Journal of Algebraic Combinatorics **10** (1999), 173–188

Combinatorics of Necklaces and "Hermite Reciprocity"

A. ELASHVILI alela@rmi.acnet.ge
M. JIBLADZE jib@rmi.acnet.ge
D. PATARAIA dito@rmi.acnet.ge
A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, M. Alexidze St. 1, Tbilisi 380093, Republic of Georgia

Received December 5, 1996; Revised March 26, 1998

Abstract. Combinatorial proof of an explicit formula for dimensions of spaces of semi-invariants of regular representations of finite cyclic groups is obtained. Using bicolored necklaces, a certain reciprocity law following from this formula is also derived combinatorially.

Keywords: partition, necklace, semi-invariant, reciprocity

Introduction

The classical Hermite Reciprocity Law asserts the isomorphism

$$S^m S^n(k^2) \cong S^n S^m(k^2)$$

of symmetric powers of representations of the Lie group $\mathrm{SL}_2(k)$ acting standardly on k^2, for a characteristic zero field k (see [6], Remark 12 by Popov in Appendix 3 of the Russian translation). In particular, the space of degree m polynomial invariants of the irreducible $(n+1)$-dimensional representation is equidimensional with the space of degree n invariants of the irreducible $(m+1)$-dimensional representation.

Recently in [3] there was obtained an explicit formula for the dimension $a_0(n, m)$ of the space of degree m homogeneous polynomial invariants of the regular representation of the nth order cyclic group. This formula implies that $a_0(n, m) = a_0(m, n)$. In the present paper, we give a combinatorial explanation of a certain generalization of this fact (see below), which we also call Hermite reciprocity.

Relationship with combinatorics stems from the observation that, as shown in [3], the number $a_0(n, m)$ coincides with the number of solutions of the system

$$\sum_{j=0}^{n-1} j\lambda_j \equiv 0 \pmod{n}; \qquad \sum_{i=0}^{n-1} \lambda_i = m. \tag{1}$$

The first author is partially supported by the INTAS Grant #93-0893.
The second and third authors are supported by the ISF Grant MXH200 and by the INTAS Grant #93-2618.

Clearly this is the same as the total number of partitions of multiples of n into no more than m parts not exceeding $n-1$. Applying combinatorial arguments one can obtain a formula for the number $a_k(n, m)$ of solutions of an even more general system

$$\sum_{j=0}^{n-1} j\lambda_j \equiv k \pmod{n}; \qquad \sum_{i=0}^{n-1} \lambda_i = m \tag{2}$$

where k is any nonnegative integer.

As a kind of illustration let us reproduce the first few values of $a_k(n, m)$ (computed using the `MAPLE` package):

(1) $a_0(n, m)$, for $1 \leq n, m \leq 10$:

1	1	1	1	1	1	1	1	1	1
1	2	2	3	3	4	4	5	5	6
1	2	4	5	7	10	12	15	19	22
1	3	5	10	14	22	30	43	55	73
1	3	7	14	26	42	66	99	143	201
1	4	10	22	42	80	132	217	335	504
1	4	12	30	66	132	246	429	715	1144
1	5	15	43	99	217	429	810	1430	2438
1	5	19	55	143	335	715	1430	2704	4862
1	6	22	73	201	504	1144	2438	4862	9252

(2) $a_1(n, m)$, for $1 \leq n, m \leq 10$:

1	1	1	1	1	1	1	1	1	1
1	1	2	2	3	3	4	4	5	5
1	2	3	5	7	9	12	15	18	22
1	2	5	8	14	20	30	40	55	70
1	3	7	14	25	42	66	99	143	200
1	3	9	20	42	75	132	212	333	497
1	4	12	30	66	132	245	429	715	1144
1	4	15	40	99	212	429	800	1430	2424
1	5	18	55	143	333	715	1430	2700	4862
1	5	22	70	200	497	1144	2424	4862	9225

Derivation of $a_k(n, m)$, given below, is analogous to a proof for the $a_0(n, m)$ communicated to us by G. Andrews. The expression obtained has all the advantages of an explicit formula, in particular it immediately implies the equality $a_k(n, m) = a_k(m, n)$, which too may be called "Hermite reciprocity". But the proof does not explain in any way the reason of this reciprocity. In the second part of the paper we give one of the possible explanations for the equality. Namely: to any solution of (1) we assign a necklace, i.e., a circular arrangement, consisting of $n+m$ beads, n of them black and m white, together with a chosen orientation and a basepoint somewhere between two adjacent beads. Thereafter, the equality $a_k(n,m) = a_k(m,n)$ turns out to follow from the existence of an involution on the set of such necklaces, acting by choosing opposite orientation and swapping black and white. We must note that in a private conversation with the first author, N. Alon communicated a proof involving necklaces, of $a_0(n,m) = a_0(m,n)$, when n and m are coprime. In the present paper this idea has been extended to the more general setting. The authors would like to express their gratitude to Alon, Andrews and Stanley for valuable information and interest to the paper. They are grateful to the referee for careful reading of the paper and finding of several misprints in important formulae.

Everywhere in the sequel, for any integers $n, m, k, \ldots$ their greatest common divisor will be denoted by $(n, m, k, \ldots)$; for $n > 0$, we denote by $(k)_n$ the residue of k modulo n, i.e., the number determined by $0 \leq (k)_n < n$, $(k)_n \equiv k (\operatorname{mod} n)$.

