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Extended Linial Hyperplane Arrangements for Root Systems and a Conjecture of Postnikov and Stanley

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Abstract. A hyperplane arrangement is said to satisfy the "Riemann hypothesis" if all roots of its characteristic polynomial have the same real part. This property was conjectured by Postnikov and Stanley for certain families of arrangements which are defined for any irreducible root system and was proved for the root system A_{n-1}. The proof is based on an explicit formula [1, 2, 11] for the characteristic polynomial, which is of independent combinatorial significance. Here our previous derivation of this formula is simplified and extended to similar formulae for all but the exceptional root systems. The conjecture follows in these cases.

Keywords: hyperplane arrangement, characteristic polynomial, root system

1. Introduction

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$, i.e. a finite collection of affine subspaces of $\mathbb{R}^n$ of codimension one. The *characteristic polynomial* [9, §2.3] of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x)\, q^{\dim x},$$

where $L_{\mathcal{A}}$ is the poset of all affine subspaces of $\mathbb{R}^n$ which can be written as intersections of some of the hyperplanes of $\mathcal{A}$, $\hat{0} = \mathbb{R}^n$ is the unique minimal element of $L_{\mathcal{A}}$ and μ stands for its Möbius function [14, §3.7]. The polynomial $\chi(\mathcal{A}, q)$ is a fundamental combinatorial and topological invariant of $\mathcal{A}$ and plays a significant role throughout the theory of hyperplane arrangements [9].

Very often the polynomial $\chi(\mathcal{A}, q)$ factors completely over the nonnegative integers. This happens, for instance, when $\mathcal{A}$ is a *Coxeter arrangement*, i.e. the arrangement of reflecting hyperplanes of a finite Coxeter group [9, p. 3]. A number of theories [8, 13, 16] have been developed to explain this phenomenon (see also the survey [12]). A different phenomenon has appeared in recent work of Postnikov and Stanley [11] and is referred to as the "Riemann hypothesis" for $\mathcal{A}$. It asserts that all roots of $\chi(\mathcal{A}, q)$ have the same real part. This property was conjectured in [11] for certain affine *deformations* of Coxeter arrangements and was proved for the Coxeter type A_{n-1}. The proof was based on explicit

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formulae for the characteristic polynomials, first obtained in [1] [2, Part II]. In this paper we improve and extend our previous arguments to treat case by case all but the exceptional Coxeter types. There is no general theory known that could give a more uniform proof.

We first state precisely the Conjecture of Postnikov and Stanley and our main result.

The main result. A *root system* Φ will be a crystallographic root system [7, §2.9] which is not necessarily reduced, i.e. if $\alpha, \beta \in \Phi$ with $\beta = c\alpha$ then we do not require that $c = \pm 1$. This includes the non-reduced system BC_n which is the union of B_n and C_n. Let $\mathcal{A}$ be the Coxeter arrangement corresponding to Φ. A deformation of $\mathcal{A}$ [11, 15] is an arrangement each of whose hyperplanes is parallel to some hyperplane of $\mathcal{A}$. Fix a system of positive roots Φ^+ and let $a \leq b$ be integers. We denote by $\hat{\mathcal{A}}^{[a,b]}(\Phi)$ the deformation of $\mathcal{A}$ which has hyperplanes

$$(\alpha, x) = k \quad \text{for } \alpha \in \Phi^+ \quad \text{and} \quad k = a, a+1, \ldots, b.$$

This reduces to $\mathcal{A}$ if $a = b = 0$. The conjecture of Postnikov and Stanley from [11, §9] is as follows.

Conjecture 1.1 *Let Φ be an irreducible root system in $\mathbb{R}^l$ and a, b be nonnegative integers, not both zero, satisfying $a \leq b$. If $h_{\hat{\mathcal{A}}}$ is the number of hyperplanes of $\hat{\mathcal{A}} = \hat{\mathcal{A}}^{[-a+1,b]}(\Phi)$ then all roots of $\chi(\hat{\mathcal{A}}, q)$ have real part equal to $h_{\hat{\mathcal{A}}}/l$.*

Note. For any arrangement $\mathcal{A}$ in $\mathbb{R}^l$, the sum of the roots of $\chi(\mathcal{A}, q)$ is equal to the number $h_{\mathcal{A}}$ of hyperplanes of $\mathcal{A}$. Hence, if all roots of $\chi(\mathcal{A}, q)$ have the same real part, this has to be $h_{\mathcal{A}}/l$.

The characteristic polynomial of $\hat{\mathcal{A}}^{[a,b]}(\Phi)$ is independent of the choice of positive roots Φ^+, so from now and on we assume that this set is as in [7, §2.10]. We then abbreviate $\hat{\mathcal{A}}^{[a,b]}(\Phi)$ as $\hat{\mathcal{B}}_n^{[a,b]}$, $\hat{\mathcal{C}}_n^{[a,b]}$, $\hat{\mathcal{D}}_n^{[a,b]}$ or $\hat{\mathcal{BC}}_n^{[a,b]}$ if $\Phi = B_n, C_n, D_n$ or BC_n respectively. For $\Phi = A_{n-1}$, this is an arrangement in $\mathbb{R}^{n-1}$. For convenience, we denote by $\hat{\mathcal{A}}_n^{[a,b]}$ the arrangement of hyperplanes in $\mathbb{R}^n$ of the form

$$x_i - x_j = a, a+1, \ldots, b \quad \text{for} \quad 1 \leq i < j \leq n,$$

so that $\hat{\mathcal{A}}_n^{[a,b]}$ is the product [9, Definition 2.13] of the empty one dimensional arrangement and $\hat{\mathcal{A}}^{[a,b]}(\Phi)$ and hence $\chi(\hat{\mathcal{A}}_n^{[a,b]}, q) = q\chi(\hat{\mathcal{A}}^{[a,b]}(\Phi), q)$, where $\Phi = A_{n-1}$. The arrangements $\hat{\mathcal{A}}_n^{[1,b]}$ are referred to as the *extended Linial arrangements*. They were studied enumeratively because of a remarkable conjecture of Linial and Stanley, first proved by Postnikov [11, Thm. 8.2] (also in [1, §4] [2, §6.4]), about the number of regions of the *Linial arrangement*, the one which corresponds to $b = 1$; see Remark 1 in Section 6. The polynomials $\chi(\hat{\mathcal{A}}_n^{[1,b]}, q)$ were first computed explicitly in [1, §4] [2, §6.4] with the "finite field method". We use the same method to find similar explicit formulae in the case of the other classical root systems and prove the following theorem.

