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GENERALIZED DOMAINS OF SEMISTABLE ATTRACTION OF NONNORMAL LAWS

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Abstract. Operator semistable laws are the natural multivariable analogue of semistable laws in one variable. Operator semistable laws occur as the limit of normalized and centered sums of i.i.d. random vectors when we consider only the sums which terminate at some k_n, with the ratio of successive k_n tending to some constant $c \geq 1$. The generalized domain of semistable attraction of an operator semistable law consists of all such underlying distributions, when we allow normalizing by linear operators. In this paper we give concise necessary and sufficient conditions for a probability distribution to belong to the generalized domain of semistable attraction of any operator semistable law having no normal component. These results, together with the case of a normal limit, constitute a more general framework in which an i.i.d. sum of random vectors can be usefully approximated by a limit distribution. We anticipate a number of applications to multivariate analysis for random vectors whose covariance matrix is undefined because of heavy tails.

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1. Introduction. Suppose that $X, X_1, X_2, X_3, \ldots$ are independent random vectors on $\mathbb{R}^d$ with common distribution μ. Suppose that Y is a random vector on $\mathbb{R}^d$ whose distribution ν is full, i.e. it cannot be supported on any $d-1$ dimensional hyperplane. If there exist linear operators A_n and nonrandom vectors b_n such that

$$A_n(X_1 + \cdots + X_{k_n}) - b_n \Rightarrow Y \tag{1.1}$$

for some sequence of natural numbers $k_n \to \infty$ with $k_{n+1}/k_n \to c \geq 1$, then we say that μ belongs to the generalized domain of semistable attraction of ν. The class of all possible limiting distributions is called the operator semistable laws. Operator semistable laws are the natural multivariable analogue of the semistable laws in one dimension, see [7] and [11]. They may also be considered a generalization of the operator stable laws. Operator stable laws form the class of all possible limiting distributions in (1.1) in the special case where $k_n = n$, see [23].

Generalized domains of semistable attraction are important because they represent a more general context in which the sum $X_1 + \cdots + X_n$ of i.i.d. random vectors can be usefully approximated by a limit distribution. This allows statistical inference on the distribution of the sum. If $E\|X\|^2$ exists then the limit is normal, and the central limit theorem applies (see Section 4 below). In the remaining case, the tails of X are too heavy to allow for the existence of a finite covariance matrix. Heavy tail distributions are important in a number of real applications. See [8] for an excellent survey of this area including applications to physics, chemistry, and economics. Convergence criteria for generalized domains of attraction are known, but their application requires the assumption of regularly varying tails. This does not allow for the tail oscillations seen in many real problems, such as seasonal time series models. The approach of this paper allows such oscillations. We believe that generalized domain of semistable attraction will prove useful in many real applications, since it allows a more robust set of statistical assumptions to be employed.

Full operator semistable laws were characterized in [7]. If $c = 1$ then the limiting distribution ν in (1.1) is operator stable. If $c > 1$ then ν is still infinitely divisible, and satisfies

$$\nu^c = B\nu * \delta(a) \tag{1.2}$$

for some invertible linear operator B on $\mathbb{R}^d$ and some $b \in \mathbb{R}^d$. Here ν^c is the c-fold convolution product of ν, $B\nu\{dx\} = \nu\{B^{-1}\,dx\}$, and $\delta(x)$ is the unit mass at $x \in \mathbb{R}^d$. We say that ν is (B, c) operator semistable. In this paper we restrict our attention to the case where ν has no normal component. In that case, the complex absolute value of every eigenvalue of

B must exceed $\sqrt{c}$. See the Theorem in [7] and [12] for more details about operator semistable laws.

Let $\mathrm{GL}(\mathbb{R}^d)$ denote the collection of all invertible linear operators on $\mathbb{R}^d$. If (1.1) holds with all of the norming operators A_n contained in some subcollection $\mathcal{D}$ of $\mathrm{GL}(\mathbb{R}^d)$, we say that μ belongs to the $\mathcal{D}$-normed domain of semistable attraction of ν. $\mathcal{D}$-normed domains of semistable attraction where $\mathcal{D}$ consists of scalar multiples of the identity, and the case where $\mathcal{D}$ consists of all linear operators whose matrix with respect to the standard basis for $\mathbb{R}^d$ is diagonal with all positive entries, were characterized in [20]. The paper [9] also characterized the domain of normal attraction of a non-normal (B, c) operator semistable law, in which we assume that $k_n = [c^n]$ and $A_n = B^{-n}$. If (1.1) holds with $k_n = n$, we say that μ belongs to the generalized domain of attraction of the operator stable limit law ν. A modern reference on operator stable laws is [10]. Necessary and sufficient conditions for a probability measure μ to belong to the generalized domain of attraction of some full operator stable law ν can be obtained from the criteria of [19] for convergence of a triangular array of random vectors. The papers [5] and [13] give alternative necessary and sufficient conditions. In this paper, we will adapt the methods of [13] to extend the results of [20] to the general case $\mathcal{D} = \mathrm{GL}(\mathbb{R}^d)$.

