ON UNAVOIDABLE OBSTRUCTIONS IN GAUSSIAN WALKS

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Abstract

We investigate a problem about certain walks in the ring of Gaussian integers. Let n and d be two natural numbers. Does there exist a sequence of Gaussian integers z_j, such that $|z_{j+1} - z_j| = 1$ and a pair of indices r and s, such that $z_r - z_s = n$ and for all indices t and u, $z_t - z_u \neq d$? If there exists such a sequence, we say that n is d *avoidable*. Let A_n be the set of all $d \in \mathbb{N}$ such that n is not d avoidable. Recently, Ledoan and Zaharescu proved that $\{d \in \mathbb{N} : d|n\} \subset A_n$. We extend this result by giving a necessary and sufficient condition for $d \in A_n$, which answers a question posed by Ledoan and Zaharescu. We also find a precise formula for the cardinality of A_n and answer three other questions raised in the same paper.

1. Introduction

Walks in Gaussian integers have been investigated in the past by several authors ([1], [2], [3], [5], [6]) to work on the question of whether one can start in the vicinity of the origin of the complex plane and walk to infinity using the Gaussian primes and only taking steps of bounded length. Recently, in [4], there has been an investigation in a different direction. In the paper, the authors have investigated walks of unit steps and demonstrated that there exists some kind of divisibility obstruction. Let n and d be two natural numbers. Does there exist a sequence (z_j) of Gaussian integers, such that $|z_{j+1} - z_j| = 1$ and a pair of indices r and s, such that $z_r - z_s = n$ and for all indices t and u, $z_t - z_u \neq d$? If there exists such a sequence, we say that n is d *avoidable*. Let A_n be the set of all $d \in \mathbb{N}$ such that n is not d avoidable. Ledoan and Zaharescu [4] prove that the set of all divisors of n is a subset of A_n. That is, if $d|n$, then n is not d avoidable.

In Section 2, we give the precise structure of A_n along with the cardinality of A_n. From this precise definition of A_n, we answer four of the six questions asked in [4] in Section 3. Before going to the main theorem of Section 2, let us consider the following example which helps in stating the main theorem of the next section.

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Example. Let $n = 20$. We consider three sequences S_1, S_2 and S_3 defined as follows: S_1 : $z_0 = 0, z_{12} = 10, z_{24} = 20, z_j = j - 1 - i$ for $1 \leq j \leq 11$, and $z_j = j - 3 + i$ for $13 \leq j \leq 23$. Here, $z_{24} - z_0 = 20$. Clearly, the set of all positive integer differences is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20\}$.

S_2 : $z_0 = 0, z_1 = i, z_2 = 2i, z_3 = 1 + 2i, z_4 = 2 + 2i, z_5 = 3 + 2i, z_6 = 4 + 2i, z_7 = 5 + 2i, z_8 = 6 + 2i, z_9 = 7 + 2i, z_{10} = 7 + i, z_{11} = 7, z_{12} = 7 - i, z_{13} = 8 - i, z_{14} = 9 - i, z_{15} = 9, z_{16} = 9 + i, z_{17} = 10 + i, z_{18} = 11 + i, z_{19} = 12 + i, z_{20} = 13 + i, z_{21} = 14 + i, z_{22} = 14, z_{23} = 14 - i, z_{24} = 14 - 2i, z_{25} = 15 - 2i, z_{26} = 16 - 2i, z_{27} = 17 - 2i, z_{28} = 18 - 2i, z_{29} = 19 - 2i, z_{30} = 20 - 2i, z_{31} = 20 - i, z_{32} = 20$. The set of all positive differences is $\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 20\}$.

S_3 : $z_0 = 0, z_1 = i, z_2 = 2i, z_3 = 1 + 2i, z_4 = 2 + 2i, z_5 = 3 + 2i, z_6 = 4 + 2i, z_7 = 5 + 2i, z_8 = 6 + 2i, z_9 = 7 + 2i, z_{10} = 8 + 2i, z_{11} = 8 + i, z_{12} = 8, z_{13} = 8 - i, z_{14} = 9 - i, z_{15} = 10 - i, z_{16} = 10, z_{17} = 10 + i, z_{18} = 11 + i, z_{19} = 12 + i, z_{20} = 13 + i, z_{21} = 14 + i, z_{22} = 15 + i, z_{23} = 16 + i, z_{24} = 16, z_{25} = 16 - i, z_{26} = 16 - 2i, z_{27} = 17 - 2i, z_{28} = 18 - 2i, z_{29} = 19 - 2i, z_{30} = 20 - 2i, z_{31} = 20 - i, z_{32} = 20$. The set of all positive differences is $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 20\}$.

The intersection of positive difference sets of S_1, S_2 and S_3 is $\{1, 2, 3, 4, 5, 6, 7, 10, 20\}$. If we try to go from 0 to 20 through any walk, we suspect that we cannot avoid any number that belongs to the intersection. We believe that $A_{20} = \{1, 2, 3, 4, 5, 6, 7, 10, 20\}$.