Theorem 1.2 *Conjecture* 1.1 *holds for the infinite families of root systems A_{n-1}, B_n, C_n, D_n and BC_n, where $n \geq 2$.*

As remarked earlier, the proof of Theorem 1.2 will be done case by case. No uniform proof is known.

The paper is organized as follows: Section 2 contains a review and refinement of the finite field method of [1] [2, Part II] and other useful background. In Section 3 we simplify substantially the derivations of the formulae for $\chi(\hat{\mathcal{A}}_n^{[1,b]}, q)$ and $\chi(\hat{\mathcal{A}}_n^{[0,b]}, q)$ given in [1, §4] [2, §6.4]. In particular, we get a simple proof of Postnikov's theorem for the number of regions of the Linial arrangement. In Section 4 we obtain similar formulae for the root systems B_n, C_n, D_n and BC_n. In Section 5 we use the results of Sections 3 and 4 and an elementary lemma, employed by Postnikov and Stanley, to complete the proof of Theorem 1.2. We conclude with some remarks in Section 6.

2. Background

We first review the finite field method of [1] [2, Part II]. This method reduces the computation of the characteristic polynomial to a simple counting problem in a vector space over a finite field. It will be more convenient here to work over the abelian group $\mathbb{Z}_q$ of integers modulo q, where q is not necessarily a power of a prime. We will naturally restrict our attention to hyperplane arrangements, as opposed to the more general *subspace* arrangements [3, 4].

Let $\mathcal{A}$ be any hyperplane arrangement in $\mathbb{R}^n$ and q be a positive integer. We call $\mathcal{A}$ a $\mathbb{Z}$*-arrangement* if its hyperplanes are given by equations with integer coefficients. Such equations define subsets of the finite set $\mathbb{Z}_q^n$ if we reduce their coefficients modulo q. We denote by $V_{\mathcal{A}}$ the union of these subsets, supressing q from the notation. The next theorem is a variation of [1, Thm. 2.2] [2, Thm. 5.2.1] (see also the original formulation in [6, §16] as well as [9, Thm. 2.69], [5, Thm. 2.1] and Proposition 3.2 and Lemma 5.1 in [4]).

Theorem 2.1 *Let $\mathcal{A}$ be a $\mathbb{Z}$-hyperplane arrangement in $\mathbb{R}^n$. There exist positive integers m, k which depend only on $\mathcal{A}$, such that for all q relatively prime to m with $q > k$,*

$$\chi(\mathcal{A}, q) = \#\big(\mathbb{Z}_q^n - V_{\mathcal{A}}\big).$$

Proof: Let $H_1, H_2, \ldots, H_r$ be some of the hyperplanes of $\mathcal{A}$, $X \subseteq \mathbb{R}^n$ be their intersection and X_q be the intersection of the corresponding subsets of $\mathbb{Z}_q^n$. It suffices to guarantee that $\#X_q = q^{\dim X}$ if X is nonempty and $X_q = \emptyset$ otherwise, for any such choice of hyperplanes. The result then follows by Möbius inversion as in [1, 2, 5, 6] or, equivalently, by the argument given in Propositions 3.1 and 3.2 of [4]. Let X be described by the linear system

$$Ax = b, \tag{1}$$

where A is an r by n $\mathbb{Z}$-matrix and b has integer entries. Since there are invertible $\mathbb{Z}$-matrices P, Q such that $P^{-1}AQ$ is diagonal, we can assume that (1) consists of the equations $d_i x_i = b_i$ for $1 \le i \le r$. It suffices to choose m, k so that $d_i \mid m$ whenever $d_i \neq 0$ and $k > |b_i|$ whenever $d_i = 0$. □

Remark We can choose m to be 1 or 2 if $\mathcal{A}$ is a $\mathbb{Z}$-deformation of $\mathcal{A}_n$ or $\mathcal{BC}_n$, respectively, i.e. if $\mathcal{A}$ is contained in some $\hat{\mathcal{A}}_n^{[a,b]}$ or $\hat{\mathcal{BC}}_n^{[a,b]}$ for integers $a \le b$. We will make use of this fact in the following sections without further comment. Also, we can choose $k = 0$ if $\mathcal{A}$ is central.

Notation. We often write

$$\phi_a(y) := 1 + y + y^2 + \cdots + y^{a-1}.$$

This polynomial will appear repeatedly in the formulae of Sections 3 and 4. The *shift operator* S acts on polynomials f of one variable by

$$Sf(y) := f(y-1).$$

The following elementary lemma will be needed in the next sections.

Lemma 2.2 *For fixed positive integers a, n let*

$$(\phi_a(y))^n := (1 + y + y^2 + \cdots + y^{a-1})^n = \sum_{k=0}^{n(a-1)} c_k y^k.$$

If $0 \le i \le a-1$ and f is a polynomial of degree less than n, then the sum

$$\Sigma_i f(y) := \left(\sum_{k \equiv i \pmod{a}} c_k S^k \right) f(y) = \sum_{k \equiv i \pmod{a}} c_k f(y-k)$$

is independent of i and hence

$$\Sigma_i f(y) = \frac{1}{a} (\phi_a(S))^n f.$$

Proof: By linearity, it suffices to prove the result for $f(y) = y^j$, where $0 \le j \le n-1$. We fix such a j and r with $0 \le r \le j$. The coefficient of y^{j-r} in $\Sigma_i y^j$ is $(-1)^r \binom{j}{r} s_i$, where

$$s_i = \sum_{k \equiv i \pmod{a}} c_k k^r.$$

Therefore, it suffices to show that $s_0 = s_1 = \cdots = s_{a-1}$. Note that

$$\sum_{k=0}^{n(a-1)} c_k k^r y^k = \left(y \frac{d}{dy} \right)^r (\phi_a(y))^n$$

is divisible by $\phi_a(y)$. Thus, setting $y = \omega$, a primitive ath root of unity, we get $s_0 + s_1\omega + s_2\omega^2 + \cdots + s_{a-1}\omega^{a-1} = 0$. The same is true if ω is replaced with ω^m for $m = 2, \ldots, a-1$. Hence the column vector $(s_0, s_1, \ldots, s_{a-1})^t$ is in the kernel of the $a-1$ by a matrix Ω whose entry in position (m, l) is equal to $\omega^{m(l-1)}$. The first $a-1$ columns of the matrix Ω are linearly independent, so it has rank $a-1$ and a one dimensional kernel. The kernel is clearly spanned by the column vector with all entries equal to 1 so indeed, $s_0 = s_1 = \cdots = s_{a-1}$. □

3. The root system A_{n-1}

In this section we consider the case of the root system A_{n-1}. We rederive the formulae for $\chi(\hat{\mathcal{A}}_n^{[0,a]}, q)$ and $\chi(\hat{\mathcal{A}}_n^{[1,a]}, q)$ along the lines of [1, §4] [2, §6.4] but use a simpler and more direct combinatorial argument. This case will serve as a prototypical example of application of the finite field method, which we will adjust in the next section to the case of other root systems.