2. Preliminaries. The basic tools we will use to study generalized domains of semistable attraction are the Lévy representation for infinitely divisible laws, and the standard convergence criteria for triangular arrays of random vectors. See [1] and [19]. If ν is an infinitely divisible probability measure on $\mathbb{R}^d$, we can write the characteristic function of ν in the form $\exp(f(t))$ where

$$f(t) = i\langle a, t\rangle - \frac{1}{2}\, Q(t) + \int_{x\neq 0} e^{i\langle t,x\rangle} - 1 - \frac{i\langle t, x\rangle}{1 + \|x\|^2}\ \phi\{dx\} \tag{2.1}$$

where $a \in \mathbb{R}^d$, $Q(t)$ is a nonnegative definite quadratic form on $\mathbb{R}^d$, and ϕ is a Lévy measure on $\Gamma = \mathbb{R}^d \setminus \{0\}$. In other words, ϕ is a σ-finite Borel measure on Γ such that

$$\int_{x\neq 0} \min\{1, \|x\|^2\}\ \phi\{dx\} < \infty. \tag{2.2}$$

The triple $[a, Q, \phi]$ is called the Lévy representation of ν. Suppose that for some sequence $k_n \to \infty$ we have for each n that $Z_{n1}, \ldots, Z_{nk_n}$ are independent and identically distributed with distribution μ_n. Then there exist $b_n \in \mathbb{R}^d$ such that $Z_{n1} + \cdots + Z_{nk_n} - b_n \Rightarrow Y$ if and only if

$$k_n\mu_n \to \phi \tag{2.3a}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty} k_n\left(\int_{0<\|x\|<\varepsilon}\langle t,x\rangle^2\ \mu_n\{dx\} - \Big(\int_{0<\|x\|<\varepsilon}\langle t,x\rangle\mu_n\{dx\}\Big)^2\right) \\
=&\lim_{\varepsilon\to 0}\liminf_{n\to\infty} k_n\left(\int_{0<\|x\|<\varepsilon}\langle t,x\rangle^2\ \mu_n\{dx\} - \Big(\int_{0<\|x\|<\varepsilon}\langle t,x\rangle\ \mu_n\{dx\}\Big)^2\right) \\
=&Q(t)
\end{aligned} \tag{2.3b}$$

for some nonnegative definite quadratic form $Q(t)$ and some Lévy measure ϕ on $\Gamma = \mathbb{R}^d \setminus \{0\}$. In this case, the distribution of the limit Y is infinitely divisible with Lévy representation $[a, Q, \phi]$, where $a \in \mathbb{R}^d$ depends on the choice of centering constant b_n. The convergence (2.3a) is vague convergence in the space of σ–finite Borel measures on Γ. In this topology we have $\nu_n \to \nu$ if $\nu_n(A) \to \nu(A)$ for all Borel sets $A \subset \Gamma$ bounded away from the origin whose topological boundary has ν–measure zero. The centering constants b_n may be chosen according to the formula

$$b_n = k_n \int_{0<\|x\|<r} x\ \mu_n\{dx\} \tag{2.4}$$

where $r > 0$ is chosen so that the domain of integration in (2.4) is a ϕ–continuity set.

