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The Greedy Algorithm and Coxeter Matroids

A. VINCE vince@math.ufl.edu
Department of Mathematics, University of Florida, P.O. Box 118105, 474 Little Hall, Gainesville, FL 32611, USA

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Abstract. The notion of matroid has been generalized to Coxeter matroid by Gelfand and Serganova. To each pair (W, P) consisting of a finite irreducible Coxeter group W and parabolic subgroup P is associated a collection of objects called Coxeter matroids. The (ordinary) matroids are a special case, the case $W = A_n$ (isomorphic to the symmetric group Sym_{n+1}) and P a maximal parabolic subgroup. The main result of this paper is that for Coxeter matroids, just as for ordinary matroids, the greedy algorithm provides a solution to a naturally associated combinatorial optimization problem. Indeed, in many important cases, Coxeter matroids are characterized by this property. This result generalizes the classical Rado-Edmonds and Gale theorems.

A corollary of our theorem is that, for Coxeter matroids L, the greedy algorithm solves the L-assignment problem. Let W be a finite group acting as linear transformations on a Euclidean space $\mathbb{E}$, and let

$$f_{\xi,\eta}(w) = \langle w\xi, \eta\rangle \quad \text{for } \xi, \eta \in \mathbb{E}, w \in W.$$

The L-assignment problem is to minimize the function $f_{\xi,\eta}$ on a given subset $L \subseteq W$.

An important tool in proving the greedy result is a bijection between the set W/P of left cosets and a "concrete" collection $\mathcal{A}$ of tuples of subsets of a certain partially ordered set. If a pair of elements of W are related in the Bruhat order, then the corresponding elements of $\mathcal{A}$ are related in the Gale (greedy) order. Indeed, in many important cases, the Bruhat order on W is isomorphic to the Gale order on $\mathcal{A}$. This bijection has an important implication for Coxeter matroids. It provides *bases* and *independent sets* for a Coxeter matroid, these notions not being inherent in the definition.

Keywords: greedy algorithm, Coxeter group, matroid, Bruhat order

1. Introduction

Perhaps the best known algorithm in combinatorial optimization is the greedy algorithm. The classical MAXIMAL (MINIMAL) SPANNNING TREE problem, for example, is solved by the greedy algorithm: Given a finite graph G with weights on the edges, find a spanning tree of G with maximum (minimum) total weight. At each step in the greedy algorithm that solves this problem, there is set of edges T comprising the partial tree; an edge e of maximum weight among the edges not in T (the greedy choice) is added to T so long as $T + e$ contains no cycle.

A natural context in which to place the greedy algorithm is that of a matroid. Consider a pair $(X, \mathcal{I})$ consisting of a finite set X together with a nonempty collection $\mathcal{I}$ of subsets of X, called *independent sets*, closed under inclusion. There is a natural combinatorial optimization problem associated with this pair.

Optimization Problem. Given a weight function $\phi : \mathrm{X} \to \mathbb{R}$, find an independent set that has the greatest total weight.

The *greedy algorithm* for this problem is simply:

$I = \emptyset$
while $X \neq \emptyset$ **do**
    remove an element $x \in X$ of largest weight
    if $I + x \in \mathcal{I}$ **then** $I = I + x$

In the spanning tree problem, the set X consists of the set of edges of G and the independent sets are the acyclic subsets of edges.

It is well known that the following statements are equivalent for a pair $M = (X, \mathcal{I})$. Here $\mathcal{B}$ denotes the set of bases of M, a *basis* being a maximal independent set.

(1) M is a matroid.
(2) The greedy algorithm correctly solves the combinatorial optimization problem associated with M for any positive weight function $\phi : X \to \mathbb{R}$.
(3) Every basis has the same cardinality and, for every linear ordering $<$ on X, there exists a $B \in \mathcal{B}$ such that for any $B' \in \mathcal{B}$, if we write $B = (b_1, b_2, \ldots, b_k)$ and $B' = (b_1', b_2', \ldots, b_k')$ with the elements of B and B' both in increasing order, then $b_i \geq b_i'$ for all i.

The componentwise ordering of bases given in statement (3) is called *Gale ordering* [8], and it is a main concern of this paper.

The primary purpose of this paper is to place the greedy algorithm into a natural setting broader than that of matroids, into the setting of Coxeter matroids. The notion of matroid has been generalized to Coxeter matroid by Gelfand and Serganova [10, 11]. To each pair (W, P) consisting of a finite irreducible Coxeter group W and parabolic subgroup P is associated a collection of objects called Coxeter matroids. The (ordinary) matroids are a special case, the case $W = A_n$ (isomorphic to the symmetric group Sym_{n+1}) and P a maximal parabolic subgroup. The other Coxeter matroids provide new families of interesting combinatorial structures analogous to the ordinary matroids. There has been a flurry of research in the area of Coxeter matroids; in particular there are several relevant articles in a recent issue of the journal Annals of Combinatorics (**1**, 1998), and a book by Borovik and White [3] is forthcoming.

The main result of this paper, Theorem 3 of Section 5, states that for Coxeter matroids, just as for ordinary matroids, the greedy algorithm furnishes a correct solution to a naturally associated combinatorial optimization problem. Indeed in many important cases, Coxeter matroids are characterized by the greedy algorithm furnishing a correct solution to the naturally associated combinatorial optimization problem. After the completion of the first draft of this paper, the preprint in Russian by Serganova and Zelevinsky [16] came to our attention. That paper deals with connections between a greedy algorithm and the classical Weyl groups. This paper generalizes and extends those results.

The organization of the paper is as follows. Section 2 gives basic definitions related to Coxeter groups and Bruhat order. Also in that section is information about the geometric interpretation of a Coxeter group in terms of its Coxeter complex. This allows for geometric insight into the mainly algebraic constructions used in the paper.

The main result of Section 3 (Theorem 1) basically states that Bruhat order is Gale (greedy) order. For a given parabolic subgroup P of a Coxeter group W, the collection W/P of left cosets can be represented as a concrete set $\mathcal{A}$ of tuples of a fixed partially ordered set. Each element $(B_1, \ldots, B_m)$ of $\mathcal{A}$ is called an *admissible set*. If P is a maximal parabolic subgroup of W, then $m = 1$ and an admissible set is a single set B. If a pair of elements of W/P are related in the Bruhat order, then the corresponding elements of $\mathcal{A}$ are related in the Gale order. Indeed, in important cases, the Bruhat order on W/P is isomorphic to the Gale order on $\mathcal{A}$.

For a given parabolic subgroup P of a Coxeter group W, the notion of *admissible function* $f : W/P \to \mathbb{R}$ is defined in Section 4. The combinatorial optimization problem associated with the pair (W, P) is, given a subset $L \subset W/P$ and an admissible function f, find an element of L that maximizes f.

The concept of *Coxeter matroid M* is defined in Section 5 and is endowed with a collection $\mathcal{B}(M)$ of bases, each basis being an admissible set. This allows for the investigation of Coxeter matroids in terms of its bases, basis being a concept not inherent in the definition of Coxeter matroid. Section 5 also contains the main result on Coxeter matroids and the greedy algorithm.

An application of the main theorem to the L-assignment problem is contained in Section 6. It provides a greedy algorithm to solve the L-assignment problem when L is a Coxeter matroid. Every finite Coxeter group W can be realized as a reflection group in some Euclidean space $\mathbb{E}$ of dimension equal to the rank of W. Consider a finite group W acting as linear transformations on a Euclidean space $\mathbb{E}$, and let

$$f_{\xi,\eta}(w) = \langle w\xi, \eta\rangle \quad \text{for } \xi, \eta \in \mathbb{E}, w \in W.$$

The L-assignment problem is to minimize the function $f_{\xi,\eta}$ on a given subset $L \subseteq W$.

2. Coxeter systems and Bruhat order

Let (W, S) be a finite Coxeter system of rank n. This means that W is a finite group with the set S consisting of n generators and with the presentation

$$\langle s \in S \mid (ss')^{m_{ss'}} = 1\rangle,$$

where $m_{ss'}$ is the order of ss', and $m_{ss} = 1$ (hence each generator is an involution). The group W is called a *Coxeter group*. The *diagram* of (W, S) is the graph where each generator is represented by a node, and nodes s and s' are joined by an edge labeled $m_{ss'}$ whenever $m_{ss'} \geq 3$. By convention, the label is omitted if $m_{ss'} = 3$. A Coxeter system is *irreducible* if its diagram is a connected graph. A reducible Coxeter group is the direct product of the Coxeter groups corresponding to the connected components of its diagram. Finite irreducible Coxeter groups have been completely classified and are usually denoted by A_n $(n \geq 1)$, $B_n (= C_n)$ $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, and $I_2(m)$ $(m \geq 5,\ m \neq 6)$, the subscript denoting the rank. The diagram of each of these groups is given in figure 1.

A *reflection* in W is a conjugate of some element of S. Let $T = T(W)$ denote the set of all reflections in W. Every finite Coxeter group W can be realized as a reflection group

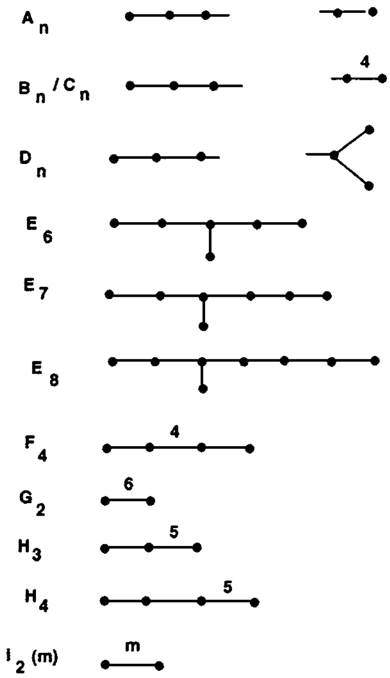

Figure 1. Irreducible finite reflection groups.

in some Euclidean space $\mathbb{E}$ of dimension equal to the rank of W. In this realization, each element of T corresponds to the orthogonal reflection through a hyperplane in $\mathbb{E}$ containing the origin. Each of the irreducible Coxeter groups listed above, except D_n, E_6, E_7, and E_8, is the symmetry group of a regular convex polytope. The group A_n is isomorphic to the symmetric group Sym_{n+1}, the set S of generators consisting of the adjacent transpositions $(i, i+1), i = 1, 2, \ldots, n$.

