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Plane Partitions and Characters of the Symmetric Group

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Abstract. In this paper we show that the existence of plane partitions, which are *minimal* in a sense to be defined, yields minimal irreducible summands in the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of two irreducible characters of the symmetric group $\mathsf{S}(n)$. The minimality of the summands refers to the dominance order of partitions of n. The multiplicity of a minimal summand χ^ν equals the number of pairs of Littlewood-Richardson multitableaux of shape (λ, μ), conjugate content and type ν. We also give lower and upper bounds for these numbers.

Keywords: Kronecker product, character of symmetric group, dominance order of partition, tableau

1. Introduction

The Kronecker product $\chi^\lambda \otimes \chi^\mu$ of two irreducible characters of the symmetric group $\mathsf{S}(n)$ is in general a reducible character of $\mathsf{S}(n)$. The multiplicity $c(\lambda, \mu, \nu)$ of an irreducible character χ^ν in the product $\chi^\lambda \otimes \chi^\mu$ can be described by a simple formula which follows from the orthogonality relations, and goes back at least to Murnaghan [16, p. 765]

$$c(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\sigma \in \mathsf{S}(n)} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma).$$

This formula has the virtue of showing the symmetry of $c(\lambda, \mu, \nu)$ in λ, μ, ν. However, it doesn't help too much when one wants to compute $c(\lambda, \mu, \nu)$ explicitly, or even to decide whether $c(\lambda, \mu, \nu)$ is different from zero for some particular choices of λ, μ, ν. Methods for computing $c(\lambda, \mu, \nu)$ are described in [16, 13, 11, 10, 5, 6, 27]. Explicit formulas for $c(\lambda, \mu, \nu)$ can be obtained, basically from the Littlewood-Richardson rule, for arbitrary λ, μ, and the simplest choices of ν, see for example [16, 13, 23, 28, 27]. Other formulas have been found when each of λ and μ is either a hook partition or a partition with two parts, and ν is arbitrary by Remmel and Whitehead [17–19]; and when λ, μ, ν are rectangular partitions by Clausen and Meier [2]. They also described an algorithm that produces the maximal sumand of $\chi^\lambda \otimes \chi^\mu$ in the lexicographic order, either by rows or by columns.

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In this paper we show that the existence of plane partitions which are *minimal* in a certain sense yields minimal summands of $\chi^\lambda \otimes \chi^\mu$ in the dominance order. The multiplicity of the minimal summands has a combinatorial description in terms of pairs of Littlewood-Richardson multitableaux. More precisely:

Let λ, μ be partitions of n. We denote by $\mathsf{M}(\lambda, \mu)$ the set of matrices A with non-negative integer coefficients of size $\ell(\lambda) \times \ell(\mu)$ such that its i-th row sums λ_i, and its j-th column sums μ_j. Given $A \in \mathsf{M}(\lambda, \mu)$, we denote by $\pi(A)$ the partition of n obtained from A by ordering its entries decreasingly. We recall that a matrix with non-negative integer coefficients is called a *plane partition* if its rows and columns are weakly decreasing. For any partition ν of n we define

$$a(\lambda, \mu; \nu) := |\{A \in \mathsf{M}(\lambda, \mu) \mid \pi(A) = \nu\}|$$

and

$$p(\lambda, \mu; \nu) := |\{A \in \mathsf{M}(\lambda, \mu) \mid \pi(A) = \nu \text{ and } A \text{ is a plane partition}\}|.$$

We denote by $\trianglelefteq$ the dominance order of partitions, see [1, 11, 22]. We say that a matrix A is *minimal* in $\mathsf{M}(\lambda, \mu)$ if $A \in \mathsf{M}(\lambda, \mu)$, and it does not exist $B \in \mathsf{M}(\lambda, \mu)$ with $\pi(B) \triangleleft \pi(A)$. We also say that ν is *minimal* for $\chi^\lambda \otimes \chi^\mu$ if $c(\lambda, \mu, \nu) \neq 0$ and $c(\lambda, \mu, \gamma) = 0$ for all $\gamma \triangleleft \nu$. Finally let $lr^*(\lambda, \mu; \nu)$ denote the number of pairs of Littlewood-Richardson multitableaux of shape (λ, μ), conjugate content and type ν, see (7). Then we have