For the following proof, we represent an n-tuple $x = (x_1, x_2, \ldots, x_n)$ of distinct elements of $\mathbb{Z}_q$ as a placement of the integers $1, 2, \ldots, n$ and $q-n$ indistinguishable balls along a line. Such a placement corresponds to the n-tuple x for which $x_i + 1$ is the position that i occupies, counting from the left. For example, for $q = 10$ and $n = 4$, the placement

$$4 \bigcirc \bigcirc \bigcirc 2\ 3 \bigcirc \bigcirc 1 \bigcirc \tag{2}$$

corresponds to the 4-tuple (8, 4, 5, 0) of elements of $\mathbb{Z}_{10}$. We denote by $[y^k]\,F(y)$ the coefficient of y^k in the formal power series $F(y)$.

Proposition 3.1 ([1, Thm. 4.4] [2, Thm. 6.4.4]) *For all $a \geq 1$ and $q > an$ we have*

$$\chi\big(\hat{\mathcal{A}}_n^{[0,a]}, q\big) = q\,[y^{q-n}](1 + y + y^2 + \cdots + y^{a-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{aj}. \tag{3}$$

Proof: Theorem 2.1 implies that, for large positive integers q, $\chi(\hat{\mathcal{A}}_n^{[0,a]}, q)$ counts the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_q^n$ which satisfy

$$x_i - x_j \neq 0, 1, \ldots, a$$

in $\mathbb{Z}_q$ for all $1 \leq i < j \leq n$. Since x satisfies these conditions if and only if $x + m := (x_1 + m, \ldots, x_n + m)$ does so, we can assume that, say, $x_n = 0$ and disregard the factor of q in the right hand side of (3).

The corresponding placements of $1, 2, \ldots, n$ and $q-n$ balls, henceforth called *valid*, are the ones in which:

(i) n occupies the first position from the left and
(ii) at least a balls separate an integer k from the leftmost integer i to the right of k, if $k > i$.

For example, the placement (2) is valid if $a \leq 2$. If a maximal string of consecutive balls has p elements, we write $p = sa + r$ with $0 \leq r < a$ and think of the string as s blocks consisting of a balls each, simply refered to as *a-blocks*, followed by r balls. If $a = 2$ then (2) has two a-blocks.

To construct the valid placements, let j be the number of a-blocks. Place j such blocks along a line and the integer n first from the left, to guarantee (i). Insert $1, 2, \ldots, n-1$ in the j spaces between the blocks and to the right of the last one, listing the integers within each space in increasing order to guarantee (ii). This can be done in j^{n-1} ways. Finally, place the remaining $q - n - aj$ balls in the n possible spaces between the integers and to the right of the last one, with at most $a - 1$ in each space. The total number of ways is the coefficient of y^{q-n} in (3). This is clearly a polynomial in q for $q > an$, hence (3) holds specifically for all $q > an$. $\square$

If $a = 2$, the three-step procedure just described to construct (2) is the following:

4 ◯ ◯ ◯ ◯
4 ◯ ◯ 2 3 ◯ ◯ 1
4 ◯ ◯ ◯ 2 3 ◯ ◯ 1 ◯.

A simple application of Lemma 2.2 yields the more explicit formula for $\chi(\hat{\mathcal{A}}_n^{[0,a]}, q)$, given in [11, Thm. 9.7].

Corollary 3.2 *For all $a \geq 1$,*

$$\chi\big(\hat{\mathcal{A}}_n^{[0,a]}, q\big) = \frac{q}{a^n} S^n (1 + S + S^2 + \cdots + S^{a-1})^n \, q^{n-1}.$$

Proof: Formula (3) can be written in the form

$$\chi\big(\hat{\mathcal{A}}_n^{[0,a]}, q\big) = \frac{q}{a^{n-1}} \sum_{k \equiv q-n (\operatorname{mod} a)} c_k (q - n - k)^{n-1},$$

where the coefficients c_k are as in Lemma 2.2. This lemma implies the proposed equality for $q > an$. Since both hand sides are polynomials in q, the equality follows for all q. $\square$

A similar formula follows for $\chi(\hat{\mathcal{A}}_n^{[1,a]}, q)$. For convenience, as in [1, 2], we use the notation $\tilde{\chi}(\mathcal{A}, q) := \frac{1}{q} \, \chi(\mathcal{A}, q)$.

Proposition 3.3 ([1, Thm. 4.3] [2, Thm. 6.4.3]) *For all $a \geq 1$,*

$$\tilde{\chi}\big(\hat{\mathcal{A}}_n^{[0,a]}, q\big) = \tilde{\chi}\big(\hat{\mathcal{A}}_n^{[1,a-1]}, q - n\big).$$

Proof: For q large, the quantity on the right counts the n-tuples $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{q-n}^n$ which satisfy $x_i - x_j \neq 1, \ldots, a-1$ for all $1 \leq i < j \leq n$ and, say, $x_n = 0$. These n-tuples can be modeled again by placements of length $q - n$ of the integers $1, 2, \ldots, n$ and balls, in which more than one integer can occupy the same position since some of the x_i may be equal.

To define an explicit bijection with the valid placements of Proposition 3.1, we start with a valid placement and remove a ball between any two consecutive integers, including the pair formed by the rightmost integer in the placement and n, which is the leftmost. If no ball lies between such a pair (i, j) then we place i and j in the same position. For example, the placement (2) becomes

$$4 \bigcirc \bigcirc \widehat{2\ 3} \bigcirc 1$$

and corresponds to the 4-tuple $(5, 3, 3, 0) \in \mathbb{Z}_6^4$. This map is clearly a bijection between the two kinds of placements. □

Corollary 3.4 ([11, Thm. 9.7]) *For all $a \geq 1$,*

$$\chi\big(\hat{\mathcal{A}}_n^{[1,a]}, q\big) = \frac{q}{(a+1)^n}(1 + S + S^2 + \cdots + S^a)^n\, q^{n-1}.$$

The special case $a = 1$ of this corollary leads to another proof of Postnikov's theorem [11, 15], initially conjectured by Linial and Stanley. We give more details in Remark 1 of Section 6.