In order to simplify the proofs in the next section, we will use regular variation. Regular variation is a powerful analytic tool which has found numerous applications in probability theory, see [3]. Regular variation in $\mathbb{R}^d$ was introduced in [14]. The papers [15], [16] contain two applications of the theory to generalized domains of attraction and operator stable laws in $\mathbb{R}^d$. In the present context we require a generalization of regular variation called R–O variation. For a modern reference see [22]. In the case of monotone functions, R–O variation is sometimes called monotone variation. A Borel measurable function $R(r)$ is R–O varying if it is real–valued and positive for $r \geq A$ and if there exist positive constants $t_0 > 1, m < 1, M > 1$ such that $m \leq R(tr)/R(r) \leq M$ whenever $1 \leq t \leq t_0$ and $r \geq A$. We will say that the function $R(r,x)$ is uniformly R–O varying if it is an R–O varying function of r for each x, and the constants A, t_0, m, M can be chosen independent of x. A necessary and sufficient condition for uniform R–O variation is that $mt^h \leq R(tr,x)/R(r,x) \leq Mt^H$ whenever $t \geq 1$ and $r \geq A$. Here we have let $h = \log\ m/\log t_0$ and $H = \log\ M/\log\ t_0$.

Suppose that μ is a probability measure on $\mathbb{R}^d$, and define the truncated moments U_b, V_a as in [4], i.e. for $a, b \geq 0$ let

$$V_a(r,x) = \int_r^\infty t^a \, R(dt,x)$$

$$U_b(r,x) = \int_0^r t^b \, R(dt,x) \tag{2.5}$$

where $R(r,x) = \mu\{y : |\langle x,y\rangle| \leq r\}$.

Theorem 2.1. *Suppose that for a probability measure μ there exist a sequence k_n of natural numbers tending to infinity with $k_{n+1}/k_n \to c > 1$ and linear operators A_n such that*

$$k_n A_n \mu \longrightarrow \phi \tag{2.6}$$

for some σ-finite Borel measure ϕ on $\Gamma = \mathbb{R}^d \backslash \{0\}$ which cannot be supported on any $d-1$ dimensional subspace of $\mathbb{R}^d$, and which satisfies $c\phi = B\phi$ with $|\beta| > \sqrt{c}$ for all eigenvalues β of B. Define $\lambda = \min\{|\beta|\}$, $\Lambda = \max\{|\beta|\}$, where β ranges over the eigenvalues of B. Then for any compact set $K \subset \Gamma$ and any $\varepsilon > 0$:

(i) If $b > \log c/\log \Lambda$ then for some $A > 0$ and $M > 0$, we have

$$\frac{U_b(tr,x)}{U_b(r,x)} \leq M t^{b+\varepsilon-\log c/\log \Lambda} \tag{2.7}$$

for all $t \geq 1$, $r \geq A$, and $x \in K$;

(ii) For some $A > 0$ and $m > 0$, we have

$$\frac{V_0(tr,x)}{V_0(r,x)} \geq m t^{-\varepsilon-\log c/\log \lambda} \tag{2.8}$$

for all $t \geq 1$, $r \geq A$, and $x \in K$.

The proof of both parts is similar, and so we will only prove part (ii). In order to make the proof easier to read, we begin with a few simple lemmas. In each of these, we assume the hypotheses of the theorem. Since the absolute value of any eigenvalue of B must exceed $\sqrt{c} > 1$, we must have $\lambda > 1$. Recall that if β is an eigenvalue of B, then β is also an eigenvalue of B^*, and β^{-1} is an eigenvalue of B^{-1}.

Lemma 2.1. *Given $\alpha \in (1,\lambda)$ and a relatively compact set $S \subset \Gamma$, there exists a positive integer k_0 such that*

$$\|(B^*)^{-k}x\| < \alpha^{-k} \tag{2.9}$$

for all $k \geq k_0$ and all $x \in S$.

Proof. If $\beta_1, \dots, \beta_p$ is the spectrum of B, then the spectrum of the linear operator $\alpha(B^*)^{-1}$ is $\alpha\beta_1^{-1}, \dots, \alpha\beta_p^{-1}$, and so the absolute value of any of these eigenvalues is no larger than $\alpha/\lambda < 1$. Thus $\alpha^k \|(B^*)^{-k}x\| \to 0$ uniformly on compact sets, which proves the lemma. □

The next lemma gives bounds on the tails of ϕ. Define

$$g(x) = \phi\{y : |\langle x, y\rangle| > 1\} \tag{2.10}$$

for all $x \in \Gamma$. Since $c\phi = B\phi$, we have $cg(x) = g(B^*x)$ for all $x \in \Gamma$. Since ϕ cannot be supported on any proper subspace of $\mathbb{R}^d$, we must also have $g(x) > 0$ for all $x \in \Gamma$, and in fact $g(x)$ is bounded away from both zero and infinity on any compact subset of Γ.