For a finite Coxeter system (W, S), let Σ denote the set of all reflecting hyperplanes in $\mathbb{E}$. Let $E' = \mathbb{E} \setminus \cup_{H \in \Sigma} H$. The connected components of E' are called *chambers*. For any chamber Γ, its closure $\bar{\Gamma}$ is a simplicial cone in $\mathbb{E}$. These simplicial cones and all their faces form a simplicial fan called the *Coxeter complex* and denoted $\Delta := \Delta(W, S)$. It is known that W acts simply transitively on the set of chambers. To identify the elements of W with chambers, we choose a fundamental chamber Γ_0 whose facets (i.e., faces of codimension one) are on reflecting hyperplanes for the simple reflections $s \in S$; then the bijective correspondence between W and the set of chambers is given by $w \mapsto w(\Gamma_0)$.

Every subset $J \subset S$ gives rise to a (standard) *parabolic subgroup* W_J generated by J. The maximal parabolic subgroups $W_{S-\{s\}}$ will be of special importance for us, and we will use the shorthand $P_s = W_{S-\{s\}}$ for $s \in S$. If $P = W_J$ is a parabolic subgroup, we denote by $\Gamma_0(P)$ the set of points in $\bar{\Gamma}_0$ whose stabilizer in W is exactly P. The closure $\overline{\Gamma_0(P)}$ is a face of the simplicial cone $\bar{\Gamma}_0$, and the correspondence $P \mapsto \overline{\Gamma_0(P)}$ is a bijection between the set of parabolic subgroups of W and the set of faces of $\bar{\Gamma}_0$. Using the action of W, we obtain the following well known description of the faces of the Coxeter complex [12].

Proposition 1 *Let (W, S) be a finite Coxeter system. The correspondence*

$$wP \mapsto w(\overline{\Gamma_0(P)})$$

is an inclusion reversing bijection between the union of left coset spaces $\cup W/P$ modulo all parabolic subgroups and the collection of all faces of $\Delta(W, S)$. Two faces $w(\overline{\Gamma_0(P)})$ and $w'(\overline{\Gamma_0(P')})$ are contained in the same chamber of Δ if and only if $wP \cap w'P' \neq \emptyset$.

In the case that W is the symmetry group of a regular polytope $Q := Q(W)$, the Coxeter complex is essentially the barycentric subdivision of the boundary complex of Q. Two faces q and q' of Q are called *incident* if either $q \subset q'$ or $q' \subset q$. The last statement in Proposition 1 implies that two faces of Q are incident if and only if the corresponding cosets have nonempty intersection.

We give two equivalent definitions of the Bruhat order on a Coxeter group; for a proof of the equivalence see e.g., [7]. We will use the notation $u \succeq w$ for the Bruhat order. For $w \in W$ a factorization $w = s_1 s_2 \cdots s_k$ into the product of simple reflections is called *reduced* if it is shortest possible. Let $l(w)$ denote the length k of a reduced factorization of w.

Definition 1 Define $u \succeq v$ if there exists a sequence $v = u_0, u_1, \ldots, u_m = u$ such that $u_i = t_i u_{i-1}$ for some reflection $t_i \in T(W)$, and $l(u_i) > l(u_{i-1})$ for $i = 1, 2 \ldots, m$.

Definition 2 If $u = s_1 s_2 \cdots s_k$ is a reduced factorization, then $u \succeq v$ if and only if there exist indices $1 \leq i_1 < \cdots < i_j \leq k$ such that $v = s_{i_1} \cdots s_{i_j}$.

The Bruhat order can be also defined on the left coset space W/P for any parabolic subgroup P of G. Again we give two definitions.

Definition 3 Define Bruhat order on W/P by $\bar{u} \succeq \bar{v}$ if there exists a $u \in \bar{u}$ and $v \in \bar{v}$ such that $u \succeq v$.

It is known (see e.g., [12]) that any coset $\bar{u} \in W/P$ has a unique representative of minimal length, denoted $\bar{u}_{\min}$.

Definition 4 We have $\bar{u} \succeq \bar{v}$ in the Bruhat order on W/P if and only if $\bar{u}_{\min} \succeq \bar{v}_{\min}$.

We will associate with each $w \in W$ a shifted version of the Bruhat order on W/P, which will be called the w-Bruhat order and denoted $\succeq_w$.

Definition 5 Define $\bar{u} \succeq_w \bar{v}$ in the *w-Bruhat order* on W/P if $w^{-1}\bar{u} \succeq w^{-1}\bar{v}$.

The Bruhat orders for many particular choices of W and P have been explicitly worked out [14]. It is instructive to keep in mind the following three classical examples, where W is of the type A_n, C_n or D_n, and $P = P_1 := W_{S-\{s\}}$ is the special maximal parabolic subgroup for which the simple reflection s corresponds to the leftmost node in the Coxeter diagram of W in figure 1.

Example 1 (ordinary case) The group $W = A_n$ is the symmetric group Sym_{n+1}, the set $S = \{s_1, \ldots, s_n\}$ of generators consisting of the adjacent transpositions $s_i = (i, i+1)$, $i = 1, \ldots, n$. The parabolic subgroup P_1 is the stabilizer in W of the element $1 \in [1, n+1] := \{1, \ldots, n+1\}$, so W/P_1 is identified with $[1, n+1]$ via $wP_1 \mapsto w(1)$. Under this identification, the Bruhat order on W/P_1 becomes the linear order on $[1, n+1]$ given by

$$1 \prec 2 \prec \cdots \prec n+1.$$

The group A_n is also isomorphic to the symmetry group of the regular n-simplex. Geometrically W/P_1 corresponds, under the bijection of Proposition 1, to the set of vertices of the regular n-simplex.

Example 2 (symplectic case) The group $W = C_n$ can be identified with the subgroup of the symmetric group Sym_{2n} consisting of all permutations that commute with the longest permutation $w_0 \in S_{2n}$. It is convenient to denote by $[1, n] \cup [1, n]^* = \{1, \ldots, n, 1^*, \ldots, n^*\}$ the set of indices permuted by Sym_{2n}, and to realize w_0 as the permutation $i \mapsto i^* \mapsto i$, $i \in [1, n]$. The standard choice of $S = \{s_1, \ldots, s_n\}$ is then the following: $s_i = (i, i+1)(i^*, (i+1)^*)$ for $i = 1, \ldots, n-1$, and $s_n = (n, n^*)$. As in the previous example, P_1 is the stabilizer in W of the element $1 \in [1, n] \cup [1, n]^*$, so W/P_1 is identified with $[1, n] \cup [1, n]^*$ via $wP_1 \mapsto w(1)$. Under this identification, the Bruhat order on W/P_1 becomes the linear order on $[1, n] \cup [1, n]^*$ given by

$$1 \prec 2 \cdots \prec n-1 \prec n \prec n^* \prec (n-1)^* \prec \cdots \prec 2^* \prec 1^*.$$

The group C_n is also isomorphic to the symmetry group of the regular n-dimensional cross polytope (general octahedron). Geometrically W/P_1 corresponds, under the bijection of Proposition 1, to the set of vertices of the regular n-dimensional cross polytope. With the notation above, vertices i and i^* are antipodal.

Example 3 (even orthogonal case) The group $W = D_n$ can be identified with the subgroup of even permutations in the Coxeter group C_n realized as in the previous example. The set $S = \{s_1, \ldots, s_n\}$ then consists of the elements $s_i = (i, i+1)(i^*, (i+1)^*)$ for $i = 1, \ldots, n-1$, and $s_n = (n-1, n^*)(n, (n-1)^*)$. As in the first two examples, P_1 is the stabilizer in W of the element $1 \in [1, n] \cup [1, n]^*$, so W/P_1 is identified with $[1, n] \cup [1, n]^*$ via $wP_1 \mapsto w(1)$. However, the Bruhat order on W/P_1 is no longer linear; it is given by

$$1 \prec 2 \prec \cdots n-1 \prec \begin{matrix} n \\ \\ n^* \end{matrix} \prec (n-1)^* \prec \cdots \prec 2^* \prec 1^*.$$

Returning to a general Coxeter group W and a parabolic subgroup P, regard W as a reflection group in Euclidean space $\mathbb{E}$ with the usual inner product $\langle \xi, \eta \rangle$. Fix any $\delta \in \Gamma_0(P)$.

Since, by definition, the stabilizer of δ in W is P, we can unambiguously define the point $\bar{u}\delta := \bar{u}(\delta) \in \mathbb{E}$ for any $\bar{u} \in W/P$. The following proposition is given in [15] as a consequence of the definition of Bruhat order.

Proposition 2 *If $\bar{u} \succ \bar{v}$ in the Bruhat order on W/P, then $\langle \bar{u}\delta, \eta \rangle < \langle \bar{v}\delta, \eta \rangle$ for any $\eta \in \Gamma_0$.*

3. The relation between Bruhat order and Gale order

Let (W, S) be a finite, irreducible, rank n Coxeter system and $P = W_J$ a parabolic subgroup in W (recall that P is generated by a subset $J \subset S$). We will provide a "concrete" realization of the Bruhat order on W/P by encoding the elements of W/P as appropriate tuples of subsets. To do this, some terminology and notation are needed.

