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Iterated Homology of Simplicial Complexes

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Abstract. We develop an iterated homology theory for simplicial complexes. This theory is a variation on one due to Kalai. For Δ a simplicial complex of dimension $d-1$, and each $r = 0, \ldots, d$, we define *rth iterated homology groups* of Δ. When $r = 0$, this corresponds to ordinary homology. If Δ is a cone over Δ', then when $r = 1$, we get the homology of Δ'. If a simplicial complex is (nonpure) shellable, then its iterated Betti numbers give the restriction numbers, $h_{k,j}$, of the shelling. Iterated Betti numbers are preserved by algebraic shifting, and may be interpreted combinatorially in terms of the algebraically shifted complex in several ways. In addition, the depth of a simplicial complex can be characterized in terms of its iterated Betti numbers.

Keywords: shellability, algebraic shifting, depth, Betti numbers, simplicial complex

1. Introduction

Let $\Delta = v * \Delta'$ be a cone over the simplicial complex Δ'. Then Δ is acyclic, i.e., all of its reduced homology vanishes, and thus any information about the reduced homology of Δ' is lost. Iterated homology is a way to algebraically recover the reduced homology of Δ' from Δ. The first iterated homology of Δ is just the ordinary homology of Δ' and subsequent iterates are gotten by "deconing" Δ'. If the complex is a "near-cone," which is almost a cone, then this deconing process makes sense. For an arbitrary complex Γ, the idea is to algebraically transform Γ into a near-cone, and then iterate the deconing process. The "zeroth" iterated homology of Γ is just the ordinary homology, and the iterates provide a combinatorial generalization of homology. However, iterated homology is not topological; that is, there are complexes with the same topological realization that have different iterated homology. The iterated homology theory that we present here is a variation on one due to Kalai [12], and we were heavily influenced by his work.

A simplicial complex is called **pure** if all of its facets have the same dimension. A pure simplicial complex is **shellable** if it can be assembled, facet by facet, in a nice way (see §5).

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Björner and Wachs [5, 6] extended the definition of shellability to include complexes that are not pure. They showed that many interesting and important nonpure complexes are shellable. In addition, they introduced a triangle of restriction numbers $h_{k,j}(\Gamma)$ $(j \leq k)$ of a shelling of Γ. When Γ is pure $(d-1)$-dimensional, the numbers $h_{d,j}(\Gamma)$ correspond to $h_j(\Gamma)$, the ordinary restriction numbers of a shelling of Γ. In the pure shellable case, it is a basic result that $\beta_{d-1}(\Gamma) = h_d(\Gamma)$, and $\beta_i(\Gamma) = 0$ for $i < d-1$. Björner and Wachs generalized this result to nonpure shellable complexes, showing that for each k, $\beta_{k-1}(\Gamma) = h_{k,k}(\Gamma)$. In this paper, we extend their result to the entire h-triangle, showing that $\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Gamma)$, where $\beta^{k-1}[r](\Gamma)$ denotes the $(k-1)$-dimensional rth iterated Betti number of Γ.

We use the method of algebraic shifting to transform Γ into a new complex $\Delta(\Gamma)$ that is much easier to work with. A full definition is in §3.

We summarize the main results in the following theorems.

Theorem 1.1 *Let Γ be a simplicial complex, and let $\Delta(\Gamma)$ denote the algebraically shifted complex obtained from Γ. Then*

$$\begin{aligned}\beta^{k-1}[r](\Gamma) &= \beta^{k-1}[r](\Delta(\Gamma))\\ &= h_{k,k-r}(\Delta(\Gamma))\\ &= \#\{\text{facets } F \in \Delta(\Gamma) : |F| = k,\ \operatorname{init}(F) = r\}.\end{aligned}$$

Proof: Theorem 4.1, Corollary 4.2, and Theorem 5.4. □

This theorem says that the iterated Betti numbers remain invariant under the operation of algebraic shifting and that they can be described combinatorially in terms of the algebraically shifted complex.

Theorem 1.2 *If Γ is a shellable simplicial complex, and $\Delta(\Gamma)$ denotes the algebraically shifted complex obtained from Γ, then*

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Gamma) = h_{k,k-r}(\Delta(\Gamma)).$$

Proof: Theorems 5.4 and 5.7, and Corollary 5.8. □

In other words, when Γ is shellable, then the h-triangle remains invariant under the operation of algebraic shifting. Moreover, the iterated Betti numbers can be computed directly from the shelling of Γ itself.