4. Other root systems

In this section we derive analogues of Corollaries 3.2 and 3.4 for the root systems B_n, C_n, D_n and BC_n. The method we use follows closely that of Section 3.

We need to adjust some of the terminology and reasoning of the previous section. Let q be an odd positive integer. If $x = (x_1, x_2, \ldots, x_n)$ is an n-tuple of elements of $\mathbb{Z}_q$ satisfying $x_i \neq 0$ for all i and $x_i \neq \pm x_j$ for $i \neq j$, then we represent x as a placement of the integers $1, 2, \ldots, n$, each with a $+$ or $-$ sign, and $\frac{q-1}{2} - n$ indistinguishable balls along a line, with an extra zero in the first position from the left. For example, omitting the $+$ signs, for $q = 27$ and $n = 6$ we have the placement

$$0 \bigcirc 2 \bigcirc 3\ -5 \bigcirc \bigcirc 4\ -1 \bigcirc \bigcirc -6 \bigcirc. \tag{4}$$

Such a placement corresponds to the n-tuple x for which $x_i + 1$ or $-x_i + 1$ is the position that i or $-i$ occupies, respectively, counting from the left. The placement (4) corresponds to the 6-tuple $(-9, 2, 4, 8, -5, -12)$ of elements of $\mathbb{Z}_{27}$.

We first derive the analogues of Corollary 3.2 in the four cases of interest. The symbol $\prec$ refers to the total order of the integers

$$1 \prec 2 \prec 3 \prec \cdots \prec 0 \prec \cdots \prec -3 \prec -2 \prec -1.$$

The root system BC_n. Recall that $\hat{\mathcal{BC}}_n^{[0,a]}$ has hyperplanes

$$\begin{aligned} x_i &= 0, 1, \ldots, a \quad \text{for } 1 \leq i \leq n,\\ 2x_i &= 0, 1, \ldots, a \quad \text{for } 1 \leq i \leq n,\\ x_i - x_j &= 0, 1, \ldots, a \quad \text{for } 1 \leq i < j \leq n,\\ x_i + x_j &= 0, 1, \ldots, a \quad \text{for } 1 \leq i < j \leq n. \end{aligned} \tag{5}$$

Proposition 4.1 *For* $a \geq 1$, $\chi(\hat{\mathcal{BC}}_n^{[0,a]}, q)$ *is equal to*

$$\frac{2}{a^{n+1}} S^{2n+1}(1 + S^2 + S^4 + \cdots + S^{2a-2})^n (1 + S^2 + S^4 + \cdots + S^{a-2})\, q^n$$

if a *is even and*

$$\frac{1}{a^{n+1}} S^{2n+1}(1 + S^2 + S^4 + \cdots + S^{2a-2})^n (1 + S + S^2 + \cdots + S^{a-1})\, q^n$$

if a *is odd.*

Proof: By Theorem 2.1, for sufficiently large odd q, $\chi(\hat{\mathcal{BC}}_n^{[0,a]}, q)$ counts the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_q^n$ for which none of the equalities (5) holds in $\mathbb{Z}_q$. The corresponding placements of integers and balls are the ones in which:

(i) at least a balls are placed between 0 and the leftmost nonzero integer, if this integer is positive,
(ii) at least $\lfloor \frac{a+1}{2} \rfloor$ balls are placed to the right of the rightmost integer, if this integer is negative and
(iii) at least a balls separate a nonzero integer k from the leftmost integer i to the right of k if $k \succ i$.

We call again these placements *valid*. The placement (4) is valid for $a = 1$ but it is not for $a \geq 2$. Conditions (i) and (ii) guarantee that no equation of the first two kinds in (5) holds. For example $2x_i \neq 1$, or equivalently $-x_i \neq \frac{q-1}{2}$, requires that the last position from the right is not occupied by $-i$. Condition (iii) takes care of the remaining two kinds of equations.

To construct the valid placements, place j a-blocks along a line, as in the proof of Proposition 3.1, and 0 to the left. Insert $1, 2, \ldots, n$, each with a sign, in one of the $j + 1$ possible spaces between 0 and the a-blocks and to the right of the last a-block. List the integers within each space in increasing order with respect to $\prec$, to guarantee (iii), and force the $-$ sign in the space immediately to the right of 0, to guarantee (i). Then distribute the remaining $\frac{q-1}{2} - n - aj$ balls between the integers, in blocks of at most $a - 1$. To take care of (ii), we distinguish two cases according to whether there is a negative integer to the right of the rightmost a-block or not. It follows that

$$\chi\big(\hat{\mathcal{BC}}_n^{[0,a]}, q\big) = [y^{p-n}](\phi_a(y))^{n+1} \sum_{j=0}^{\infty} (2j)^n y^{aj}$$
$$+ [y^{p-n}]\big(y^{\lfloor \frac{a+1}{2} \rfloor} + \cdots + y^{a-1}\big)(\phi_a(y))^n \sum_{j=0}^{\infty} ((2j+1)^n - (2j)^n) y^{aj},$$

where $p = \frac{q-1}{2}$. The quantity $(2j + 1)^n - (2j)^n$ in the second summand stands for the number of ways to insert the integers $1, 2, \ldots, n$ with signs in $j + 1$ possible spaces with the $-$ sign forced in the first space and at least one $-$ sign in the last.

We now extract the coefficients of y^{p-n} and use Lemma 2.2 as in Corollary 3.2 to get the proposed expressions, after some straightforward algebraic manipulations. Note that $(2j+1)^n - (2j)^n$ has degree $n-1$ in j, so Lemma 2.2 applies to the second summand as well. □

The derivations in the other three cases involve some complications but are treated in a similar way, so we will omit most of the details. We let $p = \frac{q-1}{2}$ until the end of this section.

The root system C_n. The arrangement $\hat{\mathcal{C}}_n^{[0,a]}$ lacks the first set of hyperplanes in (5).

Proposition 4.2 *For $a \geq 1$, $\chi(\hat{\mathcal{C}}_n^{[0,a]}, q)$ is equal to*

$$\frac{4}{a^{n+1}} S^{2n+1}(1 + S^2 + S^4 + \cdots + S^{2a-2})^{n-1}(1 + S^2 + S^4 + \cdots + S^{a-2})^2\, q^n$$

if a is even and

$$\frac{1}{a^{n+1}} S^{2n}(1 + S^2 + S^4 + \cdots + S^{2a-2})^{n-1}(1 + S + S^2 + \cdots + S^{a-1})^2 q^n$$

if a is odd.

