\left(\sum \sigma(p_i)\right) + \Delta p = \left(\sum \sigma(p_i)\right) + n - \sum \sigma(p_i) = n.$$

Thus $\bar{\sigma}$ is an operator on P_n. The intuition for our purposes states that the piles of n cards represented by p are each reduced according to rule σ, and the collected cards are deposited in a new pile, Δp. Each rule for n can be thought of as defining one possible variant of Bulgarian Solitaire. (From now on, when we speak of a *variant* of Bulgarian Solitaire, we mean one defined this way.) The original version is given by the rule which we call β, defined as $\beta(i) = i - 1$. It is easy to see that there are $n!$ distinct rules for n. However, many rules which are distinct as functions produce identical operators. For example, if σ and τ are rules for n which are identical except that $\sigma(n) = n - \tau(n)$, it follows that $\bar{\sigma} = \bar{\tau}$.

Example 1. Consider the rule for $n = 8$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 1 & 3 & 0 & 3 & 6 \end{pmatrix},$$

that is, $\sigma(1) = \sigma(6) = 0$, $\sigma(2) = \sigma(4) = 1$, and so on. Beginning with the partition $p = (4, 4)$, we have $\bar{\sigma}\, p = (\sigma(4), \sigma(4), \Delta\, p) = (6, 1, 1)$. Since the parts 1 and 6 are sent to zero by σ, we have $\bar{\sigma}\,(6, 1, 1) = (8)$. One may check that the following sequence of partitions results from successive applications of $\bar{\sigma}$ (indicated by $\rightarrow$):

$$\begin{aligned} &(4,4) \rightarrow (6,1,1) \rightarrow (8) \rightarrow (6,2) \rightarrow (7,1) \rightarrow (5,3) \\ &\qquad\qquad\qquad\qquad \rightarrow (3,3,2) \rightarrow (3,2,2,1) \rightarrow (4,2,1,1) \rightarrow (6,1,1) \end{aligned}$$

The sequence of positions following $(6, 1, 1)$ will continue repeating with further applications of $\bar{\sigma}$.

For $p = (p_1, \ldots, p_m)$, let $\ell(p) = m$, i.e., $\ell(p)$ is the number of parts in p.

Proposition 2. *Let $p \in P_n$, and let σ be a rule for n. Then*

$$(\bar{\sigma}\, p)_1 \geq \Delta p \geq \ell(p) \geq \ell(\bar{\sigma}\, p) - 1.$$

Proof. First, Δp is a part of $\bar{\sigma}\, p$, and so it is not more that the maximum of those parts, which by convention is $(\bar{\sigma}\, p)_1$. Because $p_i > \sigma(p_i)$, we have $\Delta p_i = p_i - \sigma(p_i) \geq 1$, and so $\Delta p = \sum \Delta p_i \geq 1 \cdot \ell(p) = \ell(p)$. It may be that $\sigma(p_i) > 0$ for all i; in this case, $\bar{\sigma}\, p$ has one part for each part of p, plus a new part, Δp. If $\sigma(p_i) = 0$ sometimes, then $\bar{\sigma}\, p$ has even fewer parts. Therefore, $\ell(\bar{\sigma}\, p) \leq \ell(p) + 1$, i.e., $\ell(p) \geq \ell(\bar{\sigma}\, p) - 1$. $\square$

2. Bounds on Cycles

If σ is a rule for n and i is a positive integer, then $\bar{\sigma}^i$ represents i-fold application of the operator to members of P_n. If $\bar{\sigma}^i\, p = p$ for some positive integer i, then p is called *cyclic* (with respect to $\bar{\sigma}$) . The orbit of a cyclic element under $\bar{\sigma}$ is called a *cycle* (with respect to $\bar{\sigma}$). The *length* of a cycle C is just $|C|$. For every $p \in P_n$, $\bar{\sigma}^i\, p$ must be cyclic for some i, because P_n is finite.