1. Explicit formula

Let us start with a purely formal expression for $a_k(n,m)$. Therefore recall that $p(N, M, s)$, for any integers N, M, s, denotes the number of partitions of s into no more than M parts, each not exceeding N. The generating function for these numbers,

$$G(N, M; t) = \sum_s p(N, M, s) t^s$$

is the Gauss polynomial (see e.g., [1], 3.2).

Then, one obviously has

$$a_k(n,m) = \sum_j p(n-1, m, jn+k). \tag{3}$$

We shall also need the definition of *Ramanujan sums* (see e.g., [4], 17.6; for applications in number theory see [5]). For any n and k, the Ramanujan sum $c_n(k)$ is the sum of kth powers of all primitive nth roots of 1. In particular, $c_n(0) = \varphi(n)$ (the Euler function), $c_n(1) = \mu(n)$ (the Möbius function). It is known (and easily seen using Möbius inversion) that

$$c_n(k) = \sum_{d|(n,k)} \mu\left(\frac{n}{d}\right) d.$$

Also note that this last equality obviously implies $c_n(k) = c_n((n,k))$, in particular, $c_n(-k) = c_n(k)$.

We then have the following:

Theorem 1 *For any integers k, n, m,*

$$a_k(n,m) = \frac{1}{n+m} \sum_{d|(n,m)} c_d(k) \binom{n/d + m/d}{n/d}; \tag{4}$$

in particular

$$a_k(n,m) = a_k(m,n).$$

Proof: By (3), $a_k(n,m)$ equals the sum of coefficients of $G(n-1, m; t)$ at those powers of t which are congruent to k modulo n. Now in general, given any polynomial $f(t) = \sum f_\nu t^\nu$, one has

$$\sum_{\nu \equiv k \pmod n} f_\nu = \frac{1}{n} \sum_{\zeta^n = 1} \zeta^{-k} f(\zeta),$$

the sum on the right running over all nth roots of 1. This fact easily follows from the equality

$$\sum_{\zeta^n=1} \zeta^\nu = \begin{cases} n & \text{if } n \mid \nu \\ 0 & \text{otherwise.} \end{cases}$$

So in our case

$$a_k(n,m) = \frac{1}{n} \sum_{\zeta^n=1} \zeta^{-k} G(n-1, m; \zeta).$$

Values of Gauss polynomials at roots of 1 are known; see e.g., [7], Chapter 3, Exercise 45(b). In particular, for any $\zeta^n = 1$ which is a primitive dth root of 1, for some $d \mid n$, one has

$$G(n-1, m; \zeta) = \begin{cases} \dbinom{m/d + n/d - 1}{m/d} & \text{if } d \mid m \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$a_k(n,m) = \frac{1}{n} \sum_{d|n} \sum_{\operatorname{ord}(\zeta)=d|m} \zeta^{-k} \binom{m/d + n/d - 1}{m/d},$$

where $\operatorname{ord}(\zeta)$ means order of the element ζ in the group of roots of 1. Now ζ has order d iff ζ is a primitive root of order d. Hence

$$\begin{aligned} a_k(n,m) &= \frac{1}{n} \sum_{d|(n,m)} c_d(-k) \binom{m/d + n/d - 1}{m/d} \\ &= \frac{1}{n} \sum_{d|(n,m)} c_d(k) \frac{\left(\frac{m+n}{d} - 1\right)!}{\left(\frac{n}{d} - 1\right)! \frac{m}{d}!} \end{aligned}$$

$$= \frac{1}{n} \sum_{d|(n,m)} c_d(k) \frac{\frac{d}{m+n} \frac{m+n}{d}!}{\frac{d}{n} \frac{n}{d}! \frac{m}{d}!}$$

$$= \frac{1}{n} \sum_{d|(n,m)} c_d(k) \frac{n}{m+n} \binom{m/d + n/d}{n/d}$$

$$= \frac{1}{m+n} \sum_{d|(n,m)} c_d(k) \binom{m/d + n/d}{n/d},$$

and the theorem follows. □

Remark Note that the formula obtained implies, due to the mentioned properties of Ramanujan sums, that there are many equalities between $a_k(n, m)$ for fixed n, m and various k. Namely, one has

$$a_k(n, m) = a_{(k,n,m)}(n, m).$$

For the sequel, let us fix two positive integers n, m and denote (n, m) by d.

2. Combinatorial proof of reciprocity

In this section, we are going to give another, purely combinatorial proof of the equality $a_k(n, m) = a_k(m, n)$.

Consider the set

$$\Lambda_{n,m} = \left\{ (\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{N}^n \mid \forall i\ \lambda_i \geq 0 \text{ and } \sum_{i=0}^{n-1} \lambda_i = m \right\}.$$

We define an action of the cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$ of order n on $\Lambda_{n,m}$ by setting, for $r \in \mathbb{Z}/n\mathbb{Z}$ and $\lambda = (\lambda_0, \dots, \lambda_{n-1}) \in \Lambda_{n,m}$,

$$r\lambda = (\lambda_{n-r}, \lambda_{n-r+1}, \dots, \lambda_{n-1}, \lambda_0, \dots, \lambda_{n-r-1}).$$

For any $\lambda \in \Lambda_{n,m}$, denote by $t(\lambda)$ the minimal positive integer with $(t(\lambda)1)\lambda = \lambda$, i.e., the number of elements in the orbit of λ under the action of C_n (1 is the element of $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$). Denote $n/t(\lambda)$, i.e., order of the stabilizer of λ under this action, by $s(\lambda)$.