Theorem 1.1 *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Suppose M is a plane partition. Then*
(1) *ν is minimal for $\chi^\lambda \otimes \chi^\mu$.*
(2) *$c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu')$.*
(3) *$p(\lambda, \mu; \nu) \leq c(\lambda, \mu, \nu) \leq a(\lambda, \mu; \nu)$.*
(4) *$c(\alpha, \beta, \gamma) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$, and $\gamma \triangleleft \nu$.*
(5) *$c(\lambda, \mu, \nu) = a(\lambda, \mu; \nu)$, if and only if $c(\alpha, \beta, \nu) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$ such that $(\alpha, \beta) \neq (\lambda, \mu)$.*

The paper is organized as follows. In Section 2 we review the definitions and results needed to prove our theorem. Section 3 contains a sequence of results which lead to the proof of Theorem 1.1; some of them may be of interest by themselves. In Section 4 we go back to the origins of this work: we show how a notion comming from discrete tomography, that of set of uniqueness, yields information about some $c(\lambda, \mu, \nu)$'s.

2. Definitions and known results

Let λ be a partition of n, in symbols $\lambda \vdash n$. We denote by $|\lambda|$ the sum of its parts, by $\ell(\lambda)$ the number of its parts, and by λ' its conjugate partition. We use the notation $\lambda \trianglerighteq \mu$ to indicate that λ is greater or equal than μ in the dominance order of partitions, see [1, 11, 22]. Let H be a subgroup of a group G. If χ is a character of H, we denote by

$\mathrm{Ind}_H^G(\chi)$ the character induced from χ. For any partition λ of n, let $\mathsf{S}(\lambda)$ denote a Young subgroup of $\mathsf{S}(n)$ corresponding to λ, χ^λ the irreducible character of $\mathsf{S}(n)$ associated to λ, and $\phi^\lambda = \mathrm{Ind}_{\mathsf{S}(\lambda)}^{\mathsf{S}(n)}(1_\lambda)$ the permutation character associated to λ. They are related by the *Young's rule*

$$\phi^\lambda = \sum_{\alpha \trianglerighteq \lambda} K_{\alpha\lambda}\, \chi^\alpha, \tag{1}$$

where $K_{\alpha\lambda}$ is a Kostka number, that is, the number of semistandard tableaux of shape α and content λ, see [11, 2.8.5], [22, Section 2.11]. Remember that if $\alpha \trianglerighteq \lambda$, then $K_{\alpha\lambda} > 0$ and that $K_{\lambda\lambda} = 1$. We use the symbol $\langle\cdot\,,\cdot\rangle$ for the inner product of characters.

Let λ, μ, ν be partitions of n. We denote by $\mathsf{M}(\lambda,\mu)$ the set of matrices with non-negative integer coefficients of size $\ell(\lambda)\times\ell(\mu)$, with row sum vector λ, and column sum vector μ; by $\mathsf{M}^*(\lambda,\mu)$ the subset of $\mathsf{M}(\lambda,\mu)$ formed by all matrices whose coefficients are zeros or ones; and by $\mathsf{M}^*(\lambda,\mu,\nu)$ the set of all 3-dimensional matrices $A = (a_{ijk})$ of size $\ell(\lambda)\times\ell(\mu)\times\ell(\nu)$, whose entries are zeros or ones, and have *plane sum vectors* λ, μ, ν, that is,

$$\sum_{jk} a_{ijk} = \lambda_i, \quad 1 \le i \le \ell(\lambda),$$
$$\sum_{ik} a_{ijk} = \mu_j, \quad 1 \le j \le \ell(\mu),$$
$$\sum_{ij} a_{ijk} = \nu_k, \quad 1 \le k \le \ell(\nu).$$