For a finite set X, we denote by 2^X the set of all subsets of X. If I is another finite set, denote by $(2^X)_I$ the set of I-tuples of subsets of X; that is, $(2^X)_I$ consists of families $A = (A_i)_{i \in I}$ of subsets of X indexed by I. Suppose now that X is a poset, i.e., is equipped with a partial order which we write simply as $a \geq b$. We introduce the corresponding *Gale order* on $(2^X)_I$ as follows.

Definition 6 If $A = (A_i)$ and $B = (B_i)$ are two I-tuples of subsets in X then $A \geq B$ in the Gale order if, for every $i \in I$, there exists a bijection $f_i : A_i \to B_i$ such that $a \geq f_i(a)$ for any $a \in A_i$.

In particular, if two I-tuples $A = (A_i)$ and $B = (B_i)$ are comparable in the Gale order then A_i and B_i have the same cardinality for any $i \in I$.

Returning to the Bruhat order on W/P for $P = W_J$, we will construct, for any proper parabolic subgroup Q in W, an embedding

$$\mathcal{B} = \mathcal{B}_J^Q : W/P \to (2^{W/Q})_{S-J} .$$

For any coset $\bar{v} \in W/P$ and any $i \in S - J$, denote by $\bar{v}(i) \in W/P_i$ the unique coset modulo the maximal parabolic subgroup P_i that contains $\bar{v}$.

Definition 7 For $\bar{v} \in W/P$, the *Q-basis* of $\bar{v}$ is an $(S-J)$-tuple $\mathcal{B}(\bar{v}) := \mathcal{B}_J^Q(\bar{v}) = (B_i)_{i \in S-J}$ of subsets of W/Q given by

$$B_i = \{\bar{u} \in W/Q \mid \bar{u} \cap \bar{v}(i) \neq \emptyset\}.$$

The rationale for the terminology "basis" will become clear in Section 5. Note that, if P is maximal, then the Q-basis of $\bar{v} \in W/P$ consists of the single set

$$B = \{\bar{u} \in W/Q \mid \bar{u} \cap \bar{v} \neq \emptyset\}.$$

In this case $\bar{v}$ corresponds to a vertex in the Coxeter complex $\Delta(W, S)$, and $\mathcal{B}(\bar{v})$ consists of the faces corresponding (by Proposition 1) to the cosets in W/Q that lie in a common chamber with this vertex. In the case that W is the symmetry group of a regular polytope and Q is also maximal, the coset $\bar{v}$ corresponds to a face σ of a certain dimension, say j, and $\mathcal{B}(\bar{v})$ is the set of all faces of another dimension, say k, incident with σ.

Not every member of $(2^{W/Q})_{S-J}$ can appear as a Q-basis of some element of W/P. Those that can are called Q-*admissible*, and the set of Q-*admissible* tuples for W/P will be denoted $\mathcal{A}(P, Q)$. If $Q = P_1$, the maximal parabolic subgroup corresponding to the first node in the Coxeter diagram, then the notation will be simply $\mathcal{A}(P)$.

Definition 8

$$\mathcal{A}(P, Q) = \mathcal{B}_J^Q(W/P)$$
$$\mathcal{A}(P) = \mathcal{B}_J^{P_1}(W/P)$$

Since the individual elements in $\mathcal{A}(P, Q)$ lie in W/Q and W/Q is a poset with respect to Bruhat order, $\mathcal{A}(P, Q)$ is itself a poset with respect to the corresponding Gale order given in Definition 6.

The following examples are a continuation of the three examples in the previous section.

Example 1 (ordinary case) Consider $W = A_n$ as the symmetric group Sym_{n+1}. The parabolic subgroup $P_k := W_{S-\{s_k\}}$ is the setwise stabilizer in W of $\{1, 2, \ldots, k\}$. To determine the P_1-admissible sets, note that if $\bar{u} \in W/P_k$ and $\bar{v} \in W/P_1$, then $\bar{u} \cap \bar{v} \neq \emptyset$ if and only if $v(1) \in \{u(1), \ldots, u(k)\}$. Since $\{u(1), \ldots, u(k)\}$ can be any k-element subset of $[n+1]$, the P_1-admissible sets are all the k-subsets of $[n+1]$.

$$\mathcal{A}(P_k) = \binom{[n+1]}{k}$$

Geometrically, the admissible sets are (the vertex sets of) the $(k-1)$-dimensional faces of the regular n-simplex. The Bruhat order on P_1, as given in Example 1 of Section 2, induces the Gale order on $\mathcal{A}(P_k)$. For example, with $n = 4$, $k = 3$ we have $2\ 3\ 5 > 1\ 2\ 5$ in the Gale order. (As is common in the matroid literature $\{2, 3, 5\}$ is simply denoted $2\ 3\ 5$.)

Example 2 (symplectic case) If $W = C_n$, an analysis similar to that in Example 1 indicates that

$$\mathcal{A}(P_k) = \left\{ \alpha \in \binom{[n] \cup [n]^*}{k} \,\middle|\, \text{both } i \text{ and } i^* \text{ cannot appear simultaneously in } \alpha \right\}.$$

For example, with $n = 4$, $k = 3$, the set $1\ 2\ 4^*$ is admissible but $1\ 2\ 2^*$ is not. Geometrically, the admissible sets are (the vertex sets of) the regular $(k-1)$-dimensional faces of the regular n-dimensional cross polytope, where a vertex and antipodal vertex pair are denoted by a number and its star. The Bruhat order on P_1 as given in Example 2 of Section 2 induces the Gale order on $\mathcal{A}(P_k)$. For example, with $n = 4$, $k = 3$ we have $1^*\ 2\ 3^* > 1\ 2^*\ 4$ in the Gale order because $1^* \succ 2^*$, $3^* \succ 4$, $2 \succ 1$.

Example 3 (orthogonal case) If $W = D_n$ and $k \leq n-2$, then, just as in the C_n case,

$$\mathcal{A}(P_k) = \left\{ \alpha \in \binom{[n] \cup [n]^*}{k} \,\middle|\, \text{both } i \text{ and } i^* \text{cannot appear simultaneously in } \alpha \right\}.$$

However, $\mathcal{A}(P_{n-1})$ consists of all sets in $\binom{[n]\cup[n]^*}{n}$ such that both i and i^* cannot appear simultaneously and there are an even number of starred elements. Similarly $\mathcal{A}(P_n)$ consists of all sets in $\binom{[n]\cup[n]^*}{n}$ such that both i and i^* cannot appear simultaneously and there are an odd number of starred elements. The Bruhat order on P_1 as given in Example 3 of Section 2 induces the Gale order on $\mathcal{A}(P_k)$. For example, with $n = 4,\ k = 3$ we have $1^*\,2\,3^*\,4 > 1\,2^*\,3\,4^*$ in the Gale order because $1^* \succ 2^*,\ 3^* \succ 4^*,\ 4 \succ 3,\ 2 \succ 1$.

A mapping f from one poset to another is called *monotone* if $u \geq v$ implies $f(u) \geq f(v)$ for all u, v. A bijection f for which both f and f^{-1} are monotone is called an *isomorphism*.

Theorem 1 *Let (W, S) be a finite, irreducible Coxeter system and P and Q parabolic subgroups. The Q-basis map*

$$\begin{aligned} \mathcal{B} : W/P &\to \mathcal{A}(P, Q) \\ \bar{v} &\mapsto \mathcal{B}(\bar{v}) \end{aligned}$$

is a monotone bijection from the set W/P with respect to Bruhat order to the set $\mathcal{A}(P, Q)$ with respect to Gale order. Moreover, $\mathcal{B}$ is an isomorphism if the Bruhat order on W/Q is linear.

Proof: It is surjective by definition of admissible. Let $P = W_J$. Recall the notation for a maximal parabolic subgroup $P_j = W_{S-\{j\}}$. Injectivity follows from the following known properties of Coxeter groups [17].

(i) $\cap_{j\notin J} P_j = W_J$.
(ii) If two elements in W/P_j have the same Q-basis, then they coincide.

Statement (i) says that $\bar{v} \in W/P$ is determined by $\{vP_j \mid j \notin J\}$, and statement (ii) says that vP_j is determined by its Q-basis.

Concerning the monotone property and isomorphism there are three things to prove.

(1) $vP_J \succeq uP_J \Leftrightarrow vP_j \succeq uP_j$ for all $j \notin J$.
(2) $vP_j \succeq uP_j \Rightarrow \mathcal{B}(vP_j) \geq \mathcal{B}(uP_j)$ for each $j \notin J$.
(3) If the Bruhat order on W/Q is linear, then $\mathcal{B}(vP_j) \geq \mathcal{B}(uP_j) \Rightarrow vP_j \succeq uP_j$ for each $j \notin J$.

Statement (1) is Lemma 3.6 in Deodhar [7]. The proof there applies to our situation without change.

In the following proof of statement (2), we will use Definition 4 (Section 2) of Bruhat order. For $J \subseteq [n]$, let $W^J = \{w \in W \mid l(ws) = l(w) + 1 \text{ for all } s \in J\}$. This is the set of all minimal representatives of cosets in W/P. It is well known [12] that for any $w \in W$ we have $w = w^J w_J$ where $w^J \in W^J$ and $w_J \in W_J$, and this expression is unique. Moreover, $l(w) = l(w^J) + l(w_J)$.