Let $\tilde{\Lambda}_{n,m}$ be the quotient $\Lambda_{n,m}/C_n$ and denote by $\pi_{n,m}$ the quotient map $\pi_{n,m} : \Lambda_{n,m} \to \tilde{\Lambda}_{n,m}$.

Take any $\alpha \in \tilde{\Lambda}_{n,m}$ and $\lambda \in \pi_{n,m}^{-1}(\alpha)$. Note that $\#(\pi_{n,m}^{-1}(\alpha)) = t(\lambda)$. Since $s(\lambda)$ does not depend on the choice of the inverse image of α, we may denote by $s(\alpha)$ the number $s(\lambda)$ for any $\lambda \in \pi_{n,m}^{-1}(\alpha)$.

Let $K_{n,m}$ be the mapping from $\Lambda_{n,m}$ to C_n determined by

$$K_{n,m}(\lambda_0, \ldots, \lambda_{n-1}) = (0 \cdot \lambda_0 + 1 \cdot \lambda_1 + \cdots + (n-1) \cdot \lambda_{n-1})_n.$$

Then it is clear that $a_k(n, m)$ equals $\#(K_{n,m}^{-1}(k))$, for any $k \in C_n$.

Proposition 1 *For any $r \in C_n$, $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \Lambda_{n,m}$ one has $K_{n,m}(r\lambda) = K_{n,m}(\lambda) + r \cdot (m)_n$.*

Proof: The case $r = 0$ is obvious.

For $r = 1$, one has

$$\begin{aligned} K_{n,m}(1 \cdot (\lambda_0, \ldots, \lambda_{n-1})) &= K_{n,m}(\lambda_{n-1}, \lambda_0, \ldots, \lambda_{n-2}) \\ &= (0 \cdot \lambda_{n-1} + 1 \cdot \lambda_0 + \cdots + (n-1)\lambda_{n-2})_n \\ &= (0 \cdot \lambda_0 + 1 \cdot \lambda_1 + \cdots + (n-2)\lambda_{n-2} + (n-1)\lambda_{n-1} \\ &\quad + (\lambda_0 + \cdots + \lambda_{n-2} - (n-1)\lambda_{n-1}))_n \\ &= K_{n,m}(\lambda) + (\lambda_0 + \cdots + \lambda_{n-2} + \lambda_{n-1} - n\lambda_{n-1})_n \\ &= K_{n,m}(\lambda) + 1 \cdot (m)_n. \end{aligned}$$

For $1 < r < n$, one has

$$\begin{aligned} K_{n,m}(r\lambda) &= K_{n,m}\,(\underbrace{1 \cdot \ldots (1 \cdot}_{r \text{ times}} \lambda) \ldots)) = K_{n,m}(\lambda) + \underbrace{(m)_n + \cdots + (m)_n}_{r \text{ times}} \\ &= K_{n,m}(\lambda) + r \cdot (m)_n \end{aligned}$$

□

Denote by p_d^n the natural projection from C_n to C_d; for $r \in \mathbb{Z}/n\mathbb{Z} = \{0, \ldots, n-1\}$, $p_d^n(r) = (r)_d$. Since d divides n, this is clearly a group homomorphism.

Define now the mapping

$$\tilde{K}_{n,m} : \tilde{\Lambda}_{n,m} \to C_d$$

by assigning to $\alpha \in \tilde{\Lambda}_{n,m}$ the element $p_d^n(K_{n,m}(\lambda)) \in C_d$, where λ is any element of $\pi_{n,m}^{-1}(\alpha)$. Since d divides m, by Proposition 1 the element $p_d^n(K_{n,m}(\lambda))$ does not depend on the choice of λ.

So one obtains a commutative diagram

$$\begin{array}{ccc} \Lambda_{n,m} & \xrightarrow{K_{n,m}} & C_n \\ \pi_{n,m}\big\downarrow & & p_d^n\big\downarrow \\ \tilde{\Lambda}_{n,m} & \xrightarrow{\tilde{K}_{n,m}} & C_d. \end{array} \tag{5}$$

Proposition 2 *Let $\alpha \in \tilde{\Lambda}_{n,m}$ and $\tilde{K}_{n,m}(\alpha) = r \in C_d$. Then for any $r' \in (p_d^n)^{-1}(r)$ there are exactly $d/s(\alpha)$ elements in $\pi_{n,m}^{-1}(\alpha) \subset \Lambda_{n,m}$ which are mapped to r' by $K_{n,m}$.*

Proof: Consider any $\lambda \in \pi_{n,m}^{-1}(\alpha)$. By commutativity of (5), clearly $p_d^n K_{n,m}(\lambda) = r$.

By Proposition 1, one has $K_{n,m}((l \cdot 1)\lambda) = K_{n,m}(\lambda) + l \cdot (m)_n$ for any l.

Those l for which $K_{n,m}((l \cdot 1)\lambda) = K_{n,m}(\lambda)$, are precisely those for which $l \cdot (m)_n = 0$ in $\mathbb{Z}/n\mathbb{Z}$. Since $(n, m) = d$, those l must be divisible by n/d. So the minimal nonzero l with $K_{n,m}((l \cdot 1)\lambda) = K_{n,m}(\lambda)$ is n/d. Hence for any distinct l_1, l_2 from the interval $[0; n/d[$ one has $K_{n,m}((l_1 \cdot 1)\lambda) \neq K_{n,m}((l_2 \cdot 1)\lambda)$.