Let $n = k(n, d)d + r(n, d)$, where $r(n, d)$ is a unique integer belonging to $\left[-\left\lfloor \frac{d}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil - 1\right]$.

d	$k(20, d)$	$r(20, d)$	A_{20}	$k(20, d) - \lvert r(20, d)\rvert$
1	20	0	$\in$	20
2	10	0	$\in$	10
3	7	-1	$\in$	6
4	5	0	$\in$	5
5	4	0	$\in$	4
6	3	2	$\in$	1
7	3	-1	$\in$	2
8	3	-4	$\notin$	-1
9	2	2	$\notin$	0
10	2	0	$\in$	2
11	2	-2	$\notin$	0
12	2	-4	$\notin$	-2
13	2	-6	$\notin$	-4
14	1	6	$\notin$	-5
15	1	5	$\notin$	-4
16	1	4	$\notin$	-3
17	1	3	$\notin$	-2
18	1	2	$\notin$	-1
19	1	1	$\notin$	0
20	1	0	$\in$	1

Table 1: Avoidable and non-avoidable cases

If we assume for a moment that $A_{20} = \{1, 2, 3, 4, 5, 6, 7, 10, 20\}$, then from Table 1 above, we observe that $d \in A_{20}$ if and only if $k(20, d) - |r(20, d)| \geq 1$. We prove that this property is true not only for $n = 20$ but for all natural numbers n. This is the main result of the paper, which is presented in the next section.

2. Main Results

Theorem 2.1. *Let $n \geq 1$ and $d \geq 1$ be integers. Then $d \in A_n$ if and only if $k(n, d) \geq |r(n, d)| + 1$.*

In order to prove one part of the theorem, we require the following lemma.

Lemma 2.2. *Let n, $d > 0$ be integers and $n = kd + r$. If $k \geq |r| + 1$, then $d \in A_n$.*

Proof. Let r be a number with the least absolute value such that there exists a $k \geq |r| + 1$ and $n = kd + r$ for some $n > 0$ and $d > 0$ with $d \notin A_n$. Then $r \neq 0$. For, if $r = 0$, then $d|n$. Hence, $d \in A_n$ (by Theorem 1 of [4]). Next, $d \notin A_n$ implies that there exists a sequence $S = (z_j)$ of Gaussian integers such that $z_0 = 0$ and $z_l = n$, where z_l is the final term and $|z_{p+1} - z_p| = 1$ for $0 \leq p \leq l - 1$, $z_j - z_{j'} \neq d$ for $0 \leq j$, $j' \leq l$.

Now we create an $l + 2$ terms sequence $S' = (z'_p)$ with $z'_p = z_p$ for $0 \leq p \leq l$ and $z'_{l+1} = z$, where

$$z := \begin{cases} n - 1, & \text{if } r \geq 1; \\ n + 1, & \text{if } r \leq -1, \end{cases}$$

$z = kd + r'$ with $|r'| = |r| - 1$. Hence, the minimality assumption implies that $d \in A_z$ as $k \geq |r'| + 2$. Thus, there should exist two points x, $y \in S'$ such that $x - y = \pm d$ and both x, y cannot be in S as $z_j - z_{j'} \neq d$ for $0 \leq j$, $j' \leq l$. Without loss of generality, let $y = z'_{l+1} = z$ and $x = z \pm d$. Hence, $x = (k \pm 1)d + r'$ with $k \pm 1 \geq |r'| + 1 \in S$ and from the minimality assumption on $|r|$, $d \in A_x$. Thus there exists two points z_{i_1} and z_{i_2} of S such that $z_{i_1} - z_{i_2} = d$, which contradicts that $z_j - z_{j'} \neq d$ for $0 \leq j$, $j' \leq l$. This completes the proof of the lemma. □

Now we prove Theorem 2.1.

Proof. Let $k(n, d) \geq |r(n, d)| + 1$. Then $n = k(n, d)d + r(n, d)$ and the proof follows from Lemma 2.2. For the converse part, we prove that if $k(n, d) \leq |r(n, d)|$, then $d \notin A_n$. For simplicity, let $k = k(n, d)$ and $r = r(n, d)$. Then $n = kd + r$. Let $h = d + 1$. From now on, we treat Gaussian integers as ordered pairs of integers. Let $m \geq 0$ be an integer. Let R_m and T_m denote the set of Gaussian integers in the vertical line segments joining $\big(m(d+1), -m\big)$, $\big(m(d+1), h-m\big)$ and $\big((m+1)(d-1), h-m\big)$, $\big((m+1)(d-1), -m-1\big)$, respectively, and let S_m and U_m denote the set of Gaussian integers in the horizontal line segments joining $\big(m(d+1), h-m\big)$,

$\big((m+1)(d-1), h-m\big)$ and $\big((m+1)(d-1), -m-1\big)$, $\big((m+1)(d+1), -m-1\big)$, respectively. Let P_1 be a set defined as follows: if d is odd,

$$P_1 = R_0 \cup S_0 \cup T_0 \cup U_0 \cdots R_{\frac{d-1}{2}-1} \cup S_{\frac{d-1}{2}-1} \cup T_{\frac{d-1}{2}-1} \cup U_{\frac{d-1}{2}-1} \cup R_{\frac{d-1}{2}};$$

if d is even,

$$P_1 = R_0 \cup S_0 \cup T_0 \cup U_0 \cdots R_{\frac{d}{2}-2} \cup S_{\frac{d}{2}-2} \cup T_{\frac{d}{2}-2} \cup U_{\frac{d}{2}-2} \cup R_{\frac{d}{2}-1} \cup S_{\frac{d}{2}-1} \cup T_{\frac{d}{2}-1}.$$

Let the sets of Gaussian integers in the line segments joining $\big(m(d+1), -m\big)$, $\big(m(d+1), h-m\big)$; $\big(m(d+1), h-m\big)$, $\big((m+1)(d-1)-1, h-m\big)$; $\big((m+1)(d-1)-1, h-m\big)$, $\big((m+1)(d-1)-1, -m-1\big)$ and $\big((m+1)(d-1)-1, -m-1\big)$, $\big((m+1)(d+1), -m-1\big)$ be R'_m, S'_m, T'_m and U'_m, respectively.