Finally let $m^*(\lambda,\mu) := |\mathsf{M}^*(\lambda,\mu)|$, and $m^*(\lambda,\mu,\nu) := |\mathsf{M}^*(\lambda,\mu,\nu)|$. These numbers can be expressed as inner products of characters:

$$m^*(\lambda,\mu) = \left\langle \phi^\lambda \otimes \phi^\mu, \chi^{(1^n)} \right\rangle, \tag{2}$$
$$m^*(\lambda,\mu,\nu) = \left\langle \phi^\lambda \otimes \phi^\mu \otimes \phi^\nu, \chi^{(1^n)} \right\rangle, \tag{3}$$

see [3, 4, 11, 24]. The Gale-Ryser theorem gives a characterization for the existence of matrices in $\mathsf{M}^*(\lambda,\mu)$:

$$m^*(\lambda,\mu) > 0 \Longleftrightarrow \lambda' \trianglerighteq \mu, \tag{4}$$

see [9, 20, 21]. We also have a characterization for uniqueness:

$$m^*(\lambda,\mu) = 1 \Longleftrightarrow \lambda' = \mu, \tag{5}$$

see [21, 24, 12]. For any matrix A with non-negative integer coefficients, let $\pi(A)$ denote the partition obtained from A by ordering its entries decreasingly. Then

$$\phi^\lambda \otimes \phi^\mu = \sum_{A\in\mathsf{M}(\lambda,\mu)} \phi^{\pi(A)},$$

see [4, 11]. We will denote $a(\lambda, \mu; \delta) := |\{A \in \mathsf{M}(\lambda, \mu) \mid \pi(A) = \delta\}|$, and so we rewrite the preceding formula in the following way

$$\phi^\lambda \otimes \phi^\mu = \sum_{\delta \vdash n} a(\lambda, \mu; \delta)\, \phi^\delta. \tag{6}$$

For any tableau T (a skew diagram filled with positive integers) there is a word $w(T)$ associated to T given by reading the numbers in T from right to left, in succesive rows, starting with the top row. Let ν be a partition of n of length r. Let $\rho(i) \vdash \nu_i$, $1 \le i \le r$. A sequence $T = (T_1, \ldots, T_r)$ of tableaux is called a *Littlewood-Richardson multitableau* of *shape* λ, *content* $(\rho(1), \ldots, \rho(r))$ and *type* ν if

(i) There exists a sequence of partitions

$$0 = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(r) = \lambda,$$

such that $|\lambda(i)/\lambda(i-1)| = \nu_i$ for all $1 \le i \le r$, and

(ii) for all $1 \le i \le r$, T_i is a semistandard tableau of shape $\lambda(i)/\lambda(i-1)$ and content $\rho(i)$ such that $w(T_i)$ is a lattice permutation, see [11, 2.8.13], [14, I.9], [22, Section 4.9].

For each partition λ of n let $c^\lambda_{(\rho(1),\ldots,\rho(r))}$ denote the number of Littlewood-Richardson multitableaux of shape λ and content $(\rho(1), \ldots, \rho(r))$. Let

$$lr^*(\lambda, \mu; \nu) := \sum_{\rho(1)\vdash\nu_1,\ldots,\rho(r)\vdash\nu_r} c^\lambda_{(\rho(1),\ldots,\rho(r))}\, c^\mu_{(\rho(1)',\ldots,\rho(r)')} \tag{7}$$

be the number of pairs (S, T) of Littlewood-Richardson multitableaux of shape (λ, μ) and type ν, such that S and T have conjugate content, that is, if S has content $(\rho(1), \ldots, \rho(r))$, then T has content $(\rho(1)', \ldots, \rho(r)')$. Then by applying Frobenius reciprocity to $\langle \chi^\lambda \otimes \chi^\mu, \phi^\nu \otimes \chi^{(1^n)} \rangle$ we obtain

$$lr^*(\lambda, \mu; \nu) = \sum_{\gamma \trianglelefteq \nu'} K_{\gamma'\nu}\, c(\lambda, \mu, \gamma), \tag{8}$$

compare with [11, 2.9.17], [27, Section 3].