Assume that $vP_j \succeq uP_j$ and let $\bar{v}_{\min}$ and $\bar{u}_{\min}$ be the minimum elements in vP_j and uP_j, respectively. The mapping $\phi : \bar{v}_{\min}x \mapsto \bar{u}_{\min}x,\ x \in P_j$, is a bijection between vP_j and uP_j such that $\bar{v}_{\min}x \succeq \phi(\bar{v}_{\min}x)$. Then the mapping $\hat{\phi} : yQ \mapsto \phi(y)Q$ induces a well defined bijection between $\mathcal{B}(vP_j)$ and $\mathcal{B}(uP_j)$ such that $yQ \succeq \hat{\phi}(yQ)$. But this is exactly Gale order $\mathcal{B}(vP_j) \geq \mathcal{B}(uP_j)$. Thus statement (2) is proved.

Concerning the proof of statement (3), assume that the Bruhat order on W/Q is linear. To prove that $\mathcal{B}$ is an isomorphism we must show that if $\mathcal{B}(vP_j) \geq \mathcal{B}(uP_j)$ then $vP_j \succeq uP_j$. This requires some preliminary properties of Bruhat order:

(a) **Property** $Z(s, w_1, w_2)$: If $w_1, w_2 \in W$ and $s \in S$ satisfy $l(w_1) \succeq l(sw_1)$ and $l(w_2) \succeq l(sw_2)$, then $w_2 \succeq w_1 \Leftrightarrow w_2 \succeq sw_1 \Leftrightarrow sw_2 \succeq sw_1$.
(b) If $w \in W^J$ and $s \in S$ satisfy $l(w) \succeq l(sw)$, then $sw \in W^J$.
(c) If $w \in W$ and $s \in S$ satisfy $wP_j \succ swP_j$, then $w \succ sw$.

Properties (a) and (b) are in [7]. Concerning (c), if $\bar{w}_{\min}$ is the minimum element of wP_j, then $\bar{w}_{\min} \succ s\bar{w}_{\min}$ because $s\bar{w}_{\min} \in swP_j$, so that $s\bar{w}_{\min} \succ \bar{w}_{\min}$ is impossible. By property (b) we have $s\bar{w}_{\min} \in W^J$, where $J = [n]\backslash\{j\}$. The decomposition of any element of W into a product of elements of W^J and W_J (discussed above), implies that $w \succ sw$. Thus property (c) is proved.

Let $\mathcal{B}(vP_j) = \{v_1Q, \ldots, v_mQ\}$ and $\mathcal{B}(uP_j) = \{u_1Q, \ldots, u_mQ\}$. Because we are assuming that $\mathcal{B}(vP_j) \geq \mathcal{B}(uP_j)$, we have $v_iQ \succeq u_iQ$ for $i = 1, \ldots, m$. Let $\bar{v}_{\min}$ be the minimum element of vP_j and $\bar{u}_{\min}$ the minimum element of uP_j. The proof of statement (3) now proceeds by induction on the length of $\bar{v}_{\min}$. If $l(\bar{v}_{\min}) = 0$, then $\bar{v}_{\min}$ is the identity 1. Consequently, the minimum elements in the cosets $v_1Q, \ldots, v_mQ$ are all elements of P_j. By Definition 2 of Bruhat order, the minimum elements of $u_1Q, \ldots, u_mQ$ must be subwords of the minimal elements of $v_1Q, \ldots, v_mQ$, hence also elements of P_j. Therefore $\mathcal{B}(vP_j) = \mathcal{B}(uP_j)$. By the injectivity of the mapping $\mathcal{B}$, we have $vP_j = uP_j$; the first instance in the induction is done. Now assume that $l(\bar{v}_{\min}) \geq 1$. Choose $s \in S$ such that $\bar{v}_{\min} \succ s\bar{v}_{\min}$. The proof is now divided into three cases.

Case 1. $\bar{u}_{\min} \succ s\bar{u}_{\min}$. In this case we claim that $\mathcal{B}(svP_j) \geq \mathcal{B}(suP_j)$; more precisely we claim that $sv_iQ \succeq su_iQ$ for all i. By the induction hypothesis, this would imply that $svP_j \succeq suP_j$. By property (b), $s\bar{v}_{\min}$ and $s\bar{u}_{\min}$ are the minimum elements of svP_j and suP_j, respectively. Therefore $s\bar{v}_{\min} \succeq s\bar{u}_{\min}$. By property $Z(s, \bar{u}_{\min}, \bar{v}_{\min})$, we have $\bar{v}_{\min} \succeq \bar{u}_{\min}$, and hence the desired result $vP_j \succeq uP_j$.

To prove the claim for Case 1, fix an index i. Let v' and u' be elements of $v_iQ \cap vP_j$ and $u_iQ \cap uP_j$, respectively. Then $\bar{v}_{\min} \succ s\bar{v}_{\min}$ implies that $v'P_j \succ sv'P_j$ since both $\bar{v}_{\min}$ and $s\bar{v}_{\min}$ are minimum elements. This implies, by property (c), that $v' \succ sv'$, which in turn implies that $v'Q \succeq sv'Q$. Similarly $\bar{u}_{\min} \succ s\bar{u}_{\min}$ implies that $u'Q \succeq su'Q$. If v'' and u'' are the minimum elements of $v'Q$ and $u'Q$, resp., then clearly $v'Q \succeq u'Q$ implies that $v'' \succeq u''$. Also $u''Q \succeq su''Q$ implies $u'' \succeq su''$ unless $u''Q = su''Q$ and $v''Q \succeq sv''Q$ implies $v'' \succeq sv''$ unless $v''Q = sv''Q$. Assuming the cases of equality do not occur, property $Z(s, u'', v'')$ implies that $sv'' \succeq su''$, i.e., $sv_iQ \succeq su_iQ$.

Now consider the cases of equality above. First, if $v''Q = sv''Q$ then $sv'Q = v'Q \succeq u'Q \succeq su'Q$, and we are done. Second, if $u''Q = su''Q$ and $v''Q \succ sv''Q$ then by previous arguments sv'' is the mimimum element of $sv'Q$. Since $v''Q \succ sv''Q$ we have $v'' \succ sv''$. If $u'' \succ su''$ the argument in the paragraph above works, but if $su'' \succ u''$ then let $w = su''$ and $sw = u''$ so that $w \succ sw$. Now $v'' \succeq u''$ implies that $v'' \succeq sw$. By property $Z(s, v'', w)$ we have that $v'' \succeq sw$ implies $sv'' \succeq sw = u''$. Therefore $sv''Q \succeq u''Q = su''Q$. (Note that the argument used in this paragraph will be referred to in Cases 2 and 3.)

Case 2. $s\bar{u}_{\min} \succ \bar{u}_{\min}$ and $suP_j \succ uP_j$. In this case we claim that $\mathcal{B}(svP_j) \geq \mathcal{B}(uP_j)$; more particularly we claim that $sv_iQ \succeq u_iQ$ for all i. By the induction hypothesis applied to $s\bar{v}_{\min}$, we have $svP_j \succeq uP_j$, which implies, by property (b), that $sv_{\min} \succeq u_{\min}$. By property $Z(s, s\bar{u}_{\min}, \bar{v}_{\min})$, we have $\bar{v}_{\min} \succeq \bar{u}_{\min}$, and hence the desired result $vP_j \succeq uP_j$.

To prove the claim for Case 2, fix an index i. With the same notation as in Case 1, this inequality implies that $su' \succ u'$ which implies that $su''Q \succeq u''Q$ which in turn implies that $su'' \succ u''$ unless $su''Q = u''Q$. If $su'' \succ u''$, then the same arguments as in Case 1 can be used to prove the claim for Case 2, that $sv_iQ \succeq u_iQ$. Moreover, $su''Q = u''Q$ is impossible because $u'' \succ su''$ implies, by property (b), that su'' is the minimum element for $su''Q$, and, since u'' is the minimum for $u''Q$, this would imply that $u''Q \succ su''Q$.

Case 3. $s\bar{u}_{\min} \succ \bar{u}_{\min}$ and $suP_j = uP_j$. We claim that $\mathcal{B}(svP_j) \geq \mathcal{B}(uP_j)$. By the induction hypothesis, we have $svP_j \succeq uP_j$, which implies $vP_j \succeq uP_j$ exactly as in Case 2.

To prove the claim for Case 3, consider any pair u_iQ and u_kQ of cosets in $\mathcal{B}(uP_j)$ where $u_kQ = su_iQ$. (It is possible that $u_iQ = u_kQ$.) Note that such pairs form a partition of the set $\mathcal{B}(uP_j)$. Our intention is to show that each pair $\{sv_iQ, sv_kQ\}$ is greater than or equal to $\{u_iQ, u_kQ\}$ in the Gale order. In other words, either $sv_iQ \succeq u_iQ$ and $sv_kQ \succeq u_kQ$ or $sv_iQ \succeq u_kQ$ and $sv_kQ \succeq u_iQ$.

Let u'' and u''_s be the minimum elements of u_iQ and su_iQ, respectively. If $su'' \succ u''$ and $su''_s \succ u''_s$, then, by the argument of Case 2 (and also Case 1), $sv_iQ \succeq u_iQ$ and $sv_kQ \succeq u_kQ$.