Further, let P_2 be another set defined as follows: if d is odd,

$$P_2 = \{(-1,0)\} \cup R'_0 \cup S'_0 \cup T'_0 \cup U'_0 \cdots R'_{\frac{d-5}{2}} \cup S'_{\frac{d-5}{2}} \cup T'_{\frac{d-5}{2}} \cup U'_{\frac{d-5}{2}} \cup R'_{\frac{d-3}{2}} \cup S'_{\frac{d-3}{2}} \cup T'_{\frac{d-3}{2}};$$

if d is even,

$$P_2 = \{(-1,0)\} \cup R'_0 \cup S'_0 \cup T'_0 \cup U'_0 \cdots R'_{\frac{d-4}{2}} \cup S'_{\frac{d-4}{2}} \cup T'_{\frac{d-4}{2}} \cup U'_{\frac{d-4}{2}} \cup R'_{\frac{d}{2}-1}.$$

It is not difficult to show that there exists two sequences S_1 and S_2 of Gaussian integers with their respective ranges P_1 and P_2 satisfying that for every two consecutive terms z_j and z_{j+1} of either S_1 or S_2, $|z_j - z_{j+1}| = 1$. In Lemma 2.3, we prove that neither the set P_1 nor P_2 has two elements (picked from the same set) whose difference is d.

One can clearly see that if d is odd, then

$$\left\{(m(d+1),0) : 0 \le m \le \frac{(d-1)}{2}\right\} \bigcup \left\{(i(d-1),0) : 0 \le i \le \frac{(d+1)}{2}\right\} \subset P_1,$$

$$\left\{(m(d+1),0) : 0 \le m \le \frac{(d-3)}{2}\right\} \bigcup \left\{(i(d-1)-1,0) : 0 \le i \le \frac{(d-1)}{2}\right\} \subset P_2;$$

if d is even, then

$$\left\{(m(d+1),0) : 0 \le m \le \frac{d}{2}-1\right\} \bigcup \left\{(i(d-1),0) : 0 \le i \le \frac{d}{2}\right\} \subset P_1,$$

$$\left\{(m(d+1),0) : 0 \le m \le \frac{d}{2}-1\right\} \bigcup \left\{(i(d-1)-1,0) : 0 \le i \le \frac{d}{2}-1\right\} \subset P_2.$$

Next, it is given that $n = kd + r$ and $k \le |r|$. Let k and r be of the same parity. If $r > 0$, let $m = \frac{r+k}{2}$ and $i = \frac{r-k}{2}$. Since P_1 passes through $(m(d+1),0)$ and $(i(d-1),0)$ and $m(d+1) - i(d-1) = n$, we have $d \notin A_n$ as P_1 has no two elements with d as a difference.

If $r < 0$, choose $m = \frac{-r-k}{2}$ and $i = \frac{-r+k}{2}$. As $(m(d+1), 0)$ and $(i(d-1), 0)$ are in P_1, and $i(d-1) - m(d+1) = n$, we have $d \notin A_n$.

Next, let k and r be of opposite parity. If $r > 0$, choose $m = \frac{r-1+k}{2}$ and $i = \frac{r-1-k}{2}$, and if $r < 0$, choose $m = \frac{-r-k-1}{2}$ and $i = \frac{k-r-1}{2}$. We observe that $(m(d+1), 0) \in P_2$, $(i(d-1)-1, 0) \in P_2$ and $m(d+1) - (i(d-1)-1) = \pm n$. Since P_2 has no two elements with d as a difference, we have $d \notin A_n$. This completes the proof of the theorem. $\square$

Lemma 2.3. *Neither the set P_1 nor P_2 has two elements (picked from the same set) whose difference is d.*

Proof. Clearly, if d is odd,

$$\begin{aligned} P_1 &= R_0 \cup S_0 \cup T_0 \cup U_0 \cdots R_{\frac{d-1}{2}-1} \cup S_{\frac{d-1}{2}-1} \cup T_{\frac{d-1}{2}-1} \cup U_{\frac{d-1}{2}-1} \cup R_{\frac{d-1}{2}} \\ &= \cup_{i=0}^{\frac{d-1}{2}} R_i \bigcup \cup_{i=0}^{\frac{d-1}{2}-1} S_i \bigcup \cup_{i=0}^{\frac{d-1}{2}-1} T_i \bigcup \cup_{i=0}^{\frac{d-1}{2}-1} U_i, \end{aligned}$$