Proof: The valid placements in this case are as for BC_n except that, in condition (i), a is replaced by $\lfloor \frac{a}{2} \rfloor$. To count these placements we now distinguish four cases, according to whether there is a positive integer between zero and the leftmost a-block and whether there is a negative integer to the right of the rightmost a-block. It follows that

$$\begin{aligned}
\chi\big(\hat{\mathcal{C}}_n^{[0,a]}, q\big) = [y^{p-n}](\phi_a(y))^{n+1} \sum_{j=0}^{\infty} (2j)^n y^{aj} \\
+ [y^{p-n}]\big(y^{\lfloor \frac{a}{2} \rfloor} + \cdots + y^{a-1}\big)(\phi_a(y))^n \sum_{j=0}^{\infty}((2j+1)^n - (2j)^n)y^{aj} \\
+ [y^{p-n}]\big(y^{\lfloor \frac{a+1}{2} \rfloor} + \cdots + y^{a-1}\big)(\phi_a(y))^n \sum_{j=0}^{\infty}((2j+1)^n - (2j)^n)y^{aj} \\
+ [y^{p-n}]\big(y^{\lfloor \frac{a}{2} \rfloor} + \cdots + y^{a-1}\big)\big(y^{\lfloor \frac{a+1}{2} \rfloor} + \cdots + y^{a-1}\big)(\phi_a(y))^{n-1} \\
\times \sum_{j=0}^{\infty} a_{2j+2} y^{aj},
\end{aligned}$$

where

$$a_j = j^n - 2(j-1)^n + (j-2)^n. \tag{6}$$

The result follows in a straightforward way, as before. Note that the degree of a_{2j+2} in j is at most $n-2$ and hence Lemma 2.2 applies to the last summand as well. □

The root system $\mathbf{B_n}$. The arrangement $\hat{\mathcal{B}}_n^{[0,a]}$ lacks the second set of hyperplanes in (5). The proof of the following proposition is indirect.

Proposition 4.3 *For $a \geq 1$,*

$$\chi\big(\hat{\mathcal{B}}_n^{[0,a]}, q\big) = \chi\big(\hat{\mathcal{C}}_n^{[0,a]}, q\big).$$

Proof: Let l, m denote the last two integers in a placement and s, t the number of balls between l and m and to the right of m, respectively. For the placement (4) we have $l = -1$, $m = -6$, $s = 2$ and $t = 1$. The valid placements for $\hat{\mathcal{B}}_n^{[0,a]}$ are the ones which satisfy conditions (i) and (iii) of the BC_n case and also:

$$(ii') \quad 2s + t \geq a - 1 \quad \text{if } l \succ -m.$$

Indeed, the conditions $x_i \pm x_j \neq 0, 1, \ldots, a$ require that (iii) holds, with the extra assumption $k \neq -i$, if we extend a placement, say (4), to the rest of the classes mod q as

$$0 \bigcirc 2 \bigcirc 3 \; -5 \bigcirc \bigcirc 4 \; -1 \bigcirc \bigcirc -6 \bigcirc \bigcirc 6 \bigcirc \bigcirc 1 \; -4 \cdots.$$

This also implies (ii'). Note that (ii') follows from (iii) if $l \succ m$ but is essential otherwise. It is redundant in the cases of BC_n and C_n because of (ii). To count the valid placements in this case, we first count those which satisfy (i) and (iii) and then subtract the ones which violate (ii'). For large odd q, it follows that $\chi(\hat{\mathcal{B}}_n^{[0,a]}, q)$ is the coefficient of y^{p-n} in the expression

$$(\phi_a(y))^{n+1} \sum_{j=0}^{\infty} (2j+1)^n \, y^{aj} - f_{a-2}(y)(\phi_a(y))^{n-1} \sum_{j=0}^{\infty} a'_j y^{aj},$$

where

$$f_k(y) := \sum_{\substack{s,\, t \geq 0 \\ 2s+t \leq k}} y^{s+t}$$

and a'_j is the number of ways to insert $1, 2, \ldots, n$ with signs in $j+1$ spaces and list the integers in each space in increasing order with respect to $\prec$ so that the last two integers l, m appear in the last space and satisfy $l \succ -m$, in addition to $l \prec m$. It is easy to check that

$$f_k(y) = 1 + 2y + 3y^2 + \cdots + 2y^{k-1} + y^k = \begin{cases} (\phi_{r+1}(y))^2, & k = 2r; \\ \phi_r(y)\phi_{r+1}(y), & k = 2r-1 \end{cases} \tag{7}$$

and that $a'_j = \sum_{k=2}^{n} \binom{n}{k}(2^k - 2)(2j-1)^{n-k} = a_{2j+1}$, defined by (6). In this sum, k stands for the number of integers in the last space. Using Lemma 2.2 as before, we arrive at the same expression for $\chi(\hat{\mathcal{B}}_n^{[0,a]}, q)$ as the one obtained earlier for $\chi(\hat{\mathcal{C}}_n^{[0,a]}, q)$. $\square$

The root system $\mathbf{D_n}$. The arrangement $\hat{\mathcal{D}}_n^{[0,a]}$ lacks the first two sets of hyperplanes in (5). Let $\mathcal{Q}_n$ be the arrangement of coordinate hyperplanes $x_i = 0$ in $\mathbb{R}^n$. Then $\hat{\mathcal{D}}_n^{[0,a]} \cup \mathcal{Q}_n$ has hyperplanes

$$\begin{aligned} x_i &= 0 && \text{for } 1 \le i \le n,\\ x_i - x_j &= 0, 1, \ldots, a && \text{for } 1 \le i < j \le n,\\ x_i + x_j &= 0, 1, \ldots, a && \text{for } 1 \le i < j \le n. \end{aligned} \tag{8}$$

We first prove the following lemma.

Lemma 4.4 *For $a \ge 1$ and $n \ge 3$, $\chi(\hat{\mathcal{D}}_n^{[0,a]} \cup \mathcal{Q}_n, q)$ is equal to*

$$\frac{4S^{2n-1}}{a^{n+1}} (\phi_a(S^2))^{n-3} (\phi_{a/2}(S^2))^4 (1 + 3S^2 - S^a + S^{a+2})\, q^n$$

if a is even and

$$\frac{S^{2n-1}}{a^{n+1}} (\phi_a(S^2))^{n-3} (\phi_a(S))^4 \frac{2 - S^{a-1} + S^a}{1 + S}\, q^n$$

if a is odd.