This implies, firstly, that $t(\lambda)$ is divisible by n/d, i.e., $t(\lambda)d/n = d/s(\lambda)$ is integer, and secondly, since the inverse image of r under p_d^n has n/d elements, that when l runs over the interval $[0; n/d]$, then $K_{n,m}((l \cdot 1)\lambda)$ will become each element of the inverse image of r under p_d^n exactly once. This means that, for each $r' \in (p_d^n)^{-1}(r)$, the number of those $0 \leq l < n$ with $K_{n,m}((l \cdot 1)\lambda) = r'$ equals d.

Hence the number of those λ from $\pi_{n,m}^{-1}(\alpha)$ with $K_{n,m}(\lambda) = r'$ equals $d/s(\alpha)$. □

Consider now the set W_{n+m}^n, whose elements are circular arrangements (“necklaces”) of $n + m$ beads, n of them black and m white. There has to be fixed orientation of the circle, as well as a “basepoint” located somewhere between two adjacent beads.

Let Υ_{n+m}^n be the subset of W_{n+m}^n consisting of those arrangements for which the first bead along the orientation after the basepoint is black.

Let us define an action of C_n on Υ_{n+m}^n as follows: for $\beta \in \Upsilon_{n+m}^n$ and $r \in C_n$, $r\beta$ will be the same arrangement as β but with the basepoint shifted counterorientationwise exactly by the amount needed for the number of passed black beads to become r.

Denote Υ_{n+m}^n/C_n by $\tilde{\Upsilon}_{n+m}^n$; so $\tilde{\Upsilon}_{n+m}^n$ is the set of arrangements as above, without any basepoint, and considered up to rotation.

The natural projection from Υ_{n+m}^n to $\tilde{\Upsilon}_{n+m}^n$ will be denoted by π_{n+m}^n.

For $\beta \in \Upsilon_{n+m}^n$, we denote by $t(\beta)$ the minimal positive integer with $(t(\beta) \cdot 1)\beta = \beta$. Clearly $t(\beta) = t(r\beta)$ for any $\beta \in \Upsilon_{n+m}^n$ and $r \in C_n$. The number $n/t(\beta)$ will be denoted by $s(\beta)$.

For any $\gamma \in \tilde{\Upsilon}_{n+m}^n$, the number $s(\beta)$, for $\beta \in (\pi_{n+m}^n)^{-1}(\gamma)$ does not depend on the choice of β; we will denote this number by $s(\gamma)$.

Let us construct a map $g_{n+m}^n : \Upsilon_{n+m}^n \to C_n$. Take $\beta \in \Upsilon_{n+m}^n$ and suppose that numbers of black beads in β, counted orientationwise from the basepoint, are $1, r_2, \ldots, r_n$ (by definition the first bead is black). Then one defines $g_{n+m}^n(\beta)$ to be the element

$$(1 + r_2 + \cdots + r_n)_n - (1 + 2 + \cdots + n)_n$$

of C_n.

Now construct the map $\tilde{g}_{n+m}^n : \tilde{\Upsilon}_{n+m}^n \to C_d$. Given any $\gamma \in \tilde{\Upsilon}_{n+m}^n$, choose a basepoint on it somewhere between two adjacent beads. Suppose the numbers of the black beads counted orientationwise w. r. t. this basepoint are $r_1, \ldots, r_n$. Consider the number $r_1 + \cdots + r_n$. If one would choose a different basepoint, each of the r_i would change to r_i' in such a way that $(r_i - r'_i)_{n+m}$ would be the same for all i; denote this residue by Δ. Then

$$(r_1 + \cdots + r_n)_{n+m} = (r'_1 + \cdots + r'_n)_{n+m} + n\Delta$$

and $p_d^{n+m}((r_1 + \cdots + r_n)_{n+m}) = p_d^{n+m}((r'_1 + \cdots + r'_n)_{n+m}) + p_d^{n+m}(n\Delta) = p_d^{n+m}((r'_1 + \cdots + r'_n)_{n+m})$.

So we may define

$$\tilde{g}^n_{n+m} = (r_1 + \cdots + r_n)_d - (1 + \cdots + n)_d.$$

It is clear that the diagram

$$\begin{array}{ccc} \Upsilon^n_{n+m} & \xrightarrow{g^n_{n+m}} & C_n \\ \pi^n_{n+m} \downarrow & & p^n_d \downarrow \\ \tilde{\Upsilon}^n_{n+m} & \xrightarrow{\tilde{g}^n_{n+m}} & C_d \end{array}$$

commutes.

Let us now construct a map

$$w : \Lambda_{n,m} \to \Upsilon^n_{n+m}.$$

For $\lambda = (\lambda_0, \dots, \lambda_{n-1}) \in \Lambda_{n,m}$ choose an orientation and a basepoint on a circle; start moving from the basepoint orientationwise and put the first black bead. Then put next λ_{n-1} white beads and the next black one; again put λ_{n-2} white beads and the next black one and so on. On the $(n-1)$-th step, when we will put λ_0 white beads there will be $n+m$ beads arranged—the next one will be the black bead we started with. So one obtains an element of Υ^n_{n+m} which we define to be $w(\lambda_0, \dots, \lambda_{n-1})$.

It is easy to see that w is a bijection compatible with the action of C_n. Moreover one has the following proposition.