$$\begin{aligned} P_2 = \{(-1, 0)\} &\cup R'_0 \cup S'_0 \cup T'_0 \cup U'_0 \cdots R'_{\frac{d-5}{2}} \cup S'_{\frac{d-5}{2}} \cup T'_{\frac{d-5}{2}} \cup U'_{\frac{d-5}{2}} \cup R'_{\frac{d-3}{2}} \\ &\cup S'_{\frac{d-3}{2}} \cup T'_{\frac{d-3}{2}} \\ = \{(-1, 0)\} &\bigcup \cup_{i=0}^{\frac{d-3}{2}} R'_i \bigcup \cup_{i=0}^{\frac{d-3}{2}} S'_i \bigcup \cup_{i=0}^{\frac{d-3}{2}} T'_i \bigcup \cup_{i=0}^{\frac{d-5}{2}} U'_i; \end{aligned}$$

and if d is even,

$$\begin{aligned} P_1 &= R_0 \cup S_0 \cup T_0 \cup U_0 \cdots R_{\frac{d}{2}-2} \cup S_{\frac{d}{2}-2} \cup T_{\frac{d}{2}-2} \cup U_{\frac{d}{2}-2} \cup R_{\frac{d}{2}-1} \cup S_{\frac{d}{2}-1} \cup T_{\frac{d}{2}-1} \\ &= \cup_{i=0}^{\frac{d}{2}-1} R_i \bigcup \cup_{i=0}^{\frac{d}{2}-1} S_i \bigcup \cup_{i=0}^{\frac{d}{2}-1} T_i \bigcup \cup_{i=0}^{\frac{d}{2}-2} U_i, \end{aligned}$$

$$\begin{aligned} P_2 &= \{(-1, 0)\} \cup R'_0 \cup S'_0 \cup T'_0 \cup U'_0 \cdots R'_{\frac{d-4}{2}} \cup S'_{\frac{d-4}{2}} \cup T'_{\frac{d-4}{2}} \cup U'_{\frac{d-4}{2}} \cup R'_{\frac{d}{2}-1} \\ &= \{(-1, 0)\} \bigcup \cup_{i=0}^{\frac{d}{2}-1} R'_i \bigcup \cup_{i=0}^{\frac{d}{2}-2} S'_i \bigcup \cup_{i=0}^{\frac{d}{2}-2} T'_i \bigcup \cup_{i=0}^{\frac{d}{2}-2} U'_i. \end{aligned}$$

Moreover, the sets R_i, T_i, R'_i and T'_i are the sets of Gaussian integers in their respective vertical segments. Similarly, S_i, U_i, S'_i and U'_i are the sets of Gaussian integers in their respective horizontal segments.

Let d be odd. Then, the x-coordinates of vertical segments of P_1 are $i(d+1)$ and $j(d-1)$ for $0 \le i \le \frac{d-1}{2}$ and $1 \le j \le \frac{d-1}{2}$. Next, let d be even. Then, the x-coordinates of vertical segments of P_1 are $i(d+1)$ and $j(d-1)$ for $0 \le i \le \frac{d}{2} - 1$ and $1 \le j \le \frac{d}{2}$. Hence, the x-coordinates of vertical segments modulo d are i and $-j$ for respective intervals of i and j depending on d is odd or even. Hence, one can observe that these are distinct modulo d. Thus there cannot be any two elements on the two vertical segments differing by d. Similarly, if d is odd, then for $0 \le i \le \frac{d-3}{2}$

and $1 \le j \le \frac{d-1}{2}$, and if d is even, then for $0 \le i \le \frac{d}{2}-1$ and $1 \le j \le \frac{d}{2}-1$, the x-coordinates for different vertical line segments of P_2 are $i(d+1)$ and $j(d-1)-1$. Hence, the x-coordinates of different vertical segments are distinct modulo d. Thus there cannot be any two elements whose difference is d. Since $(-1 \pm d, 0) \notin P_2$, we can ignore about $(-1,0)$ in P_2. Further, d cannot be achieved as a difference of any two elements of the same horizontal segment as in both P_1 and P_2 the length of horizontal segments are strictly less than d. As the heights of different horizontal segments do not match, there cannot be any two elements differing by d from any two distinct horizontal segments. The only case to be taken into consideration is an element from a vertical segment and an element from a horizontal segment. Thus, the remaining cases left to consider are points on R_i, S_j; R_i, U_j; T_i, S_j, T_i, U_j, R'_i, S'_j; R'_i, U'_j; T'_i, S'_j and T'_i, U'_j. We show below that there cannot be any two elements with d as a difference in all above eight cases.

`Case 1:` (R_i, S_j). To have d as a difference, we require $(i(d+1) \pm d, l_1) = (l_2, h-j)$, where

$$\begin{aligned}
&0 \le i \le \frac{d-1}{2} \ \text{ if } d \text{ is odd,}\\
&0 \le i \le \frac{d}{2}-1 \ \text{ if } d \text{ is even,}\\
&0 \le j \le \frac{d-1}{2}-1 \ \text{ if } d \text{ is odd,}\\
&0 \le j \le \frac{d}{2}-1 \ \text{ if } d \text{ is even,}
\end{aligned}$$

and $l_1 \in [-i, h-i]$, $l_2 \in [j(d+1), (j+1)(d-1)]$. This implies that $h-j \in [-i, h-i]$ or $j \in [i, i+h]$. But for $j \in [i, i+h]$, $i(d+1) \pm d \notin [j(d+1), (j+1)(d-1)]$.