Proof: Let l', m' denote the first two integers in a placement and s', t' the number of balls to the left of l' and between l' and m', respectively. For the placement (4) we have $l' = 2$, $m' = 3$ and $s' = t' = 1$. The valid placements for $\hat{\mathcal{D}}_n^{[0,a]} \cup \mathcal{Q}_n$ are the ones which satisfy conditions (ii') and (iii) of the B_n case (see the proof of Proposition 4.1 for (iii)) and also:

(i') $\quad 2s' + t' \ge a - 2 \quad$ if $-l' \succ m'$.

This is implied by (iii) if $l' \succ m'$ but is essential otherwise. It is redundant in the cases of BC_n, C_n and B_n because of (i) and its C_n analogue. We count these valid placements as in the B_n case, using a simple inclusion-exclusion to handle both (i') and (ii'). It follows that, for large odd q, $\chi(\hat{\mathcal{D}}_n^{[0,a]} \cup \mathcal{Q}_n, q)$ is the coefficient of y^{p-n} in the expression

$$\begin{aligned} &(\phi_a(y))^{n+1} \sum_{j=0}^{\infty} (2j+2)^n y^{aj} - f_{a-3}(y)(\phi_a(y))^{n-1} \sum_{j=0}^{\infty} a_{2j+2} y^{aj} - f_{a-2}(y)(\phi_a(y))^{n-1}\\ &\quad \times \sum_{j=0}^{\infty} a_{2j+2} y^{aj} + f_{a-3}(y) f_{a-2}(y) (\phi_a(y))^{n-3} \sum_{j=0}^{\infty} b_j y^{aj}, \end{aligned}$$

where we have used the notation in (6) and (7) and

$$b_j = (2j+2)^n - 4(2j+1)^n + 6(2j)^n - 4(2j-1)^n + (2j-2)^n,$$

by a computation similar to the one for a'_j in the proof of Proposition 4.3. We extract this coefficient and factor the resulting expression appropriately to get the result. $\square$

We now compute $\chi(\hat{\mathcal{D}}_n^{[0,a]}, q)$ for $n \geq 3$. It is easy to check that $\chi(\hat{\mathcal{D}}_2^{[0,a]}, q) = (q - a - 1)^2$ for all a.

Proposition 4.5 *For $a \geq 1$ and $n \geq 3$, $\chi(\hat{\mathcal{D}}_n^{[0,a]}, q)$ is equal to*

$$\frac{8S^{2n-1}}{a^{n+1}}(1 + S^2)(1 + S^2 + S^4 + \cdots + S^{2a-2})^{n-3}(1 + S^2 + S^4 + \cdots + S^{a-2})^4\, q^n$$

if a is even and

$$\frac{1}{a^{n+1}} S^{2n-2}(1 + S^2 + S^4 + \cdots + S^{2a-2})^{n-3}(1 + S + S^2 + \cdots + S^{a-1})^4\, q^n$$

if a is odd.

Proof: By Theorem 2.1, for large odd q, $\chi(\hat{\mathcal{D}}_n^{[0,a]}, q)$ counts the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_q^n$ which satisfy

$$x_i \pm x_j \neq 0, 1, \ldots, a \tag{9}$$

in $\mathbb{Z}_q$ for all $1 \leq i < j \leq n$. The ones which also satisfy $x_i \neq 0$ for all i were counted in the previous lemma. Therefore, the characteristic polynomial of $\hat{\mathcal{D}}_n^{[0,a]}$ is the sum of that of $\hat{\mathcal{D}}_n^{[0,a]} \cup \mathcal{Q}_n$ and $\psi(q)$, where $\psi(q)$ is the number of n-tuples x for which (9) holds and $x_i = 0$ for at least one, and hence exactly one i. These can be modeled by placements which satisfy conditions (ii') and (iii) of the B_n case but have a negative integer in the leftmost position, instead of 0. For example,

$$-2 \ \bigcirc\ 3\ -5\ \bigcirc\ \bigcirc\ 4\ -1\ \bigcirc\ \bigcirc\ -6\ \bigcirc$$

corresponds to the 6-tuple $(-7, 0, 2, 6, -3, -10) \in \mathbb{Z}_{23}^6$. Thus, when constructing these placements, at least one negative integer is inserted to the left of the leftmost a-block but no positive one. The argument in the B_n case shows that

$$\begin{aligned}\psi(q) = {} & [y^{p-n+1}](\phi_a(y))^n \sum_{j=0}^{\infty}((2j+1)^n - (2j)^n)y^{aj} \\ & - [y^{p-n+1}]\, f_{a-2}(y)(\phi_a(y))^{n-2} \sum_{j=0}^{\infty} d_j y^{aj},\end{aligned}$$

where

$$d_j = a_{2j+1} - a_{2j} = (2j+1)^n - 3(2j)^n + 3(2j-1)^n - (2j-2)^n.$$

It follows that $\psi(q)$ is equal to

$$\frac{4S^{2n-1}}{a^{n+1}}(1 - S^a)(\phi_a(S^2))^{n-2}(\phi_{a/2}(S^2))^2 q^n$$

if a is even and

$$\frac{S^{2n-2}}{a^{n+1}}(1 - S^a)(\phi_a(S^2))^{n-2}(\phi_{a/2}(S^2))^2 q^n$$

if a is odd. These expressions and Lemma 4.4 imply the result. $\square$

The analogue of Proposition 3.3 was derived for most of the cases of interest in [2].

Proposition 4.6 ([2, Thm. 7.2.4 and Thm. 7.2.7]) *If $\Phi = B_n$ or D_n and $a \geq 1$ or $\Phi = C_n$ or BC_n and $a \geq 2$ is even, then*

$$\chi\big(\hat{\mathcal{A}}^{[0,a]}(\Phi), q\big) = \chi\big(\hat{\mathcal{A}}^{[1,a-1]}(\Phi), q - h\big),$$

where

$$h = \begin{cases} 2n - 2, & \textit{if } \Phi = D_n; \\ 2n, & \textit{otherwise}. \end{cases}$$

Proof: For large odd q, the quantities on the right hand side count the n-tuples $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{q-h}^n$ which satisfy $x_i \pm x_j \neq 1, \ldots, a - 1$ for all $1 \leq i < j \leq n$ and some of the conditions $x_i \neq 1, \ldots, a - 1$ and $2x_i \neq 1, \ldots, a - 1$, depending on the case. These n-tuples can be modeled by placements of length $\frac{q+1}{2}$, as described in the beginning of this section, except that more than one integer can occupy the same position, possibly the leftmost, labeled with a zero otherwise.