Much of our interest in variants of Bulgarian Solitaire is directed at cycles. Length of cycles in many variants differs from that of cycles found in the original version. In particular, cycles of significantly greater length can be found. In order to compare, let us briefly review the relevant results for Bulgarian Solitaire. Define $t_k = (k, k-1, \ldots, 2, 1)$ (so obviously $t_k \in P_{T_k}$). Also, let $\bar{d} \in \{0,1\}^{k+1}$, and define $t_k + \bar{d}$ to be the partition which results from coordinate-wise addition, where t_k is "padded" with a zero on the right if necessary. For example, $t_4 + \langle 1,0,0,1,1\rangle = (5,3,2,2,1)$. If $\bar{d} = \langle d_1, d_2, \ldots, d_{k+1}\rangle$, then $\bar{d}^r$ is defined as the sequence resulting from applying a rightward, wrap-around shift to $\bar{d}$, i.e., $\bar{d}^r = \langle d_{k+1}, d_1, \ldots, d_k\rangle$. Similarly, $\bar{d}^\ell = \langle d_2, d_3, \ldots, d_{k+1}, d_1\rangle$.

Theorem 3 ([1])**.** *Let $p \in P_n$, and let k be greatest such that $n \geq T_k$. Then for Bulgarian Solitaire with n cards (with rule $\beta(i) = i-1$),*

1. *p is cyclic if and only if $p = t_k + \bar{d}$ for some $\bar{d} \in \{0,1\}^{k+1}$.*
2. *In this case, $\bar{\beta}\,(t_k + \bar{d}) = t_k + \bar{d}^r$ and $\bar{\beta}\,(t_k + \bar{d}^\ell) = t_k + \bar{d}$.*
3. *As a consequence, $\bar{\beta}^{k+1}\,(t_k + \bar{d}) = t_k + \bar{d}$, and*
4. *$\bar{\beta}\, p = p$ if and only if $n = T_k$ and $p = t_k$.*

Example 4. Since 12 lies between $T_4 = 10$ and $T_5 = 15$, the cyclic positions in Bulgarian Solitaire for $n = 12$ have the form $t_4 + \bar{d}$, where $\sum d_i = 2$; there are $\binom{5}{2} = 10$ distinct $\bar{d} \in \{0,1\}^5$ with this property. With a little scratch work, one finds that the cyclic partitions form two cycles, both of length 5, which are the orbits of $(5,4,2,1) = t_4 + \langle 1,1,0,0,0\rangle$ and $(5,3,3,1) = t_4 + \langle 1,0,1,0,0\rangle$.

Since $T_k = \frac{1}{2}(k^2 + k)$, we can deduce that maximum cycle length in Bulgarian Solitaire with n cards is bounded above by $1 + \sqrt{2n}$, using Theorem 3. Of course, cycle length should be compared to $|P_n|$, which due to results of Hardy, Ramanujan, and Rademacher [10] is known to be asymptotic to $\exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$. Another point of interest in Bulgarian Solitaire is the difference in these two rates of growth; cyclic partitions are a striking minority for large values of n.

Let μ_n be the maximum cycle length found as σ varies among all rules for n. Example 1 demonstrates both that $\mu_8 \geq 8$, and that μ_n in general may be greater than maximum cycle length in original Bulgarian Solitaire; by Theorem 3, the

n	μ_n	$\mu_n/\lvert P_n\rvert$	Example rule
3	2	0.66666	$\left(\begin{smallmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \end{smallmatrix}\right)$
4	4	0.80000	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \end{smallmatrix}\right)$
5	4	0.57142	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 0 & 2 \end{smallmatrix}\right)$
6	7	0.63636	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 1 & 0 & 3 \end{smallmatrix}\right)$
7	8	0.53333	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 0 & 3 & 0 & 4 \end{smallmatrix}\right)$
8	10	0.45455	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 2 & 4 & 1 & 0 & 3 \end{smallmatrix}\right)$
9	12	0.40000	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 & 2 & 0 & 4 & 0 & 3 \end{smallmatrix}\right)$
10	18	0.42857	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 2 & 1 & 3 & 4 & 2 & 0 & 7 & 5 \end{smallmatrix}\right)$
11	23	0.41071	$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 & 0 & 4 & 3 & 6 & 1 & 0 & 6 \end{smallmatrix}\right)$

Table 1: Maximal cycle length and an example of a rule producing one.

longest cycle for $n = 8$ has four partitions. Table 1 shows the values of μ_n for $n \leq 11$, based on exhaustive computer-based searches. It also gives an example of a rule producing a cycle of maximal length. We have determined with computer search that μ_n exceeds n for all $6 \leq n \leq 100$; exceeds $2n$ for $11 \leq n \leq 100$; exceeds $3n$ for $22 \leq n \leq 100$; and exceeds $4n$ for $28 \leq n \leq 100$. Many conjectures are suggested by these findings, in particular, ones concerning upper and lower bounds on maximum cycle length. The remainder of this section provides such bounds, although we feel confident that better ones may be found. We begin by showing that growth in μ_n is at least linear.