`Case 2:` (R_i, U_j). To have d as a difference, we require $(i(d+1) \pm d, l_1) = (l_2, -j-1)$, where

$$\begin{aligned}
&0 \le i \le \frac{d-1}{2} \ \text{ if } d \text{ is odd,}\\
&0 \le i \le \frac{d}{2}-1 \ \text{ if } d \text{ is even,}\\
&0 \le j \le \frac{d-1}{2}-1 \ \text{ if } d \text{ is odd,}\\
&0 \le j \le \frac{d}{2}-2 \ \text{ if } d \text{ is even,}
\end{aligned}$$

and $l_1 \in [-i, h-i]$, $l_2 \in [(j+1)(d-1), (j+1)(d+1)]$. This implies that $j \in [i-h-1, i-1]$. But $i(d+1) \pm d \notin [(j+1)(d-1), (j+1)(d+1)]$ for $j \le (i-1)$.

`Case 3:` $(T_i,\ S_j)$. To have d as a difference, we require $((i+1)(d-1) \pm d, l_1) = (l_2, h-j)$, where

$$0 \le i \le \frac{d-1}{2} - 1 \text{ if } d \text{ is odd,}$$
$$0 \le i \le \frac{d}{2} - 1 \text{ if } d \text{ is even,}$$
$$0 \le j \le \frac{d-1}{2} - 1 \text{ if } d \text{ is odd,}$$
$$0 \le j \le \frac{d}{2} - 1 \text{ if } d \text{ is even,}$$

and $l_1 \in [-i-1, h-i]$, $l_2 \in [j(d+1), (j+1)(d-1)]$. This implies that $j \in [i, h+i+1]$. But $(i+1)(d-1) \pm d \notin [j(d+1), (j+1)(d-1)]$ for $j \in [i, h+i+1]$.

`Case 4:` $(T_i,\ U_j)$. To have d as a difference, we require $((i+1)(d-1) \pm d, l_1) = (l_2, -j-1)$, where

$$0 \le i \le \frac{d-1}{2} - 1 \text{ if } d \text{ is odd,}$$
$$0 \le i \le \frac{d}{2} - 1 \text{ if } d \text{ is even,}$$
$$0 \le j \le \frac{d-1}{2} - 1 \text{ if } d \text{ is odd,}$$
$$0 \le j \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$

and $l_1 \in [-i-1, h-i]$, $l_2 \in [(j+1)(d-1), (j+1)(d+1)]$. This implies that $j \in [i-h-1, i]$. But $(i+1)(d-1) \pm d \notin [(j+1)(d-1),)(j+1)(d+1)]$.

`Case 5:` $(R'_i,\ S'_j)$. To have d as a difference, we require $(i(d+1) \pm d, l_1) = (l_2, h-j)$, where

$$0 \le i \le \frac{d-3}{2} \text{ if } d \text{ is odd,}$$
$$0 \le i \le \frac{d}{2} - 1 \text{ if } d \text{ is even,}$$
$$0 \le j \le \frac{d-3}{2} \text{ if } d \text{ is odd,}$$
$$0 \le j \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$

and $l_1 \in [-i, h-i]$, $l_2 \in [j(d+1), (j+1)(d-1)-1]$. This implies that $j \in [i, h+i]$. But $i(d+1) \pm d \notin [j(d+1), (j+1)(d-1)-1]$ for $j \in [i, h+i]$.

`Case 6:` (R'_i, U'_j). To have d as a difference, we require $(i(d+1)\pm d, l_1) = (l_2, -j-1)$, where

$$0 \le i \le \frac{d-3}{2} \text{ if } d \text{ is odd,}$$
$$0 \le i \le \frac{d}{2} - 1 \text{ if } d \text{ is even,}$$
$$0 \le j \le \frac{d-5}{2} \text{ if } d \text{ is odd,}$$
$$0 \le j \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$

and $l_1 \in [-i, h-i]$, $l_2 \in [(j+1)(d-1)-1, (j+1)(d+1)]$. This implies that $j \in [i-h-1, i-1]$. But $i(d+1) \pm d \notin [(j+1)(d-1)-1, (j+1)(d+1)]$ for $j \le i-1$.

`Case 7:` (T'_i, S'_j). To have d as a difference, we require $((i+1)(d-1)-1 \pm d, l_1) = (l_2, h-j)$, where

$$0 \le i \le \frac{d-3}{2} \text{ if } d \text{ is odd,}$$
$$0 \le i \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$
$$0 \le j \le \frac{d-3}{2} \text{ if } d \text{ is odd,}$$
$$0 \le j \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$

and $l_1 \in [-i-1, h-i]$, $l_2 \in [j(d+1), (j+1)(d-1)-1]$, This implies that $j \in [i, h+i+1]$. But $(i+1)(d-1)-1 \pm d \notin [j(d+1), (j+1)(d-1)-1]$ for $j \ge i$.