Proposition 3 *The diagram*

$$\begin{array}{ccc} \Lambda_{n,m} & \xrightarrow{w} & \Upsilon^n_{n+m} \\ K_{n,m} \searrow & & \swarrow g^n_{n+m} \\ & \mathbb{Z}/n\mathbb{Z} & \end{array}$$

commutes.

Proof: Take $(\lambda_0, \dots, \lambda_{n-1}) \in \Lambda_{n,m}$. Then in $w(\lambda_0, \dots, \lambda_{n-1})$, numbers of black beads counted from the basepoint orientationwise will be

$$1,\ \lambda_{n-1} + 2,\ \lambda_{n-2} + \lambda_{n-1} + 3, \dots, \lambda_1 + \cdots + \lambda_{n-1} + n.$$

Hence $g^n_{n+m}(w(\lambda_0, \ldots, \lambda_{n-1})) =$

$$\begin{aligned}
&= (1 + (\lambda_{n-1} + 2) + (\lambda_{n-2} + \lambda_{n-1} + 3) + \cdots + (\lambda_1 + \cdots + \lambda_{n-1} + n))_n \\
&\quad - (1 + \cdots + n)_n \\
&= ((n-1)\lambda_{n-1} + (n-2)\lambda_{n-2} + \cdots + \lambda_1)_n \\
&= (0 \cdot \lambda_0 + \cdots + (n-1)\lambda_{n-1})_n \\
&= K_{n,m}(\lambda_0, \ldots, \lambda_{n-1}).
\end{aligned}$$

□

Since the bijection w commutes with the action of C_n, it induces a bijection $\tilde{w} : \tilde{\Lambda}_{n,m} \to \tilde{\Upsilon}^n_{n+m}$, and moreover by Proposition 3 there is a commutative diagram

$$\begin{tikzcd}
\Lambda_{n,m} \arrow[rrr, "\pi_{n,m}"] \arrow[dr, "w", "\approx"'] \arrow[dd, "K_{n,m}"'] & & & \tilde{\Lambda}_{n,m} \arrow[dl, "\tilde{w}"', "\approx"] \arrow[dd, "\tilde{K}_{n,m}"] \\
& \Upsilon^n_{n+m} \arrow[r, "\pi^n_{n+m}"] \arrow[dl, "g^n_{n+m}"] & \tilde{\Upsilon}^n_{n+m} \arrow[dr, "\tilde{g}^n_{n+m}"'] & \\
C_n \arrow[rrr, "p^n_d"] & & & C_d
\end{tikzcd} \qquad (*)$$

Proposition 4 *For any $r \in C_n$,*

$$\#\big(K^{-1}_{n,m}(r)\big) = \sum_{\gamma \in (\tilde{g}^n_{n+m})^{-1}(p^n_d(r))} \frac{d}{s(\gamma)}.$$

Proof: This is obvious from Proposition 2 and commutativity of (*). □

Let us now construct the map

$$x : \tilde{\Upsilon}^n_{n+m} \to \tilde{\Upsilon}^m_{n+m}$$

as follows: for $\gamma \in \tilde{\Upsilon}^n_{n+m}$, define the element $x(\gamma) \in \tilde{\Upsilon}^m_{n+m}$ by reversing the orientation and changing black beads by white ones and vice versa.

Proposition 5 *The diagram*

$$\begin{array}{ccc} \tilde{\Upsilon}^n_{n+m} & \xrightarrow{x} & \tilde{\Upsilon}^m_{n+m} \\ {\scriptstyle \tilde{g}^n_{n+m}} \searrow & & \swarrow {\scriptstyle \tilde{g}^m_{n+m}} \\ & \mathbb{Z}/d\mathbb{Z} & \end{array}$$

commutes.

Proof: Take any $\gamma \in \tilde{\Upsilon}^n_{n+m}$ and choose in it a basepoint between some adjacent beads. Choose the same basepoint in $x(\gamma)$. Suppose that in γ the numbers of black beads are $b_1, \ldots, b_n$ and those of white ones are $c_1, \ldots, c_m$. Since each bead is either white or black,

$$b_1 + \cdots + b_n + c_1 + \cdots + c_m = 1 + \cdots + (n+m),$$

i.e.,

$$c_1 + \cdots + c_m = 1 + \cdots + (n+m) - (b_1 + \cdots + b_n). \tag{6}$$

Numbers of black beads in $x(\gamma)$ w.r.t. the new orientation will be $n+m+1-c_1, \ldots, n+m+1-c_m$. Hence

$$\begin{aligned}
\tilde{g}^m_{n+m}(x(\gamma)) &= (n+m+1-c_1+\cdots+n+m+1-c_m)_d - (1+\cdots+m)_d \\
&= (m(n+m+1) - c_1 - \cdots - c_m)_d - (1+\cdots+m)_d \\
&= -(c_1+\cdots+c_m)_d - (1+\cdots+m)_d && (d \mid m) \\
&= (b_1+\cdots+b_n)_d - (1+\cdots+(n+m))_d - (1+\cdots+m)_d && \text{(by (6))} \\
&= (b_1+\cdots+b_n)_d - \left(\frac{(n+m)(n+m+1)}{2} + \frac{m(m+1)}{2}\right)_d \\
&= (b_1+\cdots+b_n)_d - \left(m(n+m+1) + \frac{n(n+1)}{2}\right)_d \\
&= (b_1+\cdots+b_n)_d - \left(\frac{n(n+1)}{2}\right)_d && (d \mid m) \\
&= (b_1+\cdots+b_n)_d - (1+\cdots+n)_d \\
&= \tilde{g}^n_{n+m}(\gamma). && \square
\end{aligned}$$

We have reached the goal of this section.