It remains to deal with the possibility that either $u'' \succ su''$ or $u''_s \succ su''_s$. Without loss of generality assume that $u'' \succ su''$. Since, by property (b), su'' must be a minimal element in its coset modulo Q, and, since $suP_j = uP_j$, both u'' and su'' represent elements of $\mathcal{B}(uP_j)$. Let $w = su''$ and $sw = u''$ and let $z_i \succeq sw$ and $z_k \succeq w$, be the minimal elements of the cosets v_iQ and v_kQ in $\mathcal{B}(vP_j)$ that, by assumption, dominate $u_iQ = swQ$ and $u_kQ = wQ$ resp., in the Bruhat order. Since $sw \succ w$, correspondingly $z_i \succ z_k$. Also since $sw \succ w$ it follows, as in Case 2, that $sv_kQ = u_kQ$. If it is also true that $sv_iQ \succeq u_iQ$, then the proof is complete. Assume that it is not the case that $sz_iQ = sv_iQ \succeq u_iQ = swQ$. Here is where we use the linearity of the Bruhat order on W/Q. Because $z_i \succeq sw$ we have $z_iQ \succeq swQ$ and by the linearity we have $swQ \succ sz_iQ$. These two inequalities imply $z_i \succ sz_i$. But $z_i \succeq sw \succ sz_i$ implies, because z_i covers sz_i in the Bruhat order (see [12]), that $z_i = sw$. Moreover, $sw = z_i \succ z_k \succeq w$ implies that $z_k = w$. Then $sv_iQ = sz_iQ = wQ = su''Q = u_kQ$ and $sv_kQ = sz_kQ = swQ = u''Q = u_iQ$. Thus the pair $\{sv_iQ, sv_kQ\}$ is equal to $\{u_iQ, u_kQ\}$ in the Gale order, and we are done. $\square$

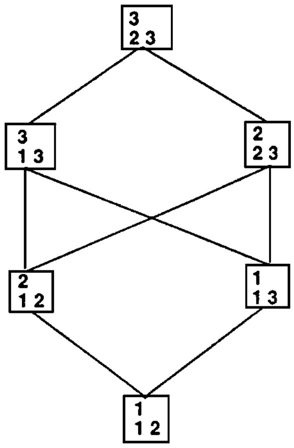

Figure 2. Gale order on $\mathcal{A}(P)$ is the Bruhat order on Sym_3.

Example 4 As an example of a collection of admissible sets with respect to a non-maximal parabolic subgroup, consider the case $W = A_2 \approx \mathrm{Sym}_3$ and the trivial parabolic subgroup $P = W_\emptyset$. Then $W/P = W$. The bijection between A_2 and $\mathcal{A}(P)$ is explicitly indicated as follows, where $S = \{s_1, s_2\}$ is the canonical set of generators of Sym_3.

$$\begin{aligned} \mathcal{B}(id) &= \mathcal{B}(123) = (1, 1\ 2), & \mathcal{B}(s_1) &= \mathcal{B}(213) = (2, 1\ 2), \\ \mathcal{B}(s_2) &= \mathcal{B}(132) = (1, 1\ 3), & \mathcal{B}(s_2 s_1) &= \mathcal{B}(312) = (3, 1\ 3), \\ \mathcal{B}(s_1 s_2) &= \mathcal{B}(231) = (2, 2\ 3), & \mathcal{B}(s_1 s_2 s_1) &= \mathcal{B}(321) = (3, 2\ 3). \end{aligned}$$

According to Theorem 1, the symmetric group Sym_3 is isomorphic to $\mathcal{A}(P)$. The Hasse diagram of $\mathcal{A}(P)$ with respect to the Gale order is given in figure 2, which is, by Theorem 1, also the Hasse diagram of the Bruhat order on Sym_3. Recall that, for the symmetric group, a permutation π covers a permutation σ in the Bruhat order if π is obtained from σ by an inversion that interchanges $\sigma(i)$ and $\sigma(j)$ for some $i < j$ with $\sigma(i) < \sigma(j)$.

Example 5 The Hasse diagram below shows the Bruhat order on the 20 elements of W/P_3, where $W = H_3$, the symmetry group of the icosahedron. Using the bijection of Theorem 1, the elements of W/P_3 have been labeled by their Q-bases in $\mathcal{A}(P_3)$, where $Q = P_1$. The Bruhat order on W/Q in this case is not linear:

$$1 \prec 2 \prec 3 \prec 4 \prec 5 \prec \begin{matrix} 6 \\ \\ 6^* \end{matrix} \prec 5^* \prec 4^* \prec 3^* \prec 2^* \prec 1^*.$$

(The * denotes the antopodal vertex if the elements of W/Q are viewed, via Proposition 1, as the 12 vertices of the icosahedron.) Nevertheless, it is easy to check that figure 3 is also the Hasse diagram for the Gale order on $\mathcal{A}(P_3)$. So H_3/P_3 and $\mathcal{A}(P_3)$ are isomorphic

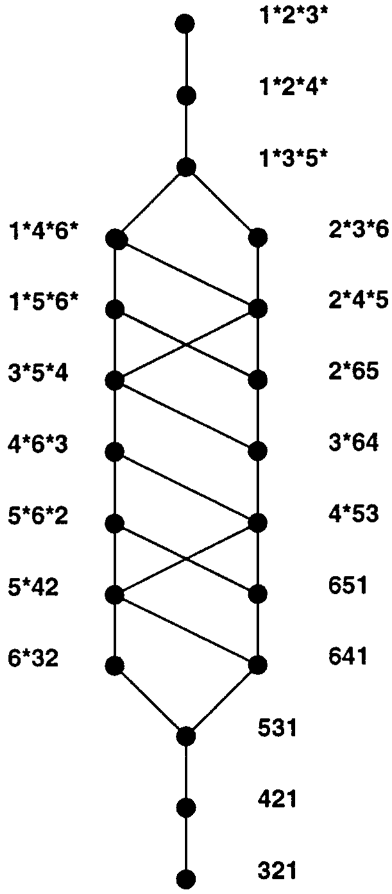

Figure 3. Gale order on $\mathcal{A}(P)$ is Bruhat order on H_3/P.

posets, although Theorem 1 only guarantees that there is a monotone bijection from H_3/P_3 to $\mathcal{A}(P_3)$. The next remark shows that it is not always the case that the Bruhat order on W/P is the Gale order on $\mathcal{A}(P, Q)$.

Remark The basis map $\mathcal{B}$ of Theorem 1 is not, in general, a poset isomorphism. For example, consider the orthogonal case (Example 3 in Sections 2 and 3) where $W = D_4$. Let $P = P_2$ and $Q = P_1$ (node 2 is the node of degree 3 in the Coxeter diagram of figure 1). The Bruhat order on W/Q is not a linear order:

$$1 \prec 2 \prec 3 \prec \begin{matrix} 4 \\ \\ 4^* \end{matrix} \prec 3^* \prec 2^* \prec 1^*.$$

The basis map $\mathcal{B}$ is a bijection between W/P and all two elements subsets of W/Q not consisting of an element and its star. Consider the two cosets uP and vP, where u and v are expressed in terms of standard generators: $u = s_2 s_1 s_4 s_2$ and $v = s_4 s_2 s_1 s_3 s_2$. Both u and v are minimal representatives of their respective cosets, and they are incomparable in the Bruhat order on W; hence uP and vP are incomparable in the Bruhat order on W/P. On the other hand $\mathcal{B}(uP) = 3\ 4$ and $\mathcal{B}(vP) = 4\ 3^*$. But $3\ 4$ is less than $4\ 3^*$ in the Gale order.

4. Admissible orders and admissible functions

Let (W, S) be a Coxeter system, P and Q parabolic subgroups, and $\mathcal{A}(P, Q)$ the corresponding collection of admissible sets. A *weight function* $\phi : W/Q \to \mathbb{R}$ is said to be *compatible* with the Gale order on W/Q if $\bar{v} \succ \bar{u}$ implies that $\phi(\bar{v}) > \phi(\bar{u})$ for any $\bar{u}, \bar{v} \in W/Q$. A function $f : \mathcal{A}(P, Q) \to \mathbb{R}$ is called a *Q-linear function* if it is of the form

$$f(B) = \sum_{i=1}^{m} c_i \sum_{b \in B_i} \phi(b),$$

where $B = (B_1, B_2, \ldots, B_m)$ and ϕ is compatible with the Gale order on W/Q and $c_i > 0$ for all i. If $c_i = 1$ for all i, then $f(B)$ is simply the *total weight*, the sum of the weights of all the entries in B, counting multiplicity. In particular, if P is maximal, then B is a single set and

$$f(B) = \sum_{b \in B} \phi(b)$$

is (up to a positive constant) the total weight of B.

Define an *admissible order* on the set W/Q of cosets as a w-Bruhat order for some $w \in W$. An *admissible weight* on W/Q is a real valued function $\phi : W/Q \to \mathbb{R}$ that is compatible with some admissible order. A *Q-admissible function* $f : \mathcal{A}(P, Q) \to \mathbb{R}$ is a Q-linear function

$$f(B) = \sum_{i=1}^{m} c_i \sum_{b \in B_i} \phi(b),$$

where ϕ is an admissible weight on W/Q.

In light of the bijection $\mathcal{B} : W/P \to \mathcal{A}(P, Q)$ of Theorem 1, it is appropriate to define a function $f : W/P \to \mathbb{R}$ to be a *Q-admissible function* if the corresponding function $\hat{f} : \mathcal{A}(P, Q) \to \mathbb{R}$ defined by $\hat{f}(A) = f(\mathcal{B}^{-1}(A))$ is Q-admissible. We will usually make no distinction between f and $\hat{f}$.

Given parabolic subgroups P and Q, there is a naturally associated combinatorial optimization problem that is the main topic of the remaining sections of this paper.

Optimization Problem. Given a subset $L \subset \mathcal{A}(P, Q)$ and a Q-admissible function $f : \mathcal{A}(P, Q) \to \mathbb{R}$, find an element of L that maximizes f.