Plane partitions. We conclude this section by recalling some facts about plane partitions which will be used in the proof of Theorem 3.4. For a positive integer m, let $[m] := \{1, \ldots, m\}$. A subset S of the 3-dimensional box $B(p, q, r) := [p] \times [q] \times [r]$ is called *pyramid* if for all $(a, b, c) \in S$ and for all $(x, y, z) \in B(p, q, r)$ the conditions $x \le a$, $y \le b$ and $z \le c$ imply $(x, y, z) \in S$. Pyramids were used in [26] to give examples of sets of uniqueness. We will say more about them in Section 4. A *plane partition* with at most p rows, at most q columns, and largest part $\le r$ is a matrix $A = (a_{ij})$ with non-negative integer coefficients of size $p \times q$, such that $0 \le a_{ij} \le r$, and whose rows and columns are weakly decreasing, see [15, Section 421]. There is a simple well-known one-to-one correspondence between pyramids $S \subseteq B(p, q, r)$ and plane partitions $A = (a_{ij})$ with at most p rows, at most q columns, and largest part $\le r$, see [15, Section 423]. For this reason a pyramid is also called the graph or the diagram of its associated plane partition. The correspondence is given by $S \longmapsto Z(S) = (z_{ij})$, where $z_{ij} := |\{k \in [r] \mid (i, j, k) \in S\}|$; its inverse is $A \longmapsto S(A)$, where $S(A) := \{(i, j, k) \in B(p, q, r) \mid 1 \leq k \leq a_{ij}\}$. If we start with a pyramid $S \in B(p, q, r)$ there are other two obvious ways of associating to S a plane partition: $Y(S) = (y_{ki})$, where $y_{ki} := |\{j \in [q] \mid (i, j, k) \in S\}|$, and $X(S) = (x_{jk})$, where $x_{jk} := |\{i \in [p] \mid (i, j, k) \in S\}|$. The plane partitions $X(S)$, $Y(S)$, $Z(S)$ are related in the following way: For all $1 \leq i \leq p$, column i of $Y(S)$ is the conjugate partition of row i of $Z(S)$, in symbols, $\mathsf{r}_i(Z(S)) = \mathsf{c}_i(Y(S))'$; and similarly for all $1 \leq k \leq r$, $\mathsf{r}_k(Y(S)) = \mathsf{c}_k(X(S))'$, and for all $1 \leq j \leq q$, $\mathsf{r}_j(X(S)) = \mathsf{c}_j(Z(S))'$. The *slice vectors* λ, μ, ν of any subset S of the box $B(p, q, r)$ are formed by the cardinalities of its slices parallel to the coordinate planes:

$$\begin{aligned} \lambda_i &:= |\{x \in S \mid x_1 = i\}|, \quad 1 \leq i \leq p, \\ \mu_j &:= |\{x \in S \mid x_2 = j\}|, \quad 1 \leq j \leq q, \\ \nu_k &:= |\{x \in S \mid x_3 = k\}|, \quad 1 \leq k \leq r. \end{aligned} \tag{9}$$

If S is a pyramid, then λ, μ, and ν are partitions of $|S|$, and the correspondence $S \longmapsto Z(S)$ satisfies: $Z(S) \in \mathsf{M}(\lambda, \mu)$, and $\pi(Z(S)) = \nu'$. Conversely, if λ, μ, ν are partitions of n, and $A \in \mathsf{M}(\lambda, \mu)$ is a plane partition with $\pi(A) = \nu$, then its associated pyramid $S(A)$ has slice vectors λ, μ, ν'.

3. Minimal matrices and plane partitions

In this section we give a proof of Theorem 1.1. It is divided in several steps. We have tried to show which consequences follow only from the minimality of M, and which use the fact that M is a plane partition. Proposition 3.1 and Theorem 3.4 may be of interest by themselves. We also give an example showing that the inequalities in Theorem 1.1.3 may be strict.

Proposition 3.1 *Let M be a matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then M is minimal if and only if $m^*(\lambda, \mu, \nu') = a(\lambda, \mu; \nu)$.*

Proof: It follows from (3), (6) and (2) that for any partitions λ, μ, ν

$$m^*(\lambda, \mu, \nu') = \sum_{\delta \vdash n} a(\lambda, \mu; \delta)\, m^*(\delta, \nu').$$

If M is minimal in $\mathsf{M}(\lambda, \mu)$ and $\nu = \pi(M)$, then $a(\lambda, \mu; \delta) = 0$ for all $\delta \lhd \nu$. Moreover, it follows from (4) that $m^*(\delta, \nu') = 0$ for all $\delta \rhd \nu$. These two equalities and (5) imply $m^*(\lambda, \mu; \nu') = a(\lambda, \mu; \nu)$. The converse is similar. □

Proposition 3.2 *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then*

$$c(\alpha, \beta, \gamma) = 0$$

for all $\alpha \unrhd \lambda$, $\beta \unrhd \mu$, and $\gamma \lhd \nu$.