Proof of $\boldsymbol{a_k(n,m) = a_k(m,n)}$: Take any $n' \in C_n$ and $m' \in C_m$ with $(n')_d = (m')_d = k$. By Proposition 4,

$$\#\big(K^{-1}_{n,m}(n')\big) = \sum_{\substack{\gamma \in \tilde{\Upsilon}^n_{n+m} \\ \tilde{g}^n_{n+m}(\gamma) = d'}} \frac{d}{s(\gamma)}.$$

Hence using the isomorphism $x : \tilde{\Upsilon}^n_{n+m} \to \tilde{\Upsilon}^m_{n+m}$ one obtains

$$\begin{aligned}\#\big(K^{-1}_{n,m}(n')\big) &= \sum_{\substack{\gamma\,\in\,\tilde{\Upsilon}^n_{n+m}\\ \tilde{g}^n_{n+m}(\gamma)=d'}} \frac{d}{s(\gamma)} = \sum_{\substack{\gamma\,\in\,\tilde{\Upsilon}^n_{n+m}\\ \tilde{g}^n_{n+m}(\gamma)=d'}} \frac{d}{s(x(\gamma))}\\ &= \sum_{\substack{\gamma\,\in\,\tilde{\Upsilon}^n_{n+m}\\ \tilde{g}^m_{n+m}(x(\gamma))=d'}} \frac{d}{s(x(\gamma))} = \sum_{\substack{\delta\,\in\,\tilde{\Upsilon}^m_{n+m}\\ \tilde{g}^m_{n+m}(\delta)=d'}} \frac{d}{s(\delta)} = \#\big(K^{-1}_{m,n}(m')\big).\end{aligned}$$

It follows that

$$a_k(n,m) = \#\big(K^{-1}_{n,m}(n')\big) = \#\big(K^{-1}_{m,n}(m')\big) = a_k(m,n). \qquad \square$$

3. Another proof of the formula

Our next task will be to obtain another derivation of the formula 4,

$$a_k(n,m) = \frac{1}{n+m}\sum_{d'|d} c_{d'}(k)\frac{\frac{n+m}{d'}!}{\frac{n}{d'}!\,\frac{m}{d'}!},$$

of a more combinatorial nature.

Consider the action of C_{n+m} on the set W^n_{n+m}, under which $r \in C_{n+m}$ acts by shifting the basepoint by r beads counterorientationwise. Let $W^n_{n+m}(k)$, for $k \in C_d$, be the subset of W^n_{n+m} consisting of those elements with

$$\left(\sum_{i=1}^n r_i - \sum_{i=1}^n i\right)_d = k,$$

where $r_1, \ldots, r_n$ are numbers of black beads counted orientationwise from the basepoint.

Proposition 6 *The subset $W^n_{n+m}(k) \subset W^n_{n+m}$ is invariant under the action of C_{n+m} on W^n_{n+m}.*

Proof: Take $r \in C_{n+m}$, and let γ be an element of $W^n_{n+m}(k)$, numbers of black beads of γ being $r_1, \ldots, r_n$. Then the numbers of black beads in $r\gamma \in W^n_{n+m}$ will be $r - \varepsilon_1(n+m) + r_1$, $r - \varepsilon_2(n+m) + r_2, \ldots, r - \varepsilon_n(n+m) + r_n$, where each ε_i is 0 or 1 depending on whether $r_i + r < n+m$ or not (in other words, numbers of those beads not passed by the basepoint will grow by r, while of those passed—by $r - (n+m)$). Hence the sum of numbers of black beads will become

$$\sum_{i=1}^n r - \varepsilon_i(n+m) + r_i = nr - (n+m)\sum_{i=1}^n \varepsilon_i + \sum_{i=1}^n r_i.$$

So since both n and $n+m$ are divisible by d, the sum of numbers of black beads will remain the same modulo d. $\square$

Denote $W^n_{n+m}(k)/C_{n+m}$ by $\tilde{W}^n_{n+m}(k)$. Clearly $\tilde{W}^n_{n+m}(k) = \{\gamma \in \tilde{\Upsilon}^n_{n+m} \mid \tilde{g}^n_{n+m}(\gamma) = k\}$. The inverse image in $W^n_{n+m}(k)$ of each $\gamma \in \tilde{W}^n_{n+m}(k)$ has $\frac{n+m}{s(\gamma)}$ elements, since the pattern of the arrangement γ is periodic with period $\frac{n+m}{s(\gamma)}$ and there are $\frac{n+m}{s(\gamma)}$ possibilities to choose the basepoint. Hence

$$\#\big(\tilde{W}^n_{n+m}(k)\big) = \sum_{\substack{\gamma\,\in\,\tilde{\Upsilon}^n_{n+m}\\ \tilde{g}^m_{n+m}(\gamma)=k}} \frac{n+m}{s(\gamma)}.$$

Comparing this with the formula from Proposition 4,

$$\#\big(K^{-1}_{n,m}(n')\big) = \sum_{\substack{\gamma\,\in\,\tilde{\Upsilon}^n_{n+m}\\ \tilde{g}^n_{n+m}(\gamma)=k}} \frac{d}{s(\gamma)},$$

one concludes

$$\#\big(K^{-1}_{n,m}(n')\big) = \frac{d}{n+m}\#\big(\tilde{W}^n_{n+m}(k)\big). \tag{7}$$

Let us calculate $\#(W^n_{n+m}(k))$.