Example 1 (ordinary case) If $W = A_n$ and $Q = P_1$, then recall that $W/Q = [n+1]$ and the Bruhat order on W/Q is $1 \prec 2 \prec \cdots \prec n+1$. If $w \in W$ then, by definition, $i \prec j$ in the w-Bruhat order on W/Q if and only if $w^{-1}(i) \prec w^{-1}(j)$ in the Bruhat order on W/Q. Letting w range over all the elements of W (all permutations of $[n+1]$), we conclude that an admissible order is *any* linear order on the set $[n+1]$. An admissible weight ϕ is therefore any weight function. Considering the case of a maximal parabolic subgroup P_k, a Q-admissible function $f : \mathcal{A}(P_k) \to \mathbb{R}$ is of the form

$$f(B) = \sum_{i=1}^{k} \phi(b_i),$$

where $B = \{b_1, \ldots, b_k\}$.

Example 2 (symplectic case) If $W = C_n$ and $Q = P_1$, then recall that $W/Q = [n] \cup [n]^*$ and the Bruhat order on W/Q is

$$1 \prec 2 \cdots \prec n-1 \prec n \prec n^* \prec (n-1)^* \prec \cdots \prec 2^* \prec 1^*.$$

Because the set $S = \{s_1, \ldots, s_n\}$ of generators of C_n is of the form $s_i = (i, i+1)(i^*, (i+1)^*)$ for $i = 1, \ldots, n-1$, and $s_n = (n, n^*)$, an admissible order is any linear order on $[n] \cup [n]^*$ of the form

$$i_1 \prec i_2 \prec \cdots i_n \prec i_n^* \prec i_{n-1}^* \prec \cdots i_1^*,$$

where the first n elements are starred or unstarred and $i^{**} = i$. Consequently the admissible weight functions include all weights ϕ such that $\phi(i^*) = -\phi(i)$ for each $i \in [n]$. A Q-admissible function $f : \mathcal{A}(P_k) \to \mathbb{R}$ has the same form as in Example 1.

Example 3 (orthogonal case) Likewise, if $W = D_n$, an admissible order is any order on $[n] \cup [n]^*$ of the form:

$$i_1 \prec i_2 \prec \cdots i_{n-1} \prec \begin{matrix} i_n \\ \\ i_n^* \end{matrix} \prec i_{n-1}^* \prec \cdots \prec i_2^* \prec i_1^*.$$

where i_1 through i_n are starred or unstarred and $i^{**} = i$. The admissible weight functions in the orthogonal case are exactly the same as the admissible weight functions in the symplectic case, because a weight function must be compatible with the ordering.

The last result in this section is that a particular function, that will be needed in the next section, is admissible. Consider the realization of a rank n Coxeter group W as a reflection group in n-dimensional Euclidean space. With the notation of Section 2, set

$E' = \mathbb{E} \setminus \cup_{H\in\Sigma} H$, where Σ is the set of all reflecting hyperplanes. Call a vector *regular* if it lies in E'. Let P be a parabolic subgroup of W. Recall that if $\xi \in \Gamma_0(P)$, then $w(\xi)$ depends only on the coset of w in W/P.

Theorem 2 *Let P and Q be parabolic subgroups of W. If $\xi \in \Gamma_0(P)$ and η is regular, then*

$$\begin{aligned} &f_{\xi,\eta} : W/P \to \mathbb{R} \\ &f_{\xi,\eta}(w) = -\langle w\xi, \eta\rangle. \end{aligned} \tag{1}$$

is a Q-admissible function.

Proof: Fix $\zeta \in \Gamma_0(Q)$. Then $w(\zeta)$ depends only on the left coset of Q to which w belongs. With $\bar{w} \in W/Q$, define

$$\phi(\bar{w}) = -\langle w\zeta, \eta\rangle. \tag{2}$$

It follows from Proposition 2 that, if η is regular, then this function $\phi : W/Q \to \mathbb{R}$ is an admissible weight function. It is admissible because it is compatible with the w_0-Bruhat order, where w_0 is the unique element of W such that $w_0^{-1}\eta \in \Gamma_0$.

Choose one nonzero vector on each of the 1-dimensional faces of $\overline{\Gamma_0(P)}$. Denote these by $\xi_1, \ldots, \xi_m$. Then

$$\xi = \sum_{i=1}^{m} c_i \xi_i, \tag{3}$$

where $c_i > 0$ for all i. For each i let P_i denote the maximum parabolic subgroup corresponding to the face ξ_i under the correspondence of Proposition 1; so $P \subseteq P_i$. Let $P_i/Q = \{\bar{u}_1, \ldots \bar{u}_t\}$; this is a Q-basis for P_i. Further let $\alpha = \sum_i u_i(\zeta)$ and let v be an arbitrary element of P_i. Then $v(\alpha) = \sum_i vu_i(\zeta) = \sum_i u_i(\zeta) = \alpha$. This implies that α is fixed by all $v \in P_i$, and therefore $\alpha = k_i\xi_i$ for some constant k_i:

$$k_i\xi_i = \sum_{j=1}^{t} u_j(\zeta). \tag{4}$$

The constant k_i is positive for the following reason. First $\langle \zeta, \xi_i\rangle > 0$ since the two vectors lie in the same closed chamber $\bar{\Gamma}_0$. Similarly $\langle u_j\zeta, \xi_i\rangle > 0$ because u_j holds ξ_i fixed and hence $u_j(\zeta)$ and ξ_i lie in the same closed chamber. Now, by statement (4) we have $k_i\langle \xi_i, \xi_i\rangle = \langle k_i\xi_i, \xi_i\rangle = \sum_{j=1}^{t} \langle u_j(\zeta), \xi_i\rangle$, which implies that $k_i > 0$.

From (1)–(4) we have

$$f_{\xi,\eta}(\bar{w}) = \sum_{i=1}^{m} \frac{c_i}{k_i} \sum_{j=1}^{t} \phi(w\bar{u}_j).$$

If the Q-basis for $\bar{w}$ is $B = (B_1, B_2, \ldots, B_m)$, then, by the definition of Q-basis, $\{w\bar{u} \mid u \in P_i/Q\} = B_i$, and hence

$$f_{\xi,\eta}(\bar{w}) = \sum_{i=1}^{m} \frac{c_i}{k_i} \sum_{b \in B_i} \phi(b),$$

which shows that $f_{\xi,\eta}$ is a Q-admissible function because ϕ is an admissible weight function. $\square$

5. Coxeter matroids

Following [10] and [11], we associate to each finite, irreducible Coxeter group W and parabolic subgroup P objects called Coxeter matroids. Let (W, S) be a finite, irreducible Coxeter system and P a parabolic subgroup of W. A subset $M \subseteq W/P$ is called a *Coxeter matroid* (for W and P) if, for each $w \in W$, there is a unique maximum element in M with respect to the w-Bruhat order. In other words, there is an element $uP \in M$ such that $w^{-1}uP \succeq w^{-1}vP$ for all $vP \in M$.

An ordinary matroid (of rank k) is a special case of a Coxeter matroid, the case where $W = A_n$ and P is the maximal parabolic subgroup P_k. Why this is so will become apparent later in this section. The Coxeter matroids associated with the families of Coxeter groups B_n/C_n and D_n have been termed *symplectic matroids* and *orthogonal matroids*, respectively, by Borovik, Gelfand and White [2].

If Q is also a parabolic subgroup of W, recall that $\mathcal{B}: W/P \to \mathcal{A}(P, Q)$ is the Q-basis map of Theorem 1 that assigns to each element of W/P its Q-basis. The set of elements $\mathcal{B}(M)$ plays an analogous role for a Coxeter matroid M as the set of bases do for an ordinary matroid. Of course this set of bases depends on the choice of Q. The choice $Q = P_1$, the maximal parabolic subgroup corresponding to the first node in the Coxeter diagram, is especially appealing because of the simple structure of the Bruhat order on W/Q, in many cases a linear order. If $B = (B_1, B_2, \ldots, B_m)$ is the Q-basis for some element of W/P and $A_i \subseteq B_i$ for each i, then $A = (A_1, A_2, \ldots, A_m)$ is called a *Q-independent set*. The number of sets A_i to which an element $x \in W/Q$ belongs is called the *multiplicity* of x in A. If $L \subset W/P$, denote by $\mathcal{I}(L)$ the collection of Q-independent sets of elements in L.

Recall the optimization problem introduced in the previous section.

Optimization Problem. Given a subset $L \subset W/P$ and a Q-admissible function $f : W/P \to \mathbb{R}$, find an element of L that maximizes f.

Theorem 3 below states that, if L is a Coxeter matroid, then there is a natural greedy algorithm that correctly solves the optimization problem. Indeed, if the Bruhat order on W/Q is a linear order, then the Coxeter matroids are characterized by the property that the greedy algorithm solves this optimization problem. The greedy algorithm proceeds in terms of the Q-bases for the elements of L rather than the cosets themselves. The algorithm returns the Q-basis for the element of L that maximizes f. Recall that, since f is Q-admissible, there is a correesponding admissible weight function on W/Q.

Greedy Algorithm.

initialize $I = (A_1, \ldots, A_m)$ to $(\emptyset, \ldots, \emptyset)$.
initialize X to W/Q.
while

there exists an $x \in X$ and an $I' = (A'_1, \ldots, A'_m) \in \mathcal{I}(L)$ such that $I' \neq I$ and, for each i, either $A'_i = A_i$ or $A'_i = A_i \cup \{x\}$,

do

From all such pairs (x, I') choose the one(s) for which x has largest weight. From all the pairs above choose one (x, I') for which x has largest multiplicity in I'.
Replace I by I'.
Remove x from X.

Note that if P is maximal in W, then there is no multiplicity of entries in I' because I' consists of a single set. In this case the Greedy Algorithm takes the simple form given in Section 1.