Proposition 5. *For each value of n, a rule τ may be found with respect to which there is a cycle of length at least $\frac{n}{2} + \frac{1}{2}$.*

Proof. For a given n, let τ be the rule defined as

$$\tau(k) = \begin{cases} 0 & \text{if } k \leq \frac{n+1}{2} \\ k-1 & \text{otherwise.} \end{cases}$$

When $\bar{\tau}$ is applied to the parition (n), the result is $(n-1, 1)$. Then $\bar{\tau}\,(n-1, 1) = (n-2, 2)$, assuming $n \geq 3$. For $i \leq \frac{n}{2}$, we have $\bar{\tau}^i\,(n) = (n-i, i)$. If n is even, then $\bar{\tau}^{\frac{n}{2}}\,(n) = (\frac{n}{2}, \frac{n}{2})$, in which case $\bar{\tau}^{\frac{n}{2}+1}\,(n) = (n)$. If n is odd, then $\bar{\tau}^{\frac{n-1}{2}}\,(n) = (\frac{n+1}{2}, \frac{n-1}{2})$, and then $\bar{\tau}^{\frac{n+1}{2}}\,(n) = (n)$. Thus the orbit of (n) under $\bar{\tau}$ is a cycle of length at least $\frac{n}{2} + \frac{1}{2}$. $\square$

Our strategy for bounding cycle length from above is to show that many partitions cannot be part of *any* cycle. One example of this approach is the following:

Corollary 6. *Let $p \in P_n$, and let σ be a rule for n. If $p_1 < \ell(p) - 1$, then there is no $q \in P_n$ such that $\bar{\sigma}\, q = p$. As a result, p is not cyclic when $p_1 < \ell(p) - 1$.*

This is a straightforward consequence of Proposition 2, and reflects the intuitive observation that partitions which are "wider" than they are "tall" tend to have few or no predecessors. (In the context of Bulgarian Solitaire, partitions with no $\bar{\beta}$-predecessor have been called "Garden of Eden" partitions, e.g., in [7].) This idea is generalized in the following results.

Proposition 7. *Let σ be a rule for n, and suppose $p \in P_n$ is cyclic with respect to $\bar{\sigma}$. Then the number of parts of p equal to 1 is not more than $2\sqrt{n+1} - 1$.*

Proof. Let k be the number of parts in p equal to 1, and to arrive at a contradiction, suppose $k > 2\sqrt{n+1} - 1$. Let $q, r \in P_n$ be such that $\bar{\sigma} q = p$ and $\bar{\sigma} r = q$. (The existence of r and q is assured because p is cyclic. Note that p, q, and r need not be distinct.) It may be that $\Delta q = 1$, so let $q_{i_1}, \ldots, q_{i_{k-1}}$ be parts of q which we are sent to 1 by σ. Since $q_{i_j} > \sigma(q_{i_j}) = 1$, each of these $k-1$ parts has size at least 2. Therefore, $\sum q_i \geq 2(k-1)$, i.e., $n \geq 2k - 2$.

Now considering r, it may be that $\Delta r = q_{i_j}$ for some j. Still, for $m \neq j$, some part of r is reduced to make q_{i_m}. Thus, we can say with certainty that r has $k-2$ parts of size at least 3, and therefore $n \geq 3k - 6$.

We can iterate this reasoning for an arbitrary predecessor of p. Let i be a positive integer with $i \leq k$, and let $s \in P_n$ be such that $\bar{\sigma}^i\, s = p$. Then s has $k - i$ parts of size at least $i + 1$. As a result, we deduce that

$$n = \sum s_i \geq (k-i)(i+1) = ik - i^2 + k - i. \tag{1}$$

Suppose k is odd. When i is equal to $\frac{k+1}{2}$, the right side of (1) takes the value $\frac{1}{4}\left(k^2 + 2k - 3\right)$. Now $k = 2\sqrt{n+1} - 1$ is readily seen to be a solution to $n = \frac{1}{4}(k^2 + 2k - 3)$; since $k > 2\sqrt{n+1} - 1$, we have shown that $n > n$, a contradiction. Now suppose k is even. Setting $i = \frac{k}{2}$, we get $n \geq \frac{1}{4}(k^2 + 2k)$ from (1). But since $\frac{1}{4}(k^2 + 2k) > \frac{1}{4}(k^2 + 2k - 3) > n$, we have a similar contradiction. By this contradiction, $k \leq 2\sqrt{n+1} - 1$. $\square$