Proof: It follows from (6) and (1) that for any partitions λ, μ, ν

$$\langle \phi^\lambda \otimes \phi^\mu, \chi^\nu \rangle = \sum_{\delta \vdash n} a(\lambda, \mu; \delta)\, K_{\nu\delta}.$$

If M is minimal in $\mathsf{M}(\lambda, \mu)$ and $\nu = \pi(M)$, then one proves, in a similar way as in Proposition 3.1, that $\langle \phi^\lambda \otimes \phi^\mu, \chi^\nu \rangle = a(\lambda, \mu; \nu)$. But, then by Proposition 3.1

$$\langle \phi^\lambda \otimes \phi^\mu, \chi^\nu \rangle = \left\langle \phi^\lambda \otimes \phi^\mu \otimes \phi^{\nu'}, \chi^{(1^n)} \right\rangle.$$

The claim now follows from (1), the fact that $\chi^\alpha \otimes \chi^{(1^n)} = \chi^{\alpha'}$ for any α, and the positivity of the Kostka numbers. $\square$

Corollary 3.3 *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then*
(1) $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu')$.
(2) $a(\lambda, \mu; \nu) = c(\lambda, \mu, \nu) + \sum K_{\alpha\lambda} K_{\beta\mu} c(\alpha, \beta, \nu)$. *Here the sum is over all pairs (α, β) such that $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$ and $(\alpha, \beta) \neq (\lambda, \mu)$.*

Under the assumptions of Proposition 3.2 we know that $c(\lambda, \mu, \gamma) = 0$ for all $\gamma \triangleleft \nu$, but still $c(\lambda, \mu, \nu)$ could be zero. For example if $\lambda = \mu = (3^2)$, the minimal matrices in $\mathsf{M}(\lambda, \mu)$ are $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$. However, for $\nu = (2^2, 1^2)$, we have $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu') = 0$. Therefore, we need to impose an extra condition in M in order to assure the positivity of $c(\lambda, \mu, \nu)$. One such condition, as we shall see below, is that M is a plane partition.

Theorem 3.4 *Let λ, μ, ν be partitions of n. Then*

$$p(\lambda, \mu; \nu) \leq lr^*(\lambda, \mu; \nu').$$

Proof: Let $r = \ell(\nu')$. We construct an injective map from the set of plane partitions $A \in \mathsf{M}(\lambda, \mu)$ with $\pi(A) = \nu$ to the set of pairs (S, T) of Littlewood-Richardson multitableaux of shape (λ, μ), and type ν', such that if S has content $(\rho(1), \ldots, \rho(r))$, then T has content $(\rho(1)', \ldots, \rho(r)')$. Let S be the pyramid associated to A, so that $A = Z(S)$, see Section 2. Let $B = Y(S)$, and $C = X(S)$. Then $B \in \mathsf{M}(\nu', \lambda)$, $C \in \mathsf{M}(\mu, \nu')$, and for all $1 \leq k \leq r$, $\mathsf{r}_k(B) = \mathsf{c}_k(C)'$. From B we construct a filtration

$$0 = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(r) = \lambda$$

as follows. Let $\lambda(k) := \sum_{1 \leq \alpha \leq k} \mathsf{r}_\alpha(B)$, for $1 \leq k \leq r$. Then $\lambda(k)$ is a partition and $|\lambda(k)/\lambda(k-1)| = \nu'_k$. The skew diagram $\lambda(k)/\lambda(k-1)$ has a natural filling S_k, which is obtained by putting l's on row l. Since B is a plane partition, S_k is a Littlewood-Richardson tableau of content $\mathsf{r}_k(B)$. In this way we have constructed a Littlewood-Richardson multitableau $S = (S_1, \ldots, S_r)$ of shape λ, type ν' and content $(\mathsf{r}_1(B), \ldots, \mathsf{r}_r(B))$. Now using C^T we construct in a similar way a Littlewood-Richardson multitableau $T = (T_1, \ldots, T_r)$ of shape μ, type ν' and content $(\mathsf{c}_1(C), \ldots, \mathsf{c}_r(C))$. The correspondence $A \longmapsto (S, T)$ yields the map we are looking for. $\square$