Lemma 2.2. *Given $\alpha \in (1, \lambda)$ and a relatively compact set $S \subset \Gamma$, there exists $t_0 > 1$ and $\delta_0 > 0$ such that*

$$\frac{g(y/t)}{g(y)} \geq \delta_0 t_0^{-\log c/\log\alpha} \tag{2.11}$$

whenever $1 \leq t \leq t_0$ and $y \in S$.

Proof. It suffices to prove the lemma for sets of the form $S = \{y \in \mathbb{R}^d : a \leq \|y\| \leq b\}$, where $0 < a < b$ such that $\Gamma = \bigcup B^k S$ where k ranges over the integers. Now apply Lemma 2.1 and set $t_0 = \alpha^{k_0} a$. Given $y \in S$ and $t \in [1, t_0]$, we can write $y/t = (B^*)^{-k} y'$ for some k and some $y' \in S$. Since $\|(B^*)^{-k}y'\| = \|y/t\| \geq a/t_0 = \alpha^{-k_0}$ we see from Lemma 2.1 that $k \leq k_0$. Then

$$\frac{g(y/t)}{g(y)} = \frac{g((B^*)^{-k}y')}{g(y)} = \frac{c^{-k}g(y')}{g(y)}$$

is bounded below by $c^{-k_0} \cdot L$, where L is a lower bound of $g(y')/g(y)$ for $y, y' \in S$. Now compute that $c^{-k_0} = (t_0/a)^{-\log c/\log\alpha}$. This proves the lemma, since now (2.11) holds with $\delta_0 = La^{\log c/\log\alpha}$. □

Define

$$F(x) = \mu\{y : |\langle x, y\rangle| > 1\} \tag{2.12}$$

for all $x \in \Gamma$. The next lemma gives bounds on the growth rate of the tail function F.

Lemma 2.3. *Given $\alpha \in (1, \lambda)$ and a relatively compact set $D \subset \Gamma$, there exists $t_1 > 1$, $\delta_1 > 0$, and $n_0 > 1$ such that*

$$\frac{F(A_n^* x/t)}{F(A_n^* x)} \geq \delta_1 t_1^{-\log c/\log\alpha} \tag{2.13}$$

whenever $n \geq n_0$, $1 \leq t \leq t_1$ and $x \in D$.

Proof. It suffices to prove the lemma for sets of the form $D = \{y \in \mathbb{R}^d : b \leq \|y\| \leq d\}$. Apply Lemma 2.2 with $S = \{y : b/2 \leq \|y\| \leq 2d\}$, and then let $t_1 = t_0/2$. It follows easily from (2.6) that $k_n F(A_n^* x_n) \to g(x)$ whenever $x_n \to x$, provided that $H(x) = \{y : |\langle x, y\rangle| > 1\}$ is a ϕ–continuity set. Let $K = \{y : b/t_1\sqrt{2} \leq \|y\| \leq \sqrt{2}d\}$ and note that $F(A_n^* x)/F(A_n^* x') \to g(x)/g(x')$ uniformly over all $x, x' \in K$ such that both $H(x)$ and $H(x')$ are ϕ–continuity sets. Given any $x \in D$ and $t \in [1, t_1]$, we may always choose $1 < s < \sqrt{2}$ such that both $H(x/st)$ and $H(sx)$ are ϕ–continuity sets. Now observe that

$$\frac{F(A_n^* x/t)}{F(A_n^* x)} \geq \frac{F(A_n^* x/st)}{F(A_n^* sx)} \to \frac{g(x/st)}{g(sx)}$$

uniformly in $x \in D$ and $1 < s < \sqrt{2}$ such that both $H(x/st)$ and $H(sx)$ are ϕ–continuity sets. Choose $\varepsilon > 0$ small and let n_0 be sufficienctly large to make $|F(A_n^* x)/F(A_n^* x') - g(x)/g(x')| < \varepsilon$ for all $n \geq n_0$ and all $x, x' \in K$ such that both $H(x)$ and $H(x')$ are ϕ–continuity sets. Since $sx \in S$ and $1 \leq s^2 t \leq t_0$, we can apply the lower bound of Lemma 2.2 along with the uniform convergence to see that the expression on the left hand side of (2.13) is bounded below by $\delta_0 t_0^{-\log c/\log\alpha} - \varepsilon$. This proves the lemma, since now (2.13) holds with δ_1 arbitrarily close to $2^{-\log c/\log\alpha}\delta_0$, depending on ε. □