Corollary 8. *Let σ be a rule for n, and let $C \subseteq P_n$ be a cycle with respect to σ. Then no member of C has more than $2\sqrt{n+1} - 1$ parts equal to 1.*

Proposition 9. *Let $C \subseteq P_n$ be a cycle with respect to rule σ. If m is any integer greater than $2\sqrt{n+1} - 1$, then $|C| \leq |P_n| - |P_{n-m}|$.*

Proof. Define $D \subseteq P_n$ as all partitions with at least m parts equal to 1. By Corollary 8, $C \subseteq P_n \setminus D$, and so $|C| \leq |P_n \setminus D| = |P_n| - |D|$. Thus it suffices to show that $|D| = |P_{n-m}|$. For every partition $p \in D$, there is a unique partition in P_{n-m} which results from dropping m copies of 1 from p. On the other hand, adding m parts equal to 1 to a partition in P_{n-m} identifies a unique member of D. Thus $|D| = |P_{n-m}|$. $\square$

Allowing m to be the least integer greater than $2\sqrt{n+1} - 1$, and using the asymptotic formula for $|P_n|$ mentioned above, the value of $|P_{n-m}|/|P_n|$ will tend towards

$$\frac{n - \sqrt{2n+1} + 1}{n} \cdot \exp\left(\pi\sqrt{\frac{2}{3}}\left(\sqrt{n - 2\sqrt{n+1} + 1} - \sqrt{n}\right)\right)$$

for large values of n. Standard analytic methods show that this expression approaches $\exp(-\pi\sqrt{2/3}) \approx 0.0769$ as n approaches infinity. Thus, Corollary 8 may be understood (roughly) as forbidding cycles which include more than 93 percent of all partitions of n. Available evidence suggests a much better bound may be found.

3. Isolated Cycles

In Bulgarian Solitaire with two cards, each turn alternates between the only two positions, (2) and $(1,1)$, and so all partitions are cyclic. For any larger value of n, Bulgarian Solitaire has both cyclic and non-cyclic partitions. However, every cycle is "lead to" by at least one partition outside of it. Stated more carefully,

Theorem 10 ([6])**.** *In Bulgarian Solitaire with $n > 2$ cards, if $p \in P_n$ is cyclic, then there is some non-cyclic $q \in P_n$ and some positive integer i such that $\bar{\beta}^i\, q = p$.*

Not surprisingly, many variants of Bulgarian Solitaire lack this property. Consider the rule $\rho = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \end{smallmatrix}\right)$. For it, the positions $(3,1)$, $(2,2)$, and $(2,1,1)$ form a cycle; since $\rho\,(1,1,1,1) = \rho\,(4) = (4)$, no other partition of 4 leads to a member of this cycle. If σ is a rule for n, we call $C \subseteq P_n$ an *isolated cycle* (with respect to $\bar{\sigma}$) if C is a cycle, and every $\bar{\sigma}$-predecessor of an element of C is also in C.

Proposition 11. *For all even n except 6, there is a rule for n which has an isolated cycle.*

Proof. The cases for $n = 2$ and $n = 4$ have been treated by the examples above. Theorem 15 below handles the case for $n = 8$.

Let $n \geq 10$ be even. It follows that $n - 4 > \frac{n}{2} - 1 > 2$. Define a rule σ by $\sigma(n-4) = \frac{n}{2} - 1$, $\sigma(\frac{n}{2} - 1) = 2$, $\sigma(2) = 1$, and otherwise $\sigma(k) = 0$. (By the above inequality, σ is well-defined as a rule.) For the partition $p = (n-4, 2, 2)$ we have $\bar{\sigma}p = (\frac{n}{2} - 1, \frac{n}{2} - 1, 1, 1)$ and $\bar{\sigma}^2 p = p$. We show that p and $\bar{\sigma}p$ have only each other for predecessors. Suppose $q \in P_n$ is such that $\bar{\sigma}q = p$. Since $\sigma(k) \neq n - 4$ for all k, it must be that $\Delta q = n-4$. It follows that q has two parts which σ sends to 2; those parts must be equal to $\frac{n}{2}-1$. As $n-2(\frac{n}{2}-1)=2$, q either has one remaining part equal to 2, or two parts equal to one. In the former case, $\bar{\sigma}q = (n-5,2,2,1) \neq p$. Therefore, q has two parts equal to one, and so $q = \bar{\sigma}p$. Next, suppose $q \in P_n$ is such that $\bar{\sigma}q = \bar{\sigma}p$. Because $n-4 > \frac{n}{2}$, q cannot have two parts equal to $n-4$, and so it has one part equal to $n-4$ (sent to $\frac{n}{2}-1$ by σ), and $\Delta q = \frac{n}{2}-1$. The remaining parts of q must be reduced to the remaining parts of $\bar{\sigma}p$, both equal to 1; the only possibility is that $q = (n-4,2,2) = p$. This shows that the two-element cycle containing p is isolated. □