Corollary 3.5 *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then*
(1) $p(\lambda, \mu; \nu) \leq c(\lambda, \mu, \nu) \leq a(\lambda, \mu; \nu)$.
(2) *If M is a plane partition, then ν is minimal for $\chi^\lambda \otimes \chi^\mu$.*

Proof of Theorem 1.1 Statements (1)–(4) have already been proved. And (5) follows from Corollary 3.3.2. □

Example 3.6 Let $\lambda = (8, 7, 4, 2)$, $\mu = (11, 6, 4)$. With the aid of a computer we generated all minimal matrices in $\mathsf{M}(\lambda, \mu)$. They are

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 2 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 3 & 3 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Since A, B, C and D are plane partitions, then by Theorem 1.1.1 the partitions $\pi(A) = (3^4, 2^4, 1)$, $\pi(B) = (4, 3, 2^7)$, $\pi(C) = (4^2, 2^4, 1^5)$, $\pi(D) = (4, 3^2, 2^4, 1^3)$ are minimal for $\chi^\lambda \otimes \chi^\mu$. If ν is any of the first three partitions, then $1 = p(\lambda, \mu; \nu) = c(\lambda, \mu, \nu) = a(\lambda, \mu; \nu)$. If $\nu = \pi(D)$, then $1 \leq c(\lambda, \mu, \nu) \leq 4$, by Theorem 1.1.3, and $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu')$, by Theorem 1.1.2. Using (7) we get easily that $c(\lambda, \mu, \nu) = 3$. This shows that the inequalities in Theorem 1.1.3 may be strict. Of course Theorem 1.1.4 applies to our four partitions. In particular, for $\sigma = (9, 6, 4, 2)$ and $\nu = \pi(D)$ we have $c(\sigma, \mu, \gamma) = 0$ for all $\gamma \lhd \nu$; therefore $c(\sigma, \mu, \nu) = lr^*(\sigma, \mu; \nu')$. This last number is easily seen to be 1. Then, it follows from Corollary 3.3.2 that $c(\alpha, \beta, \nu) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$ such that $(\alpha, \beta) \neq (\lambda, \mu), (\sigma, \mu)$.

Remark 3.7 We consider again the example after Corollary 3.3: let $\lambda = \mu = (3^2)$, then no minimal matrix in $\mathsf{M}(\lambda, \mu)$ is a plane partition, so we cannot apply Theorem 1.1 to obtain minimal summands in $\chi^\lambda \otimes \chi^\mu$. The only partition associated to minimal matrices in $\mathsf{M}(\lambda, \mu)$ is $\nu = (2^2, 1^2)$ and we have $c(\lambda, \mu, \nu) = 0$. It turns out that the partitions covering ν, namely $\sigma = (2^3)$ and $\tau = (3, 1^3)$ are the only partitions corresponding to minimal summands in $\chi^\lambda \otimes \chi^\mu$. This leads us to the following natural questions: When do all minimal summands in $\chi^\lambda \otimes \chi^\mu$ come from minimal plane partitions? How do we determine all minimal summands not coming from a minimal plane partition? These are questions we will address in a future paper.

4. Sets of uniqueness and minimal matrices

In this section we show that the existence of sets of uniqueness puts severe restrictions on some $c(\lambda, \mu, \nu)$'s.

Let S be a subset of $B(p, q, r)$. Its slice vectors $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$, and $\nu = (\nu_1, \ldots, \nu_r)$ are compositions of $|S|$, that is, vectors of non-negative integers whose coordinates sum $|S|$, see (9). The set S is called a *set of uniqueness* if it is the only set with slice vectors λ, μ, ν. Sets of uniqueness were introduced in [8], where a geometric characterization of them was given by the absence of certain configurations in $B(p, q, r)$. Note that, as long as we are concerned with properties of S which depend on the cardinalities of its slices, we may and will assume that λ, μ, and ν are weakly decreasing, namely, that they are partitions of $|S|$. If this were not the case, we just permute some slices of S. Thus a set S is a set of uniqueness if and only if $m^*(\lambda, \mu, \nu) = 1$.