Example Consider the case $W = A_2$ with P the trivial parabolic subgroup. The Bruhat order on $W = W/P$ is shown in figure 2 in terms of the P_1-bases. Take as admissible weight $\phi(1) = 1$; $\phi(2) = 3$; $\phi(3) = 4$ and as admissible function $f(w) = \phi(a) + \phi(b_1) + \phi(b_2)$, where $(\{a\}, \{b_1, b_2\})$ is the basis of w. Let L be the Coxeter matroid with bases $\{(2, 23), (2, 12), (1, 13), (1, 12)\}$. The greedy algorithm maximizes f in two steps:

$$I = \emptyset$$
$$I = (\cdot, 3)$$
$$I = (2, 23).$$

On the other hand $L = \{(3, 13), (2, 23)\}$ is not a Coxeter matroid. Using the same admissible function, the greedy algorithm returns $(3, 13)$, although $f(3, 13) = 9 < 10 = f(2, 23)$.

For parabolic subgroups P and Q of a Coxeter group W, the w-Gale order on the collection $\mathcal{A}(P, Q)$ of admissible sets is defined in an analogous manner as the Gale order on $\mathcal{A}(P, Q)$. Consider the poset W/Q with respect to w-Bruhat order. Since the individual entries in $\mathcal{A}(P, Q)$ lie in this poset, $\mathcal{A}(P, Q)$ is itself a poset with respect to the corresponding Gale order given in Definition 6. This Gale order is called the *w-Gale order* on $\mathcal{A}(P, Q)$. If $L \subset \mathcal{A}(P, Q)$ and $w \in W$, then $B \in L$ is said to be a *w-Gale maximum* element of L if $B \succeq_w A$ for all $A \in L$ with respect to the w-Gale order.

Theorem 3 *Let $L \subseteq W/P$, where P is a parabolic subgroup of the finite, irreducible, rank n Coxeter group W. Let Q also be a parabolic subgroup of W. The following statements are equivalent.*

(1) *L is a Coxeter matroid.*
(2) *The set $\mathcal{B}(L)$ of Q-bases has a w-Gale maximum for every $w \in W$.*

(3) *Every Q-admissible function $f : W/P \to \mathbb{R}$ attains a unique maximum on L.*
Moreover any of the statements (1), (2) *or* (3) *implies* (4), *and statement* (4) *implies statements* (1), (2) *and* (3) *if the Bruhat order on W/Q is a linear order.*

(4) *The greedy algorithm solves the optimization problem for any Q-admissible function $f : W/P \to \mathbb{R}$.*

Proof: (1) $\Rightarrow$ (2). Assume that L is a Coxeter matroid and $\mathcal{B}(L)$ its set of Q-bases. Given $w \in W$, let $\bar{v} \in L$ be the unique maximum in L with respect to the w-Bruhat order. Thus $\overline{w^{-1}v} \succeq \overline{w^{-1}u}$ for all $u \in W$. By Theorem 1, this implies that $\mathcal{B}(\overline{w^{-1}v}) \geq \mathcal{B}(\overline{w^{-1}u})$, which, in turn, implies that $\mathcal{B}(\bar{v}) \geq_w \mathcal{B}(\bar{v})$.

(2) $\Rightarrow$ (3). Consider any Q-admissible function $f : W/P \to \mathbb{R}$. Then f has the form

$$f(\bar{v}) = \sum_{i=1}^{m} c_i \sum_{b \in B_i} \phi(b),$$

where $(B_1, B_2, \ldots, B_m) = \mathcal{B}(\bar{v})$ is the Q-basis of $\bar{v}$, $c_i > 0$ for all i, and ϕ is a weight function compatible with the w-Bruhat order on W/Q for some $w \in W$. Let $(A_1, A_2, \ldots, A_m) = \mathcal{B}(\bar{v}_0)$ be the w-Gale maximum Q-basis in $\mathcal{B}(L)$, and let $(B_1, B_2, \ldots, B_m)$ be any other Q-basis in $\mathcal{B}(L)$. Then, for each i, the elements of $A_i = \{a_{ij}\}$ and $B_i = \{b_{ij}\}$ can be arranged so that $a_{ij} \succeq_w b_{ij}$. For at least one pair (i_0, j_0) the above inequality is strict. By the compatibility of ϕ, we have $\phi(a_{ij}) \geq \phi(b_{ij})$ for all i, j and $\phi(a_{i_0 j_0}) > \phi(b_{i_0 j_0})$. Hence the function f attains a unique maximum on $\mathcal{B}(L)$ at $(A_1, A_2, \ldots, A_m)$ and hence, by the bijection of Theorem 1, a unique maximum at $\bar{v}_0$ on L.

(2) $\Rightarrow$ (4). By the paragraph above, the solution to the optimization problem is the Gale maximum $(A_1, A_2, \ldots, A_m)$. We claim that the greedy algorithm finds this Gale maximum. To see this, replace each element $(B_1, B_2, \ldots, B_m)$ of $\mathcal{B}(L)$ by the multiset B that is the concatenation of the sets in $(B_1, B_2, \ldots, B_m)$. (For example, replace (1, 12) by (112).) Call the resulting collection $\mathcal{B}'(L)$. An independent set in $\mathcal{B}(L)$ can be considered as just a multisubset of such a multiset in $\mathcal{B}'(L)$. Since $(A_1, A_2, \ldots, A_m)$ is the unique w-Gale maximum of $\mathcal{B}(L)$, it is easy to check that its concatenation A is the unique w-Gale maximum of $\mathcal{B}'(L)$. So, to simplify the exposition we now consider the greedy algorithm on $\mathcal{B}'(L)$ instead of on $\mathcal{B}(L)$.

To prove the claim let $A = \{a_1, \ldots, a_k\}$ be the w-Gale maximum, and assume that the greedy algorithm has output $B = \{b_1, \ldots, b_j\}$, $j \leq k$, on termination. Because A is the w-Gale maximum, the elements of A and B can be assumed ordered such that $a_i \succeq b_i$, $i = 1, \ldots, j$ and such that, in the greedy algorithm, b_1 is chosen before b_2 is chosen before b_3, etc. (In case of a repeated entry, they are assumed chosen consecutively. For simplicity we denote $\succeq_w$ by $\succeq$.) Because the algorithm is greedy, it must be the case that $\phi(b_1) \geq \phi(a_1)$. If $a_1 \succ b_1$, then, by compatibility, $\phi(a_1) > \phi(b_1)$, a contradiction. Because $a_1 \succeq b_1$ we have $a_1 = b_1$. Proceeding by induction, assume that $a_i = b_i$ for $i = 1, \ldots, m-1 < j$. The same argument just used implies that $a_m = b_m$. Therefore $a_i = b_i$, $i = 1, \ldots, j$. Also $j = k$; otherwise the greedy algorithm could continue by choosing, for example, $b_{j+1} = a_{j+1}$.

(3) $\Rightarrow$ (1). Assume that L is not a Coxeter matroid. To provide a Q-admissible function f which does not have a unique maximum on L, fix $\xi \in \Gamma_0(P)$. Part (4) of Theorem 3

in [15] states that if L is not a Coxeter matroid, then there exists a regular η such that $f(\bar{w}) = -\langle w\xi, \eta\rangle$ attains its maximum (or minimum) on L on at least two points. But, by Theorem 2 of Section 4, this function f is Q-admissible.

(4) $\Rightarrow$ (1). Assume that L is not a Coxeter matroid. By the paragraph above, there is a Q-admissible function

$$f(\bar{w}) = \sum_{i=1}^{m} c_i \sum_{d\in D_i} \phi(d),$$

where $(D_1, D_2, \ldots, D_m)$ is the Q-basis of $\bar{w}$ and ϕ is a Q-admissible weight, and such that f has at least two maxima on L. Assume that the greedy algorithm returns an element $\bar{u} \in L$ and that $\bar{v} \neq \bar{u}$ is one of the maxima of f on L. Let $(B_1, \ldots, B_m)$ be the Q-basis of $\bar{u}$ and $(A_1, \ldots, A_m)$ the Q-basis of $\bar{v}$. It is impossible that $\sum_{b\in B_i} \phi(b) \geq \sum_{a\in A_i} \phi(a)$ for all i with at least one inequality strict because Q-basis $(A_1, \ldots, A_m)$ is maximal. Therefore there are just two cases.

Case 1. If there exists a j such that $\sum_{a\in A_j} \phi(a) > \sum_{b\in B_j} \phi(b)$, then consider the Q-admissible function defined by

$$\bar{f}(\bar{w}) = \sum_{i=1}^{m} \bar{c}_i \sum_{b\in D_i} \phi(d),$$

where $\bar{c}_j = 1$ and $\bar{c}_i$ is sufficiently small for $i \neq j$ so that $\bar{f}(\bar{v}) > \bar{f}(\bar{u})$. The greedy algorithm finds $\bar{u}$, which cannot be the maximum of this admissible $\bar{f}$ on L. Hence the greedy algorithm fails.