Thus we can find isolated length-2 cycles for arbitrarily large even n. We next develop another approach which identifies isolated cycles of arbitrary length, many of which are for odd values of n.

In the remainder, let k be a fixed positive integer greater than 2. Define a rule α_k for n as follows:

$$\alpha_k(i) = \begin{cases} i-1 & \text{if } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 12. *Let $p = t_{k-1} + \bar{d}$ for some $\bar{d} \in \{0,1\}^k$. There is a unique q such that $\bar{\alpha}_k\, q = p$ if any of the following hold:*

1. *k is a part of p,*
2. *$\ell(p) = k$, or*
3. *$k-2$ is not a part of p.*

Proof. Begin by observing that α_k agrees with β on the parts of $t_{k-1} + \bar{d}^{\ell}$, because they are all equal to k or less. Therefore by Theorem 3, $\bar{\alpha}_k\,(t_{k-1} + \bar{d}^{\ell}) = \bar{\beta}\,(t_{k-1} + \bar{d}^{\ell}) = p$, and we may conclude that p has at least one $\bar{\alpha}_k$-predecessor. For the remainder, let q be any $\bar{\alpha}_k$-predecessor of p.

First suppose k is a part of p. Since $\alpha_k(i) \neq k$ always, it must be that $\Delta q = k$. It follows also that no other part of p can be equal to k, so all other parts of p are strictly less than k. Thus the remaining parts of p have unique preimages with respect to α_k, so q must contain these preimages. All other parts of q are sent to 0 by α_k, and so these parts are either equal to 1, or greater than k. However, if any of them were greater than k, then Δq would be as well; as it is not, all remaining parts are equal to 1. The number of such parts (which may be zero) is determined by $\Delta q = k$ and the other parts of q. This shows that q is uniquely determined, i.e., that p only has one predecessor.

Second, suppose $\ell(p) = k$. We may assume that k is not a part of p, as that case has already been handled. Then Δq must be $k-1$, since $\Delta q \geq \ell(\bar{\alpha}_k\,\bar{q}) - 1 = \ell(p) - 1 = k-1$. As in the previous case, all other parts of p have unique preimages, and all parts sent to 0 by α_k must be equal to 1. Therefore, q is determined uniquely.

Finally suppose $k-2$ is not a part of p, and assume that k is not a part of p and that $\ell(p) \neq k$ (from which we may infer that $\ell(p) = k-1$). As before, $\Delta q \geq \ell(p) - 1 = k-2$, so $\Delta q = k-1$. All other parts of q are uniquely determined as in the previous cases. $\square$

Proposition 13. *Suppose $p = t_{k-1} + \bar{d}$ where $\bar{d} \in \{0,1\}^k$. Also suppose that the three conditions of Proposition 12 fail, i.e., k is not a part of p, $k-2$ is, and $\ell(p) < k$. Then p has at least two predecessors with respect to $\bar{\alpha}_k$.*

Proof. As already observed, $\bar{\alpha}_k\,(t_{k-1} + \bar{d}^\ell) = p$, so we need only identify another $\bar{\alpha}_k$-predecessor of p. Let q be the partition resulting from dropping the part equal to $k-2$ from p, and adding one to the remaining non-zero parts. Since $\ell(p) = k-1$, $\ell(q) = k-2$. All parts of p are less than k, and so all parts of q are less than $k+1$; clearly then we have $\alpha_k(i) = i-1$ for all parts of q. As there are $k-2$ such parts, $\Delta q = k-2$, and so $\bar{\alpha}_k\, q = p$. Finally, $q \neq t_{k-1} + \bar{d}^\ell$, because $\ell(q) = k-2 < k-1 \leq \ell(t_{k-1} + \bar{d}^\ell)$. $\square$

Suppose $p = t_{k-1} + \bar{d}$ fails the three conditions of Proposition 12. We observe some consequences for $\bar{d}$. Note that $p_i = (t_{k-1} + \bar{d})_i = (k-i) + d_i$.