In §§2–3, we present background material on shifted complexes, near-cones, and algebraic shifting. We also show that shifted complexes are "iterated near-cones," extending a result of Björner and Kalai.

We define iterated homology in §4, and prove basic results. We also show that our definition of iterated homology is distinct from Kalai's, and that iterated homology is not topological. In §5, we discuss generalized or nonpure shelling, and complete the proofs of Theorems 1.1 and 1.2.

In §6, we show how the depth of Γ can be described in terms of its iterated Betti numbers.

2. Shifted complexes and near-cones

We start with basic definitions that are used throughout this paper. Let Δ be a finite (abstract) simplicial complex. We allow the possibility that Δ is the empty simplicial complex $\emptyset$ consisting of no faces, or the simplicial complex $\{\emptyset\}$ consisting of just the empty face, but we do distinguish between these two cases. The **dimension** of $F \in \Delta$ is $\dim F = |F| - 1$, and the **dimension** of Δ is $\dim \Delta = \max\{\dim F : F \in \Delta\}$. The maximal faces of Δ are called **facets**, and Δ is **pure** if all the facets have the same dimension. Let Δ_k denote the set of k-faces (i.e., k-dimensional faces) of Δ. The **f-vector** of Δ is the sequence $(f_0, \ldots, f_{d-1})$, where $f_k = \#\Delta_k$ and $d - 1 = \dim(\Delta)$. The same notion of $f_k(\Delta)$ and the f-vector will apply to every finite collection of sets.

We call $\beta_i(\Delta) = \dim_K \tilde{H}^i(\Delta; K)$ the **ith reduced Betti number** of Δ with respect to the field K, where $\tilde{H}^i(\Delta; K)$ is the ith reduced cohomology group with respect to i. The **Betti sequence** of Δ is $\beta(\Delta) = (\beta_0, \ldots, \beta_{d-1})$. Recall that over a field $\dim_K \tilde{H}^i(\Delta; K) = \dim_K \tilde{H}_i(\Delta; K)$, so that the Betti sequence measures reduced homology as well as reduced cohomology of Δ.

Let $[r] = \{1, 2, \ldots, r\}$, for any $r \geq 1$, and let $[0] = \emptyset$.

Definition If $S = \{i_1 < \cdots < i_k\}$ and $T = \{j_i < \cdots < j_k\}$ are k-subsets of integers, then:

1. $S \leq_P T$ under the **componentwise partial order** if $i_p \leq j_p$ for all p.
2. $S <_L T$ under the **lexicographic order** if there is a q such that $i_q < j_q$ and $i_p = j_p$ for $p < q$.

Lexicographic order is a total order which refines the componentwise partial order.

Definition A collection $\mathcal{C}$ of k-subsets of integers is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex Δ with vertices labelled by distinct integers is **shifted** if Δ_k is shifted for every k.

Shifted complexes are central to the development of iterated homology. We will need the following lemma in §4 and §5.

Lemma 2.1 *Let F be a face of a shifted complex Δ. If $[r] \subseteq F$, but $F \cup \{r+1\} \notin \Delta$, then F is a facet of Δ.*

Proof: Assume that F is not maximal; i.e., assume there is some j such that $j \notin F$ and $F \cup \{j\} \in \Delta$. Then $j \geq r + 1$, so, since Δ is shifted, $F \cup \{r+1\} \in \Delta$, which is a contradiction. $\square$

Björner and Kalai showed in [4] that shifted complexes are near-cones, defined below.

Definition A **near-cone** with **apex** v_0 is a simplicial complex Δ satisfying the following property: For each face $F \in \Delta$, if $v_0 \notin F$ and $w \in F$ then

$$(F - \{w\}) \cup \{v_0\} \in \Delta. \tag{1}$$

For every near-cone Δ with apex v_0, let

$$B(\Delta) = \{F \in \Delta : F \cup \{v_0\} \notin \Delta\},$$

and let

$$\Delta' = \mathrm{lk}_\Delta(v_0) = \{F \in \Delta : v_0 \notin F,\ F \cup \{v_0\} \in \Delta\}. \tag{2}$$

If $B(\Delta) = \emptyset$, then Δ is a **cone**.