The starting point of this work was the attempt to use identities (3) and (1), and some knowledge on the numbers $c(\lambda, \mu, \nu)$ in order to find conditions on λ, μ, ν which would imply that S is a set of uniqueness. However, these numbers are hard to compute; it proved more fruitful to try to get information about the $c(\lambda, \mu, \nu)$'s from the existence of sets of uniqueness. This is the content of Corollary 4.2 which eventually developed into Theorem 1.1. In [25] an algebraic characterization of sets of uniqueness was given. Here, we need only one implication, which can be reformulated in the following way:

Theorem 4.1 *Let S be a set of uniqueness and suppose that its slice vectors λ, μ, ν are partitions of $|S|$. Then S is a pyramid, its associated plane partition $Z(S)$ is minimal in $\mathsf{M}(\lambda, \mu)$, and $a(\lambda, \mu; \nu') = 1$.*

From this and from Theorem 1.1 we obtain

Corollary 4.2 *Let S be a set of uniqueness and suppose its slice vectors λ, μ, ν are partitions of $|S|$. Then*

(1) *ν' is minimal for $\chi^\lambda \otimes \chi^\mu$.*
(2) $c(\lambda, \mu, \nu') = 1$.
(3) *For all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$, $\gamma \trianglelefteq \nu'$ such that $(\alpha, \beta, \gamma) \neq (\lambda, \mu, \nu')$, we have $c(\alpha, \beta, \gamma) = 0$.*

Examples 4.3 *The simplest example of set of uniqueness is the box $B(a, b, c)$. It has slice vectors $\lambda = ((bc)^a)$, $\mu = ((ac)^b)$, and $\nu = ((ab)^c)$. Then from the previous corollary we recover Satz 2.3 in [2]: $c(\lambda, \mu, \nu') = 1$ and obtain new identitites $c(\alpha, \beta, \gamma) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$, $\gamma \trianglelefteq \nu'$ such that $(\alpha, \beta, \gamma) \neq (\lambda, \mu, \nu')$. Another simple example of set* *of uniqueness is the* hook set $H = H(a, b, c)$ *associated to the plane partition of size* $(a + 1) \times (b + 1)$

$$Z(H) = \begin{bmatrix} c+1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix},$$

see Section 1 in [25]. It has slice vectors $\lambda = (b + c + 1, 1^a)$, $\mu = \big(a + c + 1, 1^b\big)$, *and* $\nu = (a+b+1, 1^c)$. *Then the previous corollary yields* $c(\lambda, \mu, \nu') = 1$, *and* $c(\lambda, \mu, \gamma) = 0$ *for all* $\gamma \lhd \nu'$, *which are contained in [17], as well as some new identities.*

Remark 4.4 *The notion of minimal matrix seems to be important. It was used in [25] to caracterize sets of uniqueness, and in this paper to obtain information about some* $c(\lambda, \mu, \nu)$*'s. Proposition 3.1 provides a characterization for minimal matrices. It would be desirable, however, to have more practical ways for deciding whether a given matrix M is minimal in* $\mathsf{M}(\lambda, \mu)$. *This is a problem we propose for further study.*

Note added in proof: *After submitting this manuscript I learned from M. Kapranov that the inequality* $p(\lambda, \mu; \nu) \leq c(\lambda, \mu, \nu)$ *in our Theorem 3.1.3 was proved for all* λ, μ, ν *by L. Manivel (see Proposition 3.1 in Ann. Inst. Fourier (Grenoble)* **47** *(1997), no. 3, 715–773). Note, however, that* $p(\lambda, \mu; \nu)$ *is not in general a good lower bound for* $c(\lambda, \mu, \nu)$*; for example if* $\lambda = \mu = \nu = (4, 2, 1^2)$, *then it follows from the tables in [11, p. 458] that* $c(\lambda, \mu, \nu) = 17$, *but one easily finds that* $p(\lambda, \mu; \nu) = 0$ *and* $a(\lambda, \mu; \nu) = 2$. *This and Example 3.6 seems to indicate that* $p(\lambda, \mu; \nu)$ *and* $c(\lambda, \mu, \nu)$ *are much* closer, *when* ν *corresponds to a minimal plane partition. We also note that Manivel's proof is very different from ours. It uses representations of general linear groups and algebraic geometry.*

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