Lemma 2.4. *The sequence $\{(A_{n+1}^*)^{-1} A_n^*\}$ is relatively compact.*

Proof. Suppose S is a ϕ–continuity set with positive ϕ measure of the form $S = \{a \leq y \leq b\}$ for $0 < a < b$. Let $\alpha_n = k_n A_n \mu(S)$, $\alpha = \phi(S)$ and let ν_n, ν denote the restriction of the measures $\alpha_n^{-1} k_n A_n \mu$, $\alpha^{-1}\phi$ to the set S. Then $\nu_n \Rightarrow \nu$ as probability measures and it is easy to see that ν is full. Define $T_n = cA_{n+1}A_n^{-1}$ and observe that $T_n \nu_n \Rightarrow \nu$. Then the convergence of types theorem of [2] implies that $\{T_n\}$ is relatively compact. Now take the inverse transpose of T_n and multiply by c. □

Proof of Theorem 2.1. Given $x \in K$, we define $n(r,x) = \sup\{n : ||(A_n^*)^{-1}x|| \le r\}$. In order that (2.6) holds where ϕ cannot be supported an any proper subspace of $\mathbb{R}^d$, we must have that μ is full, and $A_n\mu \to \delta(0)$. Then $\|A_n\| \to 0$, and so $\|A_n^*\| \to 0$ as well. It follows that $\|(A_n^*)^{-1}x\| \to \infty$ uniformly on compact subsets of Γ, and so $n = n(r,x)$ is well–defined, and tends to infinity uniformly in $x \in K$. Define $y_n = (A_n^*)^{-1}(x/r)$. Then for some positive reals a and b we have by virtue of Lemma 2.4 that $a\|y\| \le \|A_{n+1}^*(A_n^*)^{-1}y\| \le b\|y\|$ for all $y \in \mathbb{R}^d$. Then $\|y_n\| \le 1$ while $\|(A_{n+1}^*)^{-1}A_n^*y_n\| > 1$, so that $y_n \in D = \{y : 1/b \le \|y\| \le 1\}$ for all n. In particular, the sequence $\{y_n\}$ is relatively compact. Note also that $V_0(r,x) = F(x/r)$, so that by Lemma 2.3 we have

$$\frac{V_0(tr,x)}{V_0(r,x)} = \frac{F(A_n^*y_n/t)}{F(A_n^*y_n)} \ge \delta_1 t_1^{-\log c/\log\alpha} \tag{2.14}$$

whenever $1 \le t \le t_1$, $x \in K$, and $r \ge r_0$, where r_0 is sufficiently large to make $n(r,x) \ge n_0$ for all $x \in K$. Since V_0 is monotone, we have now shown that V_0 is R–O varying, uniformly on $x \in K$. Then (2.8) follows by a standard argument, see for example [22, Theorem A.2]. This concludes the proof of part (ii). The proof of part (i) is similar, using results analogous to those of Lemmas 2.1 through 2.4. □

3. Results. In this section we will characterize the generalized domain of semistable attraction of an arbitrary nonnormal (B,c) operator semistable law. As usual nonnormal means a law without a Gaussian component, i.e. with Lévy representation $[a,0,\phi]$. We will use the basic theory of infinitely divisible laws and triangular arrays, along with the regular variation theory developed in the previous section. The following theorem generalizes the results of [9], [13], and [20].

Theorem 3.1. *A probability measure μ on $\mathbb{R}^d$ belongs to the generalized domain of semistable attraction of a full nonnormal (B,c) operator semistable law ν on $\mathbb{R}^d$ if and only if there exist linear operators A_n and natural numbers $k_n \to \infty$ with $k_{n+1}/k_n \to c > 1$ such that*

$$k_n A_n \mu \to \phi \tag{3.1}$$

for some σ–finite Borel measure ϕ on $\mathbb{R}^d \setminus \{0\}$ which cannot be supported on any $d-1$ dimensional subspace of $\mathbb{R}^d$, and which satisfies

$$c\phi = B\phi \tag{3.2}$$

with $|\beta| > \sqrt{c}$ for all eigenvalues β of B.