It follows from the definition of Δ' and $B(\Delta)$ that

$$\Delta = (v_0 * \Delta') \mathbin{\dot\cup} B(\Delta), \tag{3}$$

where $*$ denotes topological join (so $v_0 * \Delta' = \Delta' \mathbin{\dot\cup} \{F \mathbin{\dot\cup} \{v_0\} : F \in \Delta'\}$). Both Δ' and $\Delta' \mathbin{\dot\cup} B(\Delta)$ are subcomplexes of Δ. Furthermore, every $F \in B(\Delta)$ is maximal in Δ, so the collection of subsets in $B(\Delta)$ forms an antichain.

We can use Eq. (3) for an alternate definition of a near-cone: Let $\Delta' \mathbin{\dot\cup} B$ be a simplicial complex such that B is a set of maximal faces in $\Delta' \mathbin{\dot\cup} B$ (so Δ' is a subcomplex and B is an antichain); then $\Delta = (v_0 * \Delta') \mathbin{\dot\cup} B$ is a near-cone (where v_0 is some new vertex not in $\Delta' \mathbin{\dot\cup} B$).

Note, in particular, that $\emptyset$ and $\{\emptyset\}$ are near-cones and that $\emptyset = v_0 * \emptyset$ and $\{\emptyset\} = (v_0 * \emptyset) \mathbin{\dot\cup} \{\emptyset\}$. If Δ is a near-cone with apex v_0, then v_0 is one of the vertices of Δ, unless $\Delta = \emptyset$ or $\{\emptyset\}$.

For a finite sequence of non-negative integers $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, an α**-wedge of spheres** is the wedge of α_i spheres of dimension i, for each i.

Proposition 2.2 (Björner-Kalai [4, Theorem 4.3]) *Let Δ be a near-cone. Then Δ is homotopy equivalent to the $f(B(\Delta))$-wedge of spheres. In particular,*

$$\beta_k(\Delta) = f_k(B(\Delta)).$$

The observation that a shifted simplicial complex Δ is a near-cone $(1 * \Delta') \mathbin{\dot\cup} B$ is crucial to the results in [4]; equally, the following observations are crucial here.

Proposition 2.3 *If Δ is a non-empty shifted simplicial complex on vertices* $\{1, 2, 3, \ldots, k\}$, *then*

(a) (*Björner-Kalai* [4]) *Δ is a near-cone with apex* 1, *so* $\Delta = (1 * \Delta') \mathbin{\dot\cup} B$;

(b) *Δ' is a shifted simplicial complex on vertices* $\{2, 3, \ldots, k\}$.

Proof: (a) Use the definition of near-cone, Eq. (1), to show that Δ is a near-cone with apex $v_0 = 1$: If $w \in F$, but $1 \notin F$, then $(F - \{w\}) \dot{\cup} \{1\} \le_P F$; therefore $(F - \{w\}) \dot{\cup} \{1\} \in \Delta$, since $F \in \Delta$ and Δ is shifted. (b) To show that Δ' is shifted on $\{2, \ldots, k\}$, assume that $S, T \subseteq \{2, \ldots, k\}$, and that $S \le_P T \in \Delta'$; we must then show $S \in \Delta'$. By the definition of Δ', equation (2), $T \in \Delta'$ means $T \dot{\cup} \{1\} \in \Delta$. Further, $1 \notin S, T$ and $S \le_P T$ imply that $S \dot{\cup} \{1\} \le_P T \dot{\cup} \{1\} \in \Delta$, so, since Δ is shifted, $S \dot{\cup} \{1\} \in \Delta$. Then by Eq. (2) again, $S \in \Delta'$. $\square$

This means, for instance, that if $\Delta = (1 * \Delta') \dot{\cup} B$ is shifted, then $\Delta'' = (2 * \Delta'') \dot{\cup} B_1$ for some B_1 and Δ'', and thus,

$$\Delta = (1 * ((2 * \Delta'') \dot{\cup} B_1)) \dot{\cup} B.$$

More generally, we have the following corollary.