In each case there is an explicit bijection with the valid placements of Propositions 4.1–4.5. Given a valid placement, we remove a ball between any two consecutive integers, as in the proof of Proposition 3.3. These pairs of integers include the one formed by 0 and the leftmost nonzero integer in the cases of B_n, C_n and BC_n but not in the case of D_n. Also, in all four cases we leave the number of balls to the right of the rightmost integer unchanged. For example, the placement (4) becomes

$$0 \;\bigcirc\; 2 \;\; \overparen{3 \;\; -5} \;\; \bigcirc \;\; \overparen{4 \;\; -1} \;\; \bigcirc \;\; -6 \;\bigcirc$$

in the case of D_n and

$$0 \;\; 2 \;\; \overparen{3 \;\; -5} \;\; \bigcirc \;\; \overparen{4 \;\; -1} \;\; \bigcirc \;\; -6 \;\bigcirc$$

in the three other cases. They correspond to the 6-tuples $(-5, 2, 3, 5, -3, -7) \in \mathbb{Z}_{17}^6$ and $(-4, 1, 2, 4, -2, -6) \in \mathbb{Z}_{15}^6$ respectively. It is easy to see that this map is indeed a bijection in each case. $\square$

The bijection just described breaks down in the cases of odd a for $\Phi = C_n$ or BC_n, which need special care. The following proposition was conjectured in [2].

Proposition 4.7 ([2, Conjecture 7.2.8]) *For all odd $a \geq 1$,*

$$\chi\big(\hat{\mathcal{C}}_n^{[0,a]}, q\big) = \chi\big(\hat{\mathcal{C}}_n^{[1,a-1]}, q - 2n\big)$$

and

$$\chi\big(\hat{\mathcal{BC}}_n^{[0,a]}, q\big) = \chi\big(\hat{\mathcal{BC}}_n^{[1,a-1]}, q - 2n - 1\big).$$

Proof: For the first statement, let q be a large odd integer. Start with a valid placement, as described in the proof of Proposition 4.2. Read it from left to right, switch the $+$ signs to $-$ and vice versa and disregard 0, to get a new placement. Finally remove a ball between consecutive integers, as in the proof of Proposition 4.6, but leave the number of balls in the far left and far right unchanged, to get a placement counted by the right hand side. For example, (4) becomes

$$\bigcirc\ 6\ \bigcirc\ \overset{\frown}{1\ -4}\ \bigcirc\ \overset{\frown}{5\ -3}\ -2\ \bigcirc,$$

which corresponds to the 6-tuple $(3, -6, -5, -3, 5, 1)$ of elements of $\mathbb{Z}_{15}^6$. It is easy to check that this map is a bijection.

Note that a direct bijective proof by Theorem 2.1 is not possible for the second statement since q and $q - 2n - 1$ cannot both be odd. Once the valid placements for $\hat{\mathcal{BC}}_n^{[1,a-1]}$ are described explicitly, an argument similar to the one in the proof of Proposition 4.1 shows that

$$\chi\big(\hat{\mathcal{BC}}_n^{[1,a-1]}, q\big) = [y^p](\phi_a(y))^{n+1} \sum_{j=0}^{\infty} (2j)^n y^{aj} + [y^p]\big(y^{\frac{a-1}{2}} + \cdots + y^{a-1}\big)(\phi_a(y))^n \sum_{j=0}^{\infty} ((2j+1)^n - (2j)^n) y^{aj}.$$

This implies the result indirectly, by comparison to the formula of Proposition 4.1. □

Analogues of Corollary 3.4 follow in all four cases. For example, in the case of BC_n we have the following corollary.

Corollary 4.8 *For all $a \geq 1$, $\chi(\hat{\mathcal{BC}}_n^{[1,a]}, q)$ is equal to*

$$\frac{2S}{(a+1)^{n+1}} (1 + S^2 + S^4 + \cdots + S^{2a})^n (1 + S^2 + S^4 + \cdots + S^{a-1})\, q^n$$

if a is odd and

$$\frac{1}{(a+1)^{n+1}}(1+S^2+S^4+\cdots+S^{2a})^n(1+S+S^2+\cdots+S^a)\,q^n$$

if a is even.

5. Proof of the main theorem

The results of Sections 3 and 4 imply a crucial case of Theorem 1.2 via the following lemma. This lemma was used by Postnikov and Stanley in [11] to prove Conjecture 1.1 for the root system A_{n-1}.

Lemma 5.1 ([11, Lemma 9.12]) *If $g,\ f \in \mathbb{C}[q]$ are such that g has degree d, all roots of g have absolute value 1 and all roots of f have real part equal to r, then all roots of $g(S)\,f$ have real part equal to $r + d/2$.*

Corollary 5.2 *Conjecture 1.1 holds for $\mathcal{A} = \hat{\mathcal{A}}^{[0,b]}\,(\Phi)$, $\hat{\mathcal{A}}^{[1,b]}\,(\Phi)$ if b is a positive integer and Φ is one of A_{n-1}, B_n, C_n, D_n or BC_n for some $n \geq 2$.*

Proof: Combine the results of Sections 3 and 4 with Lemma 5.1. □

To complete the proof of Theorem 1.2 we need one last result. The first statement in the following proposition is the content of [2, Thm. 7.2.1]. We note that the argument in the case of C_n, given there, was oversimplified.

Proposition 5.3 *Let a, b be integers satisfying $0 \leq a \leq b$. If Φ is one of A_{n-1}, B_n, C_n or D_n then*

$$\chi\big(\hat{\mathcal{A}}^{[-a,b]}(\Phi), q\big) = \chi\big(\hat{\mathcal{A}}^{[0,b-a]}(\Phi), q - ah\big),$$

where

$$h = \begin{cases} n, & \text{if } \Phi = A_{n-1}; \\ 2n, & \text{if } \Phi = B_n \text{ or } C_n; \\ 2n-2, & \text{if } \Phi = D_n. \end{cases}$$

For $\Phi = BC_n$,

$$\chi\big(\hat{\mathcal{BC}}_n^{[-a,b]}, q\big) = \begin{cases} \chi\big(\hat{\mathcal{BC}}_n^{[0,b-a]}, q-(2n+1)a-1\big), & \text{if both } a \text{ and } b \text{ are odd}; \\ \chi\big(\hat{\mathcal{BC}}_n^{[0,b-a]}, q-(2n+1)a\big), & \text{otherwise}. \end{cases}$$