Proof. Suppose that μ belongs to the generalized domain of semistable attraction of ν full, and apply the standard convergence criteria to the triangular array $Z_{ni} = A_n X_i$. The convergence (3.1) follows immediately from (2.3a), and (3.2) can be obtained by substituting (1.2) into the Lévy representation (2.1). Since ν is full and $Q(t) = 0$ for all $t \in \mathbb{R}^d$, it follows easily that the Lévy measure ϕ cannot be supported on any proper subspace of $\mathbb{R}^d$, and we know from [7] that $|\beta| > \sqrt{c}$ for all eigenvalues β of B. This proves the direct half of the theorem. As to the converse, the condition (3.1) is again equivalent to the first condition (2.3a) for convergence of the triangular array. It remains to show that (2.3b) holds with $Q(t) = 0$ for all $t \in \mathbb{R}^d$.

The Schwartz inequality implies that both expressions under the limit in (2.3b) are nonnegative, and so the first (lim sup) convergence implies the second. For this it is sufficient to establish the convergence with the squared integral term deleted. Note that it is sufficient to show this for $\|t\| = 1$. Finally, there is no harm in enlarging the domain of integration. All together, we see that it will suffice to show that for any unit vector $t \in \mathbb{R}^d$ we have

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} k_n \int_{0 < |\langle t, x \rangle| \le \varepsilon} |\langle t, x \rangle|^2 \; A_n \mu\{dx\} = 0. \tag{3.3}$$

Write $A_n^* t = r_n \theta_n$ where $\|\theta_n\| = 1$ and $r_n > 0$. Then after a change of variable the expression under the limit in (3.3) becomes $k_n r_n^2 U_2(\varepsilon / r_n, \theta_n)$. Apply Theorem 2.1 (ii) to see that $V_0(r, \theta)$ is uniformly R–O varying on the unit sphere $\{\|\theta\| = 1\}$, and recall that $\lambda > \sqrt{c}$, so that we can write $-\varepsilon - \log c / \log \lambda = \delta - 2$ for some $\delta > 0$. Now apply a result of [4, p. 289] to see that for $V_0(r, \theta)$ uniformly R–O varying we must have $r^2 V_0(r, \theta) / U_2(r, \theta)$ bounded away from zero independent of θ and r sufficiently large. (The result of [4, p. 289] is only for R–O variation, but the lower bound can be explicitly computed, and depends only on the constants A, a, m, M in the definition of R–O variation, and so the same proof shows that the result holds for uniform R–O variation as well.) Then we have $k_n r_n^2 U_2(\varepsilon / r_n, \theta_n) \le c_1 k_n \varepsilon^2 V_0(\varepsilon / r_n, \theta_n)$ for some $c_1 > 0$. Now use (2.8) to see that this is bounded above by $c_2 \varepsilon^\delta k_n V_0(r_n^{-1}, \theta_n) = c_2 \varepsilon^\delta k_n \mu(A_n^{-1} H(t))$ for all large n, where $H(t) = \{x : |\langle t, x \rangle| > 1\}$. Use the portmanteau theorem for finite measures to bound this quantity above by $c_2 \varepsilon^\delta \phi(\bar{H}(t))$, where $\bar{H}(t)$ denotes the closure. This quantity tends to zero as $\varepsilon \to 0$, which concludes the proof. □

One advantadge of the regular variation methods used in the proof of Theorem 3.1 is that centering results can be obtained with little additional effort. The following corollary extends the results of [4, p. 289] in $\mathbb{R}^1$ and [15] in $\mathbb{R}^d$, who only considered the (generalized) domain of attraction. For domains of semistable attraction, the following result is new even in $\mathbb{R}^1$.

Corollary 3.1. *Suppose X is a random vector whose distribution μ belongs to the generalized domain of semistable attraction of a nonnormal (B, c) semistable law. Then the centering constants b_n in (1.1) may be taken as:*

(a) $b_n = k_n E(A_n X_1)$ if $\Lambda < c$, so we center to zero expectation;
and

(b) $b_n = 0$ if $\lambda > c$.