Corollary 2.4 *Let $\Delta = \Delta^{(0)}$ be a shifted simplicial complex of dimension $d - 1$. Then we may inductively define $\Delta^{(r+1)} = (\Delta^{(r)})'$, i.e.,*

$$\Delta^{(r)} = \big((r + 1) * \Delta^{(r+1)}\big) \dot{\cup} B_r \ (0 \le r \le d - 1), \tag{4}$$

for some B_r. Furthermore,

$$\Delta = 1 * (2 * (3 * (\cdots (d - 1) * ((d * B_d) \dot{\cup} B_{d-1}) \dot{\cup} B_{d-2} \cdots) \dot{\cup} B_2) \dot{\cup} B_1) \dot{\cup} B_0, \tag{5}$$

where $B_d = \{\emptyset\} = \Delta^{(d)}$.

Proof: Proposition 2.3 shows, inductively, that $\Delta^{(r)}$ is a near-cone with apex $r + 1$, allowing $\Delta^{(r+1)}$ to be defined by Eq. (4). Equation (5) then follows from iterating Eq. (4). $\square$

By Proposition 2.2, we have

$$\beta_k\big(\Delta^{(r)}\big) = f_k(B_r). \tag{6}$$

Iterated homology will give us an algebraic way to recover these Betti numbers, even when the simplicial complex is not shifted.

Example We illustrate Corollary 2.4 for the shifted complex Δ in figure 1, whose facets are (omitting commas and set brackets): 123, 124, 15, 16, 34, 7. The complexes $\Delta' = \Delta^{(1)}$ and $\Delta'' = \Delta^{(2)}$ are pictured along with Δ in figure 1.

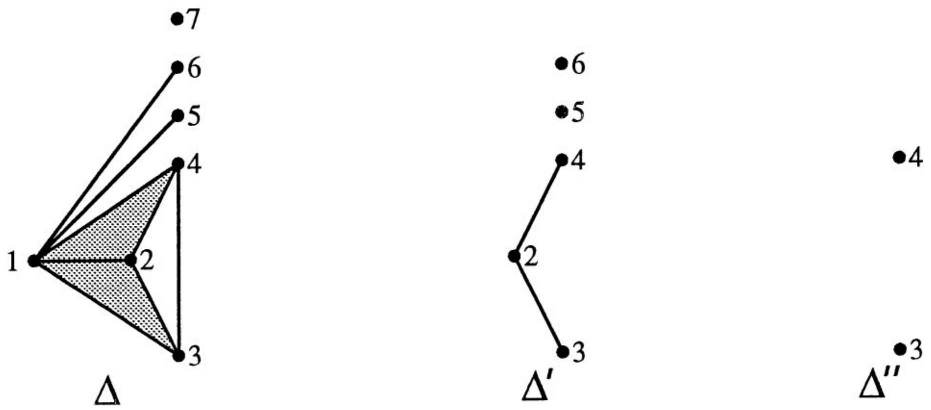


Figure 1. A shifted complex.

r	facets of $\Delta^{(r+1)}$	B_r
3	–	∅
2	∅	4
1	3, 4	5, 6
0	23, 24, 5, 6	34, 7

We tabulate the data $f_k(B_r)$, indexing rows by r and columns by k.

r, k	-1	0	1	2
0	0	1	1	0
1	0	2	0	
2	0	1		
3	1			

3. Algebraic shifting

Algebraic shifting transforms a simplicial complex into a shifted simplicial complex with the same f-vector and Betti numbers. It also preserves many algebraic properties of the original complex. Algebraic shifting was introduced by Kalai in [10]; our exposition is summarized from [4] and included for completeness (see also [3, 12]). We start with the exterior face ring.

Definition Let Γ be a $(d-1)$-dimensional simplicial complex with vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. Let $\Lambda(KV)$ denote the exterior algebra of the vector space KV; it has a K-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $S = \{e_{i_1} < \cdots < e_{i_k}\} \subseteq V$ (and $e_\emptyset = 1$). Note that $\Lambda(KV) = \bigoplus_{k=0}^{n} \Lambda^k(KV)$ is a graded K-algebra, and that $\Lambda^k(KV)$ has basis $\{e_S : |S| = k\}$. Let $(I_\Gamma)_k$ be the subspace of $\Lambda^{k+1}(KV)$ generated by the basis $\{e_S : |S| = k+1,\ S \notin \Gamma\}$. Then $I_\Gamma := \bigoplus_{k=-1}^{d-1}(I_\Gamma)_k$ is the homogeneous graded ideal of $\Lambda(KV)$ generated by $\{e_S : S \notin \Gamma\}$. Let $\Lambda^k[\Gamma] := \Lambda^{k+1}(KV)/(I_\Gamma)_k$. Then the graded quotient algebra $\Lambda[\Gamma] := \bigoplus_{k=-1}^{d-1}\Lambda^k[\Gamma] = \Lambda(KV)/I_\Gamma$ is called the **exterior face ring** of Γ (over K).