Proposition 7

$$\#\big(W^n_{n+m}(k)\big) = \frac{1}{d}\sum_{d'|d} c_{d'}(k)\begin{pmatrix}\frac{n+m}{d'}\\ \frac{n}{d'}\end{pmatrix}.$$

Proof: Consider the polynomial

$$P(x,t) = x^{\frac{n(n+1)}{2}-k}(t+x)(t+x^2)\cdots(t+x^{n+m}).$$

After expanding $P(x,t)$ and collecting the terms, the coefficient at the monomial $t^m x^l$ will be the number of representations of $t^m x^l = x^{\frac{n(n+1)}{2}-k} t^m x^{r_1} x^{r_2}\cdots x^{r_n}$, where $0 < r_1 < \cdots < r_n \le n+m$ with

$$\frac{n(n+1)}{2} - k + \sum_j r_j = l.$$

Rewrite the last equality as

$$\sum_j r_j - \frac{n(n+1)}{2} - k = l - n(n+1);$$

since d divides $n(n+1)$, it is clear that the sum of these coefficients at $t^m x^l$ over all l with $d \mid l$ will equal the number of those sequences $0 < r_1 < \cdots < r_n \le n+m$ with

$$\left(\sum_j r_j - \frac{n(n+1)}{2}\right)_d = k.$$

Assigning to such a sequence an arrangement from W^n_{n+m} with $r_1, \ldots, r_n$ as numbers of black beads, shows that the sum of these coefficients equals $\#(W^n_{n+m}(k))$.

Let $\zeta = e^{\frac{2\pi i}{d}}$ (or any other primitive dth root of 1). Since for any integer l one has

$$\sum_{j=1}^{d} \zeta^{lj} = \begin{cases} d & \text{if } d \mid l \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $\#(W^n_{n+m}(k))$ equals the coefficient at t^m of the polynomial

$$P(t) = \frac{1}{d} \sum_{j=1}^{d} P(\zeta^j, t).$$

Consider now the polynomials $P(\zeta^j, t)$ separately.

Since $d \mid n$, one has

$$\zeta^{j\frac{n(n+1)}{2}} = \begin{cases} 1 & \text{if } d \text{ is odd or both } d \text{ and } \frac{nj}{d} \text{ are even,} \\ -1 & \text{if } d \text{ is even and } \frac{nj}{d} \text{ is odd,} \end{cases}$$

i.e., $\zeta^{j\frac{n(n+1)}{2}} = (-1)^{\frac{(d-1)nj}{d}}$. Hence

$$P(\zeta^j, t) = (-1)^{\frac{(d-1)nj}{d}} \zeta^{-jk} (t + \zeta^j)(t + \zeta^{2j}) \cdots (t + \zeta^{(n+m)j}).$$

Denoting $-t$ by s, one obtains

$$\begin{aligned} P(\zeta^j, t) &= (-1)^{\frac{(d-1)nj}{d}} \zeta^{-jk} (-1)^{n+m} \Big((s - \zeta^j) \cdots \big(s - \zeta^{\frac{d}{(d,j)} j}\big)\Big)^{\frac{(n+m)(d,j)}{d}} \\ &= (-1)^{\frac{(d-1)nj}{d} + n + m} \zeta^{-jk} \big(s^{\frac{d}{(d,j)}} - 1\big)^{\frac{(n+m)(d,j)}{d}}. \end{aligned}$$

Now collect together the terms of $P(t) = \frac{1}{d} \sum P(\zeta^j, t)$ whose j’s have the same gcd with d. One obtains

$$P(t) = \frac{(-1)^{n+m}}{d} \sum_{d' \mid d} \big(s^{\frac{d}{d'}} - 1\big)^{\frac{(n+m)d'}{d}} \cdot \sum_{(j,d)=d'} (-1)^{\frac{(d-1)nj}{d}} \zeta^{-jk}.$$

The sign term in the last sum is

$$\begin{aligned}
(-1)^{\frac{(d-1)nj}{d}} &= \begin{cases} -1 & \text{if } d \text{ is even and both } \frac{n}{d} \text{ and } j \text{ are odd,} \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} -1 & \text{if } d \text{ is even and both } \frac{n}{d} \text{ and } d' = (j,d) \text{ are odd,} \\ 1 & \text{otherwise,} \end{cases} \\
&= (-1)^{(d-1)\frac{n}{d/d'}}.
\end{aligned}$$

Hence

$$\begin{aligned}
P(t) &= \frac{(-1)^{n+m}}{d} \sum_{d'|d} (-1)^{\frac{(d-1)n}{d'}} (s^{d'}-1)^{\frac{(n+m)}{d'}} \cdot \sum_{(j,d)=\frac{d}{d'}} \zeta^{-jk} \\
&= \frac{(-1)^{n+m}}{d} \sum_{d'|d} (-1)^{\frac{(d-1)n}{d'}} (s^{d'}-1)^{\frac{(n+m)}{d'}} c_{d'}(-k) \\
&= \frac{(-1)^{n+m}}{d} \sum_{d'|d} (-1)^{\frac{(d-1)n}{d'}} c_{d'}(-k)((-1)^{d'} t^{d'} - 1)^{\frac{(n+m)}{d'}}.
\end{aligned}$$