`Case 8:` (T'_i, U'_j). To have d as a difference, we require $((i+1)(d-1)-1 \pm d, l_1) = (l_2, -j-1)$, where

$$0 \le i \le \frac{d-3}{2} \text{ if } d \text{ is odd,}$$
$$0 \le i \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$
$$0 \le j \le \frac{d-5}{2} \text{ if } d \text{ is odd,}$$
$$0 \le j \le \frac{d}{2} - 2 \text{ if } d \text{ is even,}$$

and $l_1 \in [-i-1, h-i]$, $l_2 \in [(j+1)(d-1)-1, (j+1)(d+1)]$. This implies that $j \in [i-h-1, i]$. But for $j \in [i-h-1, i]$, $(i+1)(d-1)-1 \pm d \notin [(j+1)(d-1)-1, (j+1)(d+1)]$. This completes the proof of the lemma. □

An immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.4. *If $n \geq \frac{d^2}{2}$, then $d \in A_n$.*

Proof. Let d be even. Clearly, $-\frac{d}{2} \leq r(n,d) \leq \frac{d}{2} - 1$. If $r(n,d) < 0$, then

$$k(n,d) > \frac{n}{d} \geq \frac{d}{2} \geq |r(n,d)|,$$

and if $r(n,d) \geq 0$, then

$$k(n,d) = \frac{n - r(n,d)}{d} \geq \frac{(n - \frac{d}{2} + 1)}{d} > \frac{\frac{d^2}{2} - \frac{d}{2}}{d} \geq |r(n,d)|.$$

Next, let d be odd. Clearly, $-\frac{d-1}{2} \leq r(n,d) \leq \frac{d-1}{2}$, and hence

$$k(n,d) = \frac{n - r(n,d)}{d} \geq \frac{n - \frac{d-1}{2}}{d} > \frac{d^2 - d}{2d} \geq |r(n,d)|.$$

Thus, in each case $k(n,d) > |r(n,d)|$, and so by Theorem 2.1, $d \in A_n$. □

Now, we generalize Lemma 2.2 in the following theorem.

Theorem 2.5. *Let S be a sequence of Gaussian integers whose every two consecutive terms z_j and z_{j+1} satisfy $|z_{j+1} - z_j| = 1$. Further, let there exists a pair of indices j_1 and j_2 such that $z_{j_2} - z_{j_1} = n + ih$ and for a natural number d, $k(n,d) \geq |r(n,d)| + |h| + 1$. Then there exists a pair of indices j_3 and j_4 such that $z_{j_3} - z_{j_4} = d$.*

Proof. Let h be a number with the least absolute value such that there exists a sequence $S = (z_j)$ with $z_0 = 0$, the last term $z_l = n + ih$, $|z_{j+1} - z_j| = 1$ for $0 \leq j \leq l - 1$, $z_r - z_s \neq d$ for $0 \leq r, s \leq l$, and $k(n,d) \geq |r(n,d)| + |h| + 1$. We have $h \neq 0$ (by Theorem 2.1). We now define a new sequence $S' = (z'_j)$, such that $z'_j = z_j$ for $0 \leq j \leq l$ and $z'_{l+1} = n + i(h + \theta)$, where θ is given by

$$\theta = \begin{cases} -1, & \text{if } h \geq 0; \\ +1, & \text{if } h < 0. \end{cases}$$

Hence, $|h + \theta| = |h| - 1$, and from the minimality assumption on $|h|$, there exists two terms $x \in S'$ and $y \in S'$ such that $x - y = \pm d$. Both of them cannot belong to S as we have assumed that there are no two terms in S whose difference is d. Hence, without loss of generality, let $x = (n, h+\theta)$ and $y = (n \pm d, h+\theta) \in S$. Since $k(n \pm d, d) \geq |r(n,d)| + |h+\theta| + 1$, $|h+\theta| = |h| - 1$ (by the minimality assumption on $|h|$), there exists two terms of S with d as a difference, which contradicts the assumption about S that $z_r - z_s \neq d$ for $0 \leq r, s \leq l$. □

We close this section by giving a formula for the cardinality of A_n.

Theorem 2.6. *The cardinality of A_n is*

$$\lfloor\sqrt{2n}\rfloor + 2\left\lfloor \frac{n+1}{\sqrt{2n}+1} \right\rfloor - \sum_{\substack{d|n \\ d<\frac{n+1}{\sqrt{2n}+1}}} 1 + \theta(n),$$

where $\theta(n) = |\{d : d > \sqrt{2n}$ and $\frac{n+1}{\sqrt{2n}+1} \leq k(n,d) < \sqrt{\frac{n}{2}} + \frac{1}{2}\}|$. Further, $\theta(n) \in \{0,1,2\}$.

Proof. If $d \leq \sqrt{2n}$, then $d \in A_n$ (by Corollary 2.4). So, we need to count the remaining $d > \sqrt{2n}$ and $d \in A_n$. If $d \in A_n$ and $d > \sqrt{2n}$, then $n = k(n,d)d + r(n,d)$ and $k(n,d) \geq |r(n,d)| + 1$. So,

$$k(n,d) = \frac{n - r(n,d)}{d} \leq \frac{n}{d} + \frac{1}{2} < \sqrt{\frac{n}{2}} + \frac{1}{2},$$

as $|r(n,d)| \leq \frac{d}{2}$. Now to count the remaining values of d, we count the number of $k(< \sqrt{\frac{n}{2}} + \frac{1}{2})$ and count the number of distinct $d(> \sqrt{2n})$ for which $k(n,d) = k$.