Case 2. Assume that $\sum_{a\in A_i} \phi(a) = \sum_{b\in B_i} \phi(b)$ for all i. Since $\bar{u} \neq \bar{v}$, the Q-bases of $\bar{u}$ and $\bar{v}$ are also unequal, and hence there exists a j and an $a \in A_j$ such that $a \notin B_j$. Consider the weight $\bar{\phi}$ on W/Q defined by $\bar{\phi}(\alpha) = \phi(\alpha)$ for all $\alpha \neq a$, and $\bar{\phi}(a) = \phi(a) + \epsilon$, where ϵ is sufficiently small so that $\bar{\phi}$ remains admissible and so that the greedy algorithm applied to $\bar{\phi}$ chooses elements in the same order as the greedy algorithm applied to ϕ. Note that, if some other element has exactly the same weight as a, then, no matter how small ϵ is chosen, it may not be possible to satisfy this last condition. Here is where the linearity of the Bruhat order on W/Q is used. Since ϕ is compatible with this Bruhat order, no two distinct elements can have the same weight. Now consider the Q-admissible function defined by

$$\bar{f}(\bar{w}) = \sum_{i=1}^{m} \bar{c}_i \sum_{b\in D_i} \bar{\phi}(d),$$

where $\bar{c}_j = c_j$. If $\bar{c}_i$, $i \neq j$, is sufficiently small and ϵ is sufficiently small, then it remains the case that $\bar{f}(\bar{v}) > \bar{f}(\bar{u})$. Again the greedy algorithm fails for this Q-admissible function $\bar{f}$ on L. $\square$

Remark In general, it is not true that statement (4) implies statements (1–3) in Theorem 3. As a simple example, consider the orthogonal case where $W = D_4$ (see Example 3 in Sections 2 and 3). Let $P = Q = P_1$. Then there are eight admissible sets corresponding to the eight elements of W/P, each admissible set consisting of a single element: $1 \prec 2 \prec 3 \prec 4,\ 4^* \prec 3^* \prec 2^* \prec 1^*$. Note that the Bruhat order on W/Q is not linear; in particular 4 and 4^* are incomparable. A Q-admissible function is simply a weight function ϕ on the set $\mathcal{A} = \{1, 2, 3, 4, 4^*, 3^*, 2^*, 1^*\}$ satisfying the required compatibility condition. Now let $L = \{4, 4^*\}$; clearly L is not an orthogonal matroid because, as mentioned, 4 and 4^* are incomparable in the Bruhat order on W/P. On the other hand, for any function $\phi : \mathcal{A} \to \mathbb{R}$, the greedy algorithm will surely pick an element from the two in the set $\{4, 4^*\}$ for which ϕ attains a maximum.

It is worthwhile considering some particular Coxeter matroids and the corresponding optimization problems. In all these examples, we take $Q = P_1$.

Ordinary matroids. Let $W = A_n$ and let P_k be the maximal parabolic subgroup generated by $\langle s_i \mid i \neq k \rangle$. Then, according to Theorem 3 and the definition of matroid in Section 1, a Coxeter matroid $M \subseteq W/P_k$ is an ordinary rank k matroid, each basis B being a k-element subset of $[n + 1]$. The relevant objective function, whose maximum is found by the greedy algorithm, is

$$\sum_{b \in B} \phi(b),$$

where ϕ is any function on $[n + 1]$ taking distinct values. For ordinary matroids the optimization problem and the corresponding greedy algorithm are essentially the classical ones discussed in Section 1. As stated in the introduction the optimization problem seeks a maximum independent set; in this section the optimization problem seeks a maximum basis. If all weights are positive, then the two problems coincide.

Flag matroids. Let $W = A_n$ and let P be an arbitrary parabolic subgroup generated by $\langle s_i \mid i \neq i_1, i_2, \ldots, i_m \rangle$. Then an admissible set is of the form $B = (B_1, B_2, \ldots, B_m)$, where $B_1 \subset B_2 \subset \cdots \subset B_m$ and $|B_j| = i_j$ for each j. Letting $A_1 = B_1$ and $A_i = B_i \setminus B_{i-1}$ for $i > 1$, the relevant objective function takes the form

$$\sum_{i=1}^{m} c_i \sum_{a \in A_i} \phi(a),$$

where $c_1 > c_2 > \cdots > c_m > 0$. The terminology *flag matroid* for a such a Coxeter matroid $M \subset W/P$ appears in [3], where the characterization below is given. If $\mathcal{B}$ is the set of bases, each basis of the form $B = (B_1, B_2, \ldots, B_m)$, let $\mathcal{B}_i = \{B_i \mid B \in \mathcal{B}\}$.

Theorem 4 *$\mathcal{B}$ is the set of bases of a flag matroid M if and only if $\mathcal{B}_i$ is the set of bases of an ordinary matroid M_i and each closed set in M_i is closed in M_{i+1}.*

Gauss greedoids. As a special case of the previous example, consider the flag matroids where $i_1 = 1, i_2 = 2, \ldots, i_m = m$. Then, with notation as above, $|A_i| = 1$; say $A_i = \{a_i\}$ for $i = 1, \ldots, m$. In this case the objective function reduces to

$$\sum_{i=1}^{m} c_i \phi(a_i),$$

where $c_1 > c_2 > \cdots > c_m > 0$. Note that it $m = n$, i.e. the parabolic subgroup P is trivial and $W/P = A_n$ then $a_1 a_2 \cdots a_n a_{n+1}$, (the remaining element a_{n+1} of $[n+1]$ is tacked on at the end) is a permutation of $[n+1]$. This gives the isomorphism $A_n \approx \mathrm{Sym}_{n+1}$. These special flag matroids were introduced as Gauss greedoids because they are greedoids with a connection to the Gaussian elimination process [13]. In general, greedoids are not a special case of Coxeter matroids, the relevant objection function for a greedoid being a generalized bottleneck function rather than a linear function.

Symplectic matroids. Let $W = C_n$ and let $P = P_k$, the maximal parabolic subgroup generated by $\langle s_i \mid i \neq k \rangle$. A Coxeter matroid in this case is called a rank k *symplectic matroid* [2]. The terminology comes from the fact that some of these Coxeter matroids can be constructed from the totally isotropic subspaces of a symplectic space. The Q-admissible functions are discussed in Section 4. The admissible sets are k-element subsets B of $[n] \cup [n]^*$ with the property that $B \cap B^* = \emptyset$. The relevant objective function is

$$\sum_{b \in B} \phi(b),$$

where ϕ is any function on $[n] \cup [n]^*$ that takes distinct values and such that $\phi(b^*) = -\phi(b)$.

Lagrangian matroids. When $W = C_n$ and $P = P_n$ we have a special case of a symplectic matroid, the rank n case. These matroids were first introduced as symmetric matroids by A. Bouchet [6] outside the context of Coxeter matroids. They are referred to as Lagrangian (symplectic) matroids in [2]. Bouchet gives several characterizations of these matroids in addition to the characterization in terms of the greedy algorithm.

6. The *L*-assignment problem

In this section the theory in Section 5 is applied to the L-assignment problem. Let W be a finite group acting as linear transformations on a Euclidean space $\mathbb{E}$, and let

$$f_{\xi,\eta}(w) = \langle w\xi, \eta \rangle \quad \text{for } \xi, \eta \in \mathbb{E},\ w \in W.$$

The *L-assignment problem* is to minimize the function $f_{\xi,\eta}$ on a given subset $L \subseteq W$. In [1], it was shown that the L-assignment problem for $W = A_n$ is, in general, *NP*-hard. The same is probably true for the other infinite families of Coxeter groups. Corollary 1 below, however, states that for Coxeter matroids, the greedy algorithm of Section 5 correctly solves the L-assignment problem.

Assume that a rank n Coxeter group W is acting as a reflection group on Euclidean space $\mathbb{E}$ of dimension n. Let $\eta \in \mathbb{E}$ be regular and $\xi \neq 0$. Recall that $P = Stab_W\, \xi$ is a parabolic subgroup of W and, without loss of generality, can be considered a standard parabolic subgroup. Moreover, $f_{\xi,\eta}$ is constant on each left coset wP. Therefore $f_{\xi,\eta}$ is actually a function on the set W/P of left cosets. Let $L \subseteq W/P$. The L-assignment problem is then to find an *optimum* $\bar{w}_0 \in L$ with respect to the pair (ξ, η):

$$f(w_0) = \min_{\bar{w} \in L} f_{\xi,\eta}(w).$$

For $L \subseteq W/P$ denote by $\Delta_{\xi,L}$ the convex hull of the set $L\xi = \{w\xi \mid \bar{w} \in L\}$. The L-assignments problem is equivalent to the problem of finding a vertex ξ_0 of the convex polytope $\Delta_{\xi,L}$ for which the linear function $\varphi_\eta(\xi) = \langle \xi, \eta \rangle$ achieves a minimum.

Given a parabolic subgroup Q and $\zeta \in \Gamma_0(Q)$, recall from Proposition 2 that the function

$$\begin{aligned} &\phi_Q : W/Q \to \mathbb{R} \\ &\phi_Q(\bar{w}) = -\langle w\zeta, \eta \rangle \end{aligned}$$

is an admissible weight function. The greedy algorithm of the previous section applies to the L-assignment problem as follows.

Corollary 1 *Let W be a finite, irreducible, rank n Coxeter group acting on n-dimensional Euclidean space $\mathbb{E}$. Let $\eta, \xi \in \mathbb{E}$ with $\xi \neq 0$ and η regular, and let P be the parabolic subgroup $\mathrm{Stab}_W\, \xi$. If $L \subseteq W/P$ is a Coxeter matroid, then the L-assignment problem for $f_{\xi,\eta}$ has a unique solution. Moreover, if Q is also a parabolic subgroup, then the greedy algorithm applied to the weight function ϕ_Q correctly finds the optimum for every such function $f_{\xi,\eta}$.*

Proof: By Theorem 2 the function $-f_{\xi,\eta}$ is Q-admissible. Therefore, by Theorem 3, if L is a Coxeter matroid, then $-f_{\xi,\eta}$ has a unique maximum and hence $f_{\xi,\eta}$ a unique minimum. Also by Theorem 3, the greedy algorithm correctly maximizes $-f_{\xi,\eta}$, hence minimizes $f_{\xi,\eta}$. □

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