So the coefficient at t^m in $P(t)$ will be

$$\begin{aligned}
\#\big(W^n_{n+m}(k)\big) &= \frac{(-1)^{n+m}}{d} \sum_{d'|d} (-1)^{\frac{(d-1)n}{d'}} c_{d'}(-k)(-1)^{d'\frac{m}{d'}} \cdot (-1)^{\frac{n}{d'}} \binom{\frac{n+m}{d'}}{\frac{n}{d'}} \\
&= \frac{(-1)^n}{d} \sum_{d'|d} (-1)^{\frac{dn}{d'}} c_{d'}(-k) \binom{\frac{n+m}{d'}}{\frac{n}{d'}}.
\end{aligned}$$

Since $\frac{d}{d'} \mid n$, the sign term in that last sum is $(-1)^{\frac{dn}{d'}} = (-1)^n$. Hence

$$\#\big(W^n_{n+m}(k)\big) = \frac{1}{d} \sum_{d'|d} c_{d'}(k) \binom{\frac{n+m}{d'}}{\frac{n}{d'}}$$

(we have used the evident equality $c_{d'}(-k) = c_{d'}(k)$). □

Remark A similar argument for deriving $a_0(n,m)$ has been communicated to us by Alon and Stanley: for the role played by our $P(x,t)$, they use the function $(1-tx)^{-1}(1-tx^2)^{-1}\cdots(1-tx^{n-1})^{-1}$.

We have reached the desired formula: indeed, from (7) and $a_k(n,m) = \#(K_{n,m}^{-1}(k))$ one immediately obtains

$$a_k(n,m) = \frac{1}{n+m}\sum_{d'|d} c_{d'}(k)\begin{pmatrix}\frac{n+m}{d'}\\ \frac{n}{d'}\end{pmatrix}.$$

4. Some Final Remarks

Generating functions for $a_0(n,m)$ mentioned in [3] can be easily generalized: for any integer k one has

$$\sum_{n,m=0}^{\infty} a_k(n,m)x^n y^m = 1 - \sum_{d=1}^{\infty}\frac{c_d(k)}{d}\log(1 - x^d - y^d)$$

and

$$\zeta_k(s+t)\sum_{n,m=1}^{\infty} a_k(n,m)n^{-s}m^{-t} = \zeta(s+t+1)\sum_{n,m=1}^{\infty}\frac{(n+m-1)!}{n!\ m!}n^{-s}m^{-t},$$

where $\zeta_k(x) = \sum_{d|k} d^{-x}$ (in particular $\zeta_0(x) = \zeta(x)$ is the Riemann zeta function).

Let us note also a relationship of $a_k(n,m)$ to free Lie algebras. Consider a free Lie algebra $\mathsf{Lie}(x,y)$ on two generators x, y over a characteristic zero field. Then Th. 2(b) in [2], Ch. II, Section 3, no. 3 immediately implies that for any n, m,

$$a_1(n,m) = \dim(\mathsf{Lie}(x,y)_{n,m}),$$

where $(\ )_{n,m}$ denotes the homogeneous component of bidegree n in x and m in y. We could not find an explicit correspondence between generators of the above Lie algebra and our “necklace” interpretation of $a_1(n,m)$. This connection looks even more interesting in view of the fact that in a sense, all the a_k may be reduced to a_1.

Proposition 8 *For any n, m, and k,*

$$a_k(n,m) = \sum_{d'|(n,m,k)} a_1(n/d', m/d').$$

Proof: One calculates directly:

$$\sum_{d'|(n,m,k)} a_1(n/d', m/d') = \sum_{d'|(n,m,k)}\frac{1}{n/d' + m/d'}\sum_{d''|(n/d',m/d')}\mu(d'')\begin{pmatrix}\frac{n+m}{d'd''}\\ \frac{n}{d'd''}\end{pmatrix}$$
$$= \sum_{d|(n,m)}\frac{1}{n+m}\sum_{d'|(d,k)} d'\mu(d/d')\begin{pmatrix}\frac{n+m}{d}\\ \frac{n}{d}\end{pmatrix}$$

$$= \frac{1}{n+m} \sum_{d|(n,m)} c_d(k) \binom{\frac{n+m}{d}}{\frac{n}{d}}$$
$$= a_k(n, m). \qquad \square$$

We do not know a combinatorial explanation of this fact, either.

References

1. G.E. Andrews, "The theory of partitions," *Encyclopedia of Mathematics and Its Applications*, G.-C. Rota, (Ed.), Addison-Wesley, Reading, MA, 1976, Vol. 2.
2. N. Bourbaki, *Groupes et algébres de Lie*, Hermann, Paris, 1971–1972.
3. A. Elashvili and M. Jibladze, "Hermite Reciprocity for regular representations of cyclic groups," *Indag Mathem.*, N.S., **9**(2), 1998, 233–238.
4. *Higher Transcendental Functions*, Bateman Manuscript Project, A. Erdélyi (dir.), McGraw-Hill, New York, 1955.
5. S. Ramanujan, "On certain trigonometrical sums and their applications in the theory of numbers," *Trans. Camb. Phil. Soc.*, **XXII**(13) (1918), 259–276
6. T. A. Springer, Invariant theory, *Lecture Notes in Math.*, Vol. 585, Springer-Verlag, Berlin and New York, 1977.
7. R. Stanley, *Enumerative Combinatorics*, Vol. I, Wadsworth & Brooks/Cole, Monterey, CA, 1986.