`Case 1:` ($k < \frac{n+1}{\sqrt{2n}+1}$ and $k \nmid n$).

There are two values of r, say, r_1 and r_2, such that $k|(n-r)$ and $-(k-1) \leq r \leq (k-1)$. For $i \in \{1,2\}$, let $d_i = \frac{n-r_i}{k}$. We have

$$d_i = \frac{n - r_i}{k} \geq \frac{n-(k-1)}{k} > \sqrt{2n},$$

and

$$|r_i| \leq k - 1 < \frac{n+1}{\sqrt{2n}+1} - 1 \leq \frac{d_i - 1}{2}.$$

This implies that $r_i \in [-\lfloor\frac{d}{2}\rfloor, \lceil\frac{d}{2} - 1\rceil]$. Hence, $k(n,d_i) = k$ and $d_1 \neq d_2$. So each k satisfying $k < \frac{n+1}{\sqrt{2n}+1}$ and $k \nmid n$ corresponds to two distinct $d > \sqrt{2n}$.

`Case 2:` ($k < \frac{n+1}{\sqrt{2n}+1}$ and $k|n$).

There exists exactly one value of $r(= 0)$ in the interval $[-(k-1), r \leq (k-1)]$ satisfying $k|(n-r)$. The corresponding d is given by

$$d = \frac{n}{k} > \sqrt{2n}.$$

Hence, each such k corresponds to exactly one value of $d > \sqrt{2n}$ such that $k(n,d) = k$.

`Case 3:` ($\frac{n+1}{\sqrt{2n}+1} \leq k < \sqrt{\frac{n}{2}} + \frac{1}{2}$).

Clearly, there is at most one k in the interval. Such a k can correspond to at most two distinct values of $d > \sqrt{2n}$. Let $\theta(n)$ correspond to the number of $d > \sqrt{2n}$ such that $\frac{n+1}{\sqrt{2n}+1} \leq k(n,d) < \sqrt{\frac{n}{2}} + \frac{1}{2}$. Then, $\theta(n) \in \{0,1,2\}$.

We claim that distinct $k_1(<\sqrt{\frac{n}{2}}+\frac{1}{2})$ and $k_2(<\sqrt{\frac{n}{2}}+\frac{1}{2})$ correspond to distinct $d>\sqrt{2n}$. Let $k(n,d_1)=k_1$ and $k(n,d_2)=k_2$. If $d_1=d_2$, then

$$\frac{n-r(n,d_1)}{k_1}=\frac{n-r(n,d_2)}{k_2} \quad \text{implies} \quad (k_2-k_1)n=k_2r(n,d_1)-k_1r(n,d_2). \qquad (2.1)$$

Since $|r(n,d_i)|\leq k_i-1$ for $i\in\{1,2\}$, we have

$$|k_2r(n,d_1)-k_1r(n,d_2)|\leq k_2(k_1-1)+k_1(k_2-1)<2\left(\sqrt{\frac{n}{2}}+\frac{1}{2}\right)\left(\sqrt{\frac{n}{2}}-\frac{1}{2}\right)<n.$$

Now, as the right-hand side of (2.1) is a multiple of n and the absolute value is strictly less than n, the right-hand side has to be zero, which implies $k_1=k_2$.

Hence, the total count of d in all cases is

$$\begin{aligned}|A_n|=&|\{d\in\mathbb{N}:d\leq\sqrt{2n}\}|+2|\{k\in\mathbb{N}:k<\frac{n+1}{\sqrt{2n}+1}\text{ and }k\nmid n\}|+\\&|\{k\in\mathbb{N}:k<\frac{n+1}{\sqrt{2n}+1}\text{ and }k|n\}|+\theta(n)\\=&\lfloor\sqrt{2n}\rfloor+2\left\lfloor\frac{n+1}{\sqrt{2n}+1}\right\rfloor-\sum_{\substack{d|n\\ d<\frac{n+1}{\sqrt{2n}+1}}}1+\theta(n).\end{aligned}$$

This completes the proof of the theorem. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.7. *The cardinality of A_n for all $\epsilon>0$ is given by*

$$|A_n|=2\sqrt{2n}+O(n^{\epsilon}).$$

3. Answers to Some Questions Raised by Ledoan and Zaharescu

Ledoan and Zaharescu ([4], Section 3) raised six questions. We answer four of the six questions below.

Question 1 asks, which positive integers belong to A_n? Theorem 2.1 answers the question.

Question 2 asks, for which positive integers n, is A_n equal to the set of all divisors of n? We claim that the only numbers n for which A_n is equal to the set of all divisors of n are $1,2,4,6,12$. One can check that $n=1,2,4,6,12$ are the only numbers ≤ 13 such that A_n is equal to the set of all divisors of n. From Corollary 2.4, both $\lfloor\sqrt{2n}\rfloor$ and $\lfloor\sqrt{2n}-1\rfloor$ are in A_n and for them to be divisors of n, $(\lfloor\sqrt{2n}\rfloor)(\lfloor\sqrt{2n}-1\rfloor)|n$.

But for $n \geq 14$, $(\sqrt{2n}-1)(\sqrt{2n}-2) > n$, and hence $n = 1, 2, 4, 6, 12$ are the only numbers.