- First, because k is not a part of p, we have $k \neq p_i = (k-i) + d_i$, and so $i \neq d_i$, for all i. This inequality is trivially true for $i > 1$; for $i = 1$, we must have $d_i \neq 1$. Therefore, $d_1 = 0$.
- From $\ell(p) \neq k$ we may immediately infer $p_k = 0$, so $d_k = 0$.
- Finally, because $k-2$ is a part of p, we must have $k-2 = p_i = (k-i) + d_i$ for some i. Solving, we have $d_i = i-2$. Either $d_2 = 0$ or $d_3 = 1$.

Together, these facts imply that either

$$\bar{d} = \langle 0, 0, d_3, \ldots, d_{k-1}, 0 \rangle \quad \text{or} \quad \bar{d} = \langle 0, d_2, 1, d_4, \ldots, d_{k-1}, 0 \rangle.$$

Other partitions in the orbit of p by $\bar{\alpha}_k$ have the form $t_{k-1} + \bar{d}'$, where $\bar{d}'$ may be derived from $\bar{d}$ be sufficiently shifting its components (with wrap-around). Thus we have:

Proposition 14. *Suppose $p = t_{k-1} + \bar{d}$ for some $\bar{d} \in \{0,1\}^k$. Then the following are equivalent:*

1. *$\bar{d}$ does not contain a subsequence of the form $\langle \ldots, 0, 0, 0, \ldots \rangle$ or $\langle \ldots, 0, 0, d_i, 1, \ldots \rangle$ (with or without wrap-around).*
2. *The orbit of p is an isolated cycle with respect to $\bar{\alpha}_k$.*

Theorem 15. *There is an isolated cycle with respect to $\bar{\alpha}_k$ when n satisfies $T_{k-1} + \frac{k}{3} \leq n < T_k$.*

Proof. Define $\bar{d} \in \{0,1\}^k$ as follows:

$$\bar{d} = \begin{cases} \langle 0,0,1,0,0,1,\ldots,0,0,1\rangle & \text{if } k \equiv 0 \pmod 3 \\ \langle 0,0,1,0,0,1,\ldots,0,0,1,0,1,0,1\rangle & \text{if } k \equiv 1 \pmod 3 \\ \langle 0,0,1,0,0,1,\ldots,0,0,1,0,1\rangle & \text{if } k \equiv 2 \pmod 3). \end{cases}$$

(Recall $k \geq 3$; $\bar{d}$ should be truncated appropriately from the left, depending on k.) It follows that $t_{k-1} + \bar{d} \in P_{n'}$ for the least $n' \geq T_{k-1} + \frac{k}{3}$. Inspection shows that $\bar{d}$ lacks either of the prohibited subsequences, so by Proposition 14, the orbit of $t_{k-1} + \bar{d}$ is an isolated cycle with respect to α_k. If $n' < n < T_k$, one may change 0 to 1 in $n - n'$ positions of $\bar{d}$ judiciously to avoid the patterns of Proposition 14. (To do so, replace any 0 appearing immediately to the right of another; once all are replaced, any 0 may be replaced thereafter.) For the resulting $\bar{d}'$, $t_{k-1} + \bar{d}' \in P_n$ and $t_{k-1} + \bar{d}'$ has an isolated cycle with respect to $\bar{\alpha}_k$. □

Corollary 16. *Isolated cycles of arbitrarily large length may be found among variants of Bulgarian Solitaire.*

Proof. Let $n = T_k - 1$, and define $\bar{d} \in \{0,1\}^k$ by $\bar{d} = \langle 1,1,\ldots,1,0\rangle$. The proof of Theorem 15 shows that $p = t_{k-1} + \bar{d}$ lies in an isolated cycle, the length of which is k, using Theorem 3. □

It remains to settle the question of isolated cycles for the values of n not covered by Proposition 11 and Theorem 15. It has been determined via computer-based search that when $n \in \{3,6,7\}$, no rule for n produces an isolated cycle. (Thus the omission of $n = 6$ from Proposition 11 is unavoidable.) However, for odd values of n lying between T_{k-1} and $T_{k-1} + \frac{k}{3}$, $k \geq 5$, the question remains open.

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