Proof: Let Φ be as above but assume that either a is even or b is odd if $\Phi = C_n$ or BC_n. For large q if $\Phi = A_{n-1}$ and large odd q otherwise, both hand sides of the proposed equalities count placements of a certain kind. To obtain a bijection, we start with a placement counted by the right hand side and simply add a balls between consecutive integers, as defined in Propositions 3.3 and 4.6, except that we only add $\lfloor \frac{a}{2} \rfloor$ balls immediately to the left of 0 if $\Phi = C_n$ and that we add $\lfloor \frac{a+1}{2} \rfloor$ balls to the right of the rightmost integer if $\Phi = C_n$ or BC_n. If $a = 1$ then (2) becomes

$$4 \bigcirc \bigcirc \bigcirc \bigcirc 2 \bigcirc 3 \bigcirc \bigcirc \bigcirc 1 \bigcirc \bigcirc$$

and (4) becomes

$$\begin{gathered} 0 \bigcirc \bigcirc 2 \bigcirc \bigcirc 3 \bigcirc -5 \bigcirc \bigcirc \bigcirc 4 \bigcirc -1 \bigcirc \bigcirc \bigcirc -6 \bigcirc, \\ 0 \bigcirc 2 \bigcirc \bigcirc 3 \bigcirc -5 \bigcirc \bigcirc \bigcirc 4 \bigcirc -1 \bigcirc \bigcirc \bigcirc -6 \bigcirc, \\ 0 \bigcirc 2 \bigcirc \bigcirc 3 \bigcirc -5 \bigcirc \bigcirc \bigcirc 4 \bigcirc -1 \bigcirc \bigcirc \bigcirc -6 \bigcirc \bigcirc \end{gathered}$$

or

$$0 \bigcirc \bigcirc 2 \bigcirc \bigcirc 3 \bigcirc -5 \bigcirc \bigcirc \bigcirc 4 \bigcirc -1 \bigcirc \bigcirc \bigcirc -6 \bigcirc \bigcirc,$$

if $\Phi = B_n$, D_n, C_n or BC_n, respectively. This map is easily seen to be a bijection in each case.

Now suppose that a is odd, b is even and $\Phi = C_n$ or BC_n. The map described above fails to be well defined in these cases. Moreover, for $\Phi = BC_n$, a direct bijective proof is not possible since q and $q - (2n+1)a$ cannot both be odd. One way to overcome this difficulty is to prove instead that

$$\chi\big(\hat{\mathcal{BC}}_n^{[-a,b]}, q\big) = \chi\big(\hat{\mathcal{BC}}_n^{[1,b-a-1]}, q - (2n+1)(a+1)\big)$$

and

$$\chi\big(\hat{\mathcal{C}}_n^{[-a,b]}, q\big) = \chi\big(\hat{\mathcal{C}}_n^{[1,b-a-1]}, q - 2n(a+1)\big),$$

which are equivalent to the desired formulae by Proposition 4.7. Note that we have the empty arrangement in $\mathbb{R}^n$ on the right if $b = a + 1$. Now a bijective proof is possible. Start with a placement counted by the left hand side and remove $a + 1$ balls between consecutive integers, but only $\frac{a+1}{2}$ in the far right if $\Phi = BC_n$ and in the far left and far right if $\Phi = C_n$. $\square$

We now return to the proof of the main thorem.

Proof of Theorem 1.2: Combine Corollary 5.2 and Proposition 5.3. $\square$

6. Remarks

1. For $a = 1$, Corollary 3.4 yields the expression

$$\frac{q}{2^n}\sum_{k=0}^{n}\binom{n}{k}(q-k)^{n-1}$$

for the characteristic polynomial of the Linial arrangement of hyperplanes in $\mathbb{R}^n$ $x_i - x_j = 1$ for $i < j$. It follows via Zaslavsky's theorem [17] that the number g_n of regions into which this arrangement dissects $\mathbb{R}^n$ is

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1},$$

which is also the number f_n of *alternating trees* on $n+1$ vertices [10]. The fact that $g_n = f_n$ was conjectured by Linial and Stanley and first proved by Postnikov [11, 15]. No bijective proof of this fact is known.

2. The results of Section 4 yield similar expressions for the number of regions of the Linial arrangement $\hat{\mathcal{A}}^{[1]}(\Phi)$ for $\Phi = B_n, C_n, D_n$ and BC_n. This expression is

$$2\sum_{k=0}^{n-1}\binom{n-1}{k}(k+1)^n$$

if $\Phi = B_n$ or C_n,

$$4\sum_{k=0}^{n-2}\binom{n-2}{k}(k+1)^n$$

if $\Phi = D_n$ and

$$\sum_{k=0}^{n}\binom{n}{k}(k+1)^n$$

if $\Phi = BC_n$. It would be interesting to find combinatorial interpretations to these numbers similar to the one in the case of A_{n-1}.

3. The reasoning in Section 4 can be applied to the more general family of deformations of the form

$$\begin{aligned} x_i &= 0, 1, 2, \ldots, b && \text{for } 1 \le i \le n, \\ 2x_i &= 1, 3, \ldots, 2c-1 && \text{for } 1 \le i \le n, \\ x_i - x_j &= 0, 1, \ldots, a && \text{for } 1 \le i < j \le n, \\ x_i + x_j &= 0, 1, \ldots, a && \text{for } 1 \le i < j \le n. \end{aligned} \tag{10}$$

We will only mention the special case $a = b = c$. The resulting arrangement is not one of the deformations of interest but the following proposition implies via Lemma 5.1 that Conjecture 1.1 still holds in this case and suggests that the conjecture is true in an even more general setting. Furthermore, the corresponding formula is easier to obtain.

Proposition 6.1 *For $a = b = c \geq 1$, the arrangement* (10) *has characteristic polynomial*

$$\frac{1}{a^{n+1}} S^{2n+1}(1 + S^2 + S^4 + \cdots + S^{2a-2})^{n+1} q^n.$$

Also, the arrangement

$$\begin{aligned} 2x_i &= 1, 2, \ldots, 2a - 1 && \text{for } 1 \leq i \leq n, \\ x_i - x_j &= 1, 2, \ldots, a - 1 && \text{for } 1 \leq i < j \leq n, \\ x_i + x_j &= 1, 2, \ldots, a - 1 && \text{for } 1 \leq i < j \leq n \end{aligned} \tag{11}$$

has characteristic polynomial

$$\frac{1}{(a+1)^{n+1}} S(1 + S^2 + S^4 + \cdots + S^{2a})^{n+1} q^n.$$

Proof: The argument in the proof of Proposition 4.1 yields the expression

$$[y^{p-n}](1 + y + y^2 + \cdots + y^{a-1})^{n+1} \sum_{j=0}^{\infty} (2j)^n y^{aj}$$

for the characteristic polynomial of the first arrangement in question. This implies the proposed formula, as well as the formula for the second arrangement by the argument in the proof of Proposition 4.6. □

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