Proof. If $\Lambda < c$, then EX_1 exists by virtue of [21, Theorem 4.1]. From (2.4) we obtain

$$b_n = k_n \int_{\|x\|<R} x \ A_n\mu\{dx\} \tag{3.4}$$

for any $R > 0$ such that the domain of integration is a ϕ-continuity set. Part (a) states that in (1.1) we can take $b_n = k_n E(A_n X_1)$. Hence we want to show that

$$d_n = k_n \int_{\|x\|>R} x \ A_n\mu\{dx\}$$

can be made arbitrary close to zero for all large n by taking $R > 0$ sufficiently large. For any $\|t\| = 1$ and r_n, θ_n as in the proof of Theorem 3.1 we have

$$|\langle d_n, t\rangle| \leq k_n r_n V_1(R/r_n, \theta_n) + k_n \int_{A(R,t)} |\langle x, t\rangle| \ A_n\mu\{dx\} \tag{3.5}$$

where $A(R, t) = \{x : \ \|x\| > R, \ |\langle x, t\rangle| \leq R\}$. The second integral on the right is bounded above by $Rk_n A_n\mu(A(R, t))$ which is again bounded by $KR\phi(A(R, t))$ for all n. Arguing as in the proof of Lemma 2.2 we see that for any $\delta > 0$ there exists a $R_0 > 0$ such that $\phi(A(R, t)) \leq \phi\{x : \ \|x\| > R\} \leq KR^{-\log c/\log \Lambda+\delta}$ for all $R \geq R_0$. Since $\Lambda < c$ we get for a suitable choice of $\delta > 0$ that the second integral in (3.5) can be made arbitrary small by taking $R > 0$ sufficiently large. Now in order to bound the first term in (3.5) we apply Theorem 2.1 to see that $U_2(r, \theta)$ is uniformly R–O varying, and recall that $\Lambda < c$, so that we can write $2+\varepsilon-\log c/\log \Lambda = 1-\delta$ for some $\delta > 0$. An application of the result of [4, p. 289] yields $k_n r_n V_1(R/r_n, \theta_n) \leq K r_n^2 R^{-1} U_2(R/r_n, \theta_n)$. Now use (2.7) to see that this is bounded above by $K_1 k_n r_n^2 R^{-\delta} U_2(r_n^{-1}, \theta_n)$. Using uniform R–O variation and the result of [4, p. 289] again this is bounded by $K_2 R^{-\delta} k_n V_0(r_n^{-1}, \theta_n) = K_2 R^{-\delta} k_n A_n\mu(H(t))$. Using (3.1) we see as in the proof of Theorem 3.1 that this can be made arbitrary small by taking R large enough.

The proof of part (b) is similar: Apply Theorem 2.1 to see that $V_0(r,\theta)$ is uniform R–O varying. Since $\lambda > c$ we can write $-\varepsilon - \log c/\log\lambda = -1+\delta$ for some $\delta > 0$. Using the result of [4, p. 289] we get

$$|\langle b_n, t\rangle| \le k_n r_n U_1(R/r_n, \theta_n) \le KRk_nV_0(R/r_n,\theta_n) \le K_1R^\delta k_n A_n \mu(H(t))$$

and the result follows as in the proof of Theorem 3.1 by taking $R > 0$ small enough. This concludes the proof. □

4. Remarks. Inherent in Theorem 3.1 is information about the tails of a probability measure μ which belongs to the generalized domain of semistable attraction of a nonnormal limit. The estimates of Theorem 2.1 show that the tails of such a distribution must tend to zero no faster than $t^{-\log c/\log\lambda}$ and no slower than $t^{-\log c/\log\Lambda}$. In order to characterize the growth rate of R–O varying functions, it is common to employ the Matuszewska index, see for example [3]. In view of Theorem 2.1 it is reasonable to believe that V_0 is R–O varying with lower index $-\log c/\log\lambda$ and that U_2 is R–O varying with upper index $2-\log c/\log\Lambda$, but we have not been able to prove this. Related to the tail behavior is the existence and nonexistence of moments. The paper [21] contains moment results for generalized domains of semistable attraction.

It is also of some interest to describe the generalized domain of semistable attraction in the case of a normal limit, or a limit with both a normal and a nonnormal component. In [17] we consider this problem. From [18], theorem 6 we get that the generalized domain of semistable attraction of an operator stable law (for example, a normal law) is exactly the same as the generalized domain of attraction. In other words, in this case there is no loss of generality in assuming that $k_n = n$ in (1.1). Using this fact, we also show that the generalized domain of semistable attraction of a mixed limit, with both a normal and a nonnormal component, can be described in terms of the generalized domain of semistable attraction of each component.

Theorem 3.1 extends the result of [13] for nonnormal operator stable laws. It may also be possible to extend the results of [5] to obtain a different characterization of the generalized domains of semistable attraction.

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