Question 4 asks, for which numbers n, does there exist a sequence of Gaussian integers S such that $n \in \mathcal{A}$ (which is the set of differences of the terms of the sequence S) and such that for each divisor d of n, with $1 < d < n$, either both $d-1$ and $d+1$ are divisors of n, or at least one of $d-1$ and $d+1$ is not in $\mathcal{A}$?

We claim that the numbers which satisfy the hypothesis are precisely all prime numbers together with $\{1, 4, 6, 12\}$.

For, if $n = 1$ or n is a prime, then the hypothesis is vacuously true. If n is a composite number ≥ 14, then since $\sqrt{2n}-1 \geq \sqrt{n}$ for $n \geq 14$, there exists a divisor d satisfying $1 < d \leq \sqrt{2n}-1$. Let d be the largest integer less than or equal to $\sqrt{2n}-1$ which divides n.

`Case 1:` At least one of $d-1$ and $d+1$ does not divide n.

By Corollary 2.4, $d-1 \in A_n$ and $d+1 \in A_n$. Clearly, n does not satisfy the hypothesis.

`Case 2:` Both $d-1$ and $d+1$ divide n. We have, $d(d+1)|n$. Since we have assumed that d is the greatest divisor $\leq \sqrt{2n}-1$, hence $d+1 > \sqrt{2n}-1$. Now, $d(d+1)|n$ implies that $(\sqrt{2n}-1)(\sqrt{2n}-2) \leq n$. But $(\sqrt{2n}-1)(\sqrt{2n}-2) > n$ for $n \geq 14$.

Hence, there is no any composite $n \geq 14$ satisfying the hypothesis. One can check that $n \leq 13$ and n not a prime (satisfying the hypothesis) are given in the set $\{1, 4, 6, 12\}$.

Question 5 asks, for which numbers n, does there exist a sequence of Gaussian integers such that $n \in \mathcal{A}$ and for each divisor d of n with $K < d < n-K$, either all the numbers $d-K, d-K+1, \ldots, d+K$ are divisors of n or at least one of $d-K, d-K+1, ..., d+K$ is not in $\mathcal{A}$?

We claim that the set of numbers satisfying the hypothesis is $\{mp : 1 \leq m \leq K$ and $p \geq (2K+1)$ a prime $\}$ together with a finite set.

`Case 1:` Let $n = mp$, where $m \leq K$ and $p \geq 2K+1$. Then $n = m(p+1) - m$ implies that $|r(n, p+1)| \geq k(n, p+1)$. By Theorem 2.1, $p+1 \notin A_n$. Therefore, there exists a sequence S, which does not contain any two terms whose difference is $p+1$ and contains two terms with difference n. Since any divisor $d > K$ of n is of the form $d = d'p$, where d' is divisor of m, it follows that for each $d'p$, $d'p + d'(d' \leq K)$ is not in the difference set of S. For, if $d'p + d' \in \mathcal{A}$, then $p+1 \in \mathcal{A}$, which is not true. Hence, the sequence S satisfies the hypothesis.

`Case 2:` Let $n \notin \{mp : 1 \leq m \leq K$ and $p \geq (2K+1)$ a prime $\}$, $n \geq (2K+1)K$, $\frac{\sqrt{2n}-K}{K} > K$, $\sqrt{2n}-K \geq \sqrt{n}$ and $(\sqrt{2n}-K)(\sqrt{2n}-K-1) > n$.

Clearly, n is not a prime. Let d be the greatest integer dividing n and satisfying $d \leq \sqrt{2n}-K$. We claim that $d > K$. If $d \leq K$, then $n = md$ for some m.

Let p be a prime dividing m. Since $pd|n$ and $pd > d$, from the maximality of d, $pd > \sqrt{2n} - K$, which implies $p > \frac{\sqrt{2n}-K}{K} > K$. Since $n \notin \{mp : 1 \leq m \leq K$ and $p \geq (2K+1)$ a prime $\}$ and m is not a prime, it follows that m has at least two prime factors each of them being greater than K and at least one of them will be less than $\sqrt{n} \leq \sqrt{2n} - K$. Hence, at least one of the prime factors of m is greater than d and $\leq \sqrt{2n} - K$, contradicting the maximality of d. Thus, $d > K$.

If $(d+1)|n$, then from the maximality of d, $(d+1) > \sqrt{2n} - K$ and $d(d+1)|n$, which implies $(\sqrt{2n} - K)(\sqrt{2n} - K - 1) \leq n$. But $(\sqrt{2n} - K)(\sqrt{2n} - K - 1) > n$. Hence, $(d+1) \nmid n$.

For all $1 \leq i \leq K$, $d \pm i \leq \sqrt{2n}$. Further, by Corollary 2.4, $d \pm i \in A_n$. Hence, n does not satisfy the hypothesis.

`Case 3:` Let $n \notin \{mp : 1 \leq m \leq K$ and $p \geq (2K+1)$ a prime $\}$ and at least one of the inequalities $n \geq (2K+1)K$, $\frac{\sqrt{2n}-K}{K} > K$, $\sqrt{2n} - K \geq \sqrt{n}$ and $(\sqrt{2n} - K)(\sqrt{2n} - K - 1) > n$ is not true. This accounts for finitely many exceptions.

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