The exterior face ring is the exterior algebra analogue to the Stanley-Reisner face ring of a simplicial complex [14, 16]. See [17] and [8] for another use of the exterior face ring. For $x \in KV$, let $\tilde{x}$ denote the image of x in $\Lambda[\Gamma]$. The set of all **face-monomials** $\{\tilde{e}_S : S \in \Gamma\}$ is a K-vector space basis for $\Lambda[\Gamma]$, so $f_k(\Gamma) = \dim_K(\Lambda^k[\Gamma])$.

We can use the exterior face ring to compute cohomology. If $f = \alpha_1 e_1 + \cdots + \alpha_n e_n$, then $\delta_f : \Lambda[\Gamma] \to \Lambda[\Gamma]$ defined by $\delta_f(x) = \tilde{f} \wedge x$ is a **weighted coboundary operator**, so-called because

$$\delta_f(\tilde{e}_S) = \tilde{f} \wedge \tilde{e}_S = \sum_{i=1}^{n} \alpha_i \tilde{e}_i \wedge \tilde{e}_S = \sum_{\substack{i \notin S \\ S\cup\{i\}\in\Gamma}} \pm\alpha_i \tilde{e}_{S\cup\{i\}}.$$

Setting every $\alpha_i = 1$ gives the usual coboundary operator. Ordinary Betti numbers may be computed using weighted coboundary operators: $\beta_{k-1}(\Gamma) = \dim_K(\ker\delta_f)_{k-1}/(\operatorname{im}\delta_f)_{k-1}$, if $f = \alpha_1 e_1 + \cdots + \alpha_n e_n$ and every α_i is non-zero [4, pp. 289–290].

To create a “generic” basis in the following definition, let $\bar{K} = K(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{nn})$ be the field extension over K by n^2 transcendentals, $\{\alpha_{ij}\}_{1\le i,j\le n}$, algebraically independent over K. We will consider $\Lambda[\Gamma]$ as being over $\bar{K}$ instead of K from now on. We are, in effect, simply adjoining these α_{ij}’s to our field of coefficients.

Definition For $1 \le i \le n$, let

$$f_i = \sum_{j=1}^{n} \alpha_{ij} e_j,$$

so $\{f_1, \ldots, f_n\}$ forms a “generic” basis of $\bar{K}V$. Define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_k}$ for $S = \{i_1 < \cdots < i_k\}$ (and set $f_\emptyset = 1$). Let

$$\Delta(\Gamma, K) := \{S \subseteq [n] : \tilde{f}_S \notin \operatorname{span}\{\tilde{f}_R : R <_L S\}\}$$

be the **algebraically shifted complex** obtained from Γ. We will write $\Delta(\Gamma)$ instead of $\Delta(\Gamma, K)$ when the field is understood to be K.

The k-subsets of $\Delta(\Gamma)$ can be chosen by listing all the k-subsets of $[n]$ in lexicographic order and omitting those that are in the span of earlier subsets on the list, modulo I_Γ and with respect to the f-basis.

We collect here the basic facts we need about algebraic shifting.

Proposition 3.1 (Kalai [4, Theorem 3.1]) *Let Γ be a simplicial complex, and let K be a field. Then $\Delta = \Delta(\Gamma, K)$ is a shifted simplicial complex such that*

(a) $f_i(\Gamma) = f_i(\Delta)$ *for* $i \geq 0$,
(b) $\beta_i(\Gamma) = \beta_i(\Delta)$ *for* $i \geq 0$ (*Betti numbers with coefficients in* K),
and Δ *is independent of the numbering of the vertices of* Γ.

Proposition 3.2 (Kalai [11, §4, Remark (4)]) *If* Γ *is shifted*, *then* $\Delta(\Gamma) = \Gamma$.

Corollary 3.3 *Let* Γ *be a simplicial complex. Then* $\Delta(\Delta(\Gamma)) = \Delta(\Gamma)$.

4. Iterated homology

Because $\Delta = \Delta(\Gamma)$ is shifted, we may write $\Delta = (1 * \Delta') \dot{\cup} B$. We wish to find the Betti numbers of Δ' from Γ algebraically, without first constructing Δ. This would in effect extend Proposition 3.1(b) to Δ'.

To simplify notation, we will from now on use $\tilde{f}$ in place of its corresponding coboundary operator $\delta_f = \tilde{f} \wedge \cdot$.

Consider the set $\Delta_1 = \{F \in \Delta : 1 \in F\}$, which has a natural bijection with Δ'. Algebraically, Δ_1 is a basis of the subspace im $\tilde{f}_1$, the space of $\tilde{f}$-monomials that are multiples of $\tilde{f}_1$. (Note that in [17] and [8], Δ' is considered directly, by examining $\Lambda[\Gamma]/\ker \tilde{f}_1$.) We need to find a coboundary operator to compute the cohomology groups of im $\tilde{f}_1$; we cannot use $\tilde{f}_1$, since it annihilates the entire subspace. Fortunately, the $\tilde{f}_i$'s are linearly independent coboundary operators, so we may use $\tilde{f}_2$ as a coboundary operator. Thus, the $(k-1)$st cohomology group of Δ' is given by

$$\left(\ker_{\operatorname{im} \tilde{f}_1} \tilde{f}_2\right) \big/ \left(\operatorname{im}_{\operatorname{im} \tilde{f}_1} \tilde{f}_2\right) = (x \in \tilde{f}_1 \wedge \Lambda^{k-1}[\Gamma] : \tilde{f}_2 \wedge x = 0) / (\tilde{f}_2 \wedge (\tilde{f}_1 \wedge \Lambda^{k-2}[\Gamma])).$$

We continue this process to find the Betti numbers of $\Delta^{(r)}$ $(r \leq d-1)$. Algebraically, $\{F \in \Delta : \{1, \ldots, r\} \subseteq F\}$ (which has a natural bijection with $\Delta^{(r)}$) is a basis of the image of $\tilde{f}_{[r]} = \tilde{f}_1 \wedge \tilde{f}_2 \wedge \cdots \wedge \tilde{f}_r$. To find the Betti numbers of im $\tilde{f}_{[r]}$, we can use the weighted coboundary operator $\tilde{f}_{r+1}$, which is linearly independent of $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_r$. We make the following definitions and notation.

Definition If Γ is a simplicial complex and $0 \leq r \leq k+1 \leq d$, we define

$$\begin{aligned}
\Lambda^k[r](\Gamma) &= \tilde{f}_{[r]} \wedge \Lambda^{k-r}[\Gamma] = \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_r \wedge \Lambda^{k-r}[\Gamma],\\
Z^k[r](\Gamma) &= \{x \in \Lambda^k[r](\Gamma) : \tilde{f}_{r+1} \wedge x = 0\},\\
B^k[r](\Gamma) &= \begin{cases} \tilde{f}_{r+1} \wedge \Lambda^{k-1}[r](\Gamma) & \text{if } r < k+1 \\ 0 & \text{if } r = k+1 \end{cases},\\
H^k[r](\Gamma) &= Z^k[r](\Gamma)/B^k[r](\Gamma).
\end{aligned}$$

Notice that $B^k[r](\Gamma) = \Lambda^k[r+1](\Gamma)$. The $H^k[r](\Gamma)$'s are called the r**th iterated cohomology groups** of Γ. We define the r**th iterated Betti numbers** by

$$\beta^k[r](\Gamma) = \dim H^k[r](\Gamma).$$

The $r = 0$ case is just ordinary reduced cohomology.

Remark Kalai [12] defined another version of iterated cohomology. We distinguish between the two definitions by putting bars over his. Assume $1 \le r \le n$. First let $F_r = \text{span}\{\tilde{f}_1, \ldots, \tilde{f}_r\}$. Then define

$$\bar{Z}^k[r](\Gamma) = \{x \in \Lambda^{k+1}[\Gamma] : \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_r \wedge x = 0\},$$
$$\bar{B}^k[r](\Gamma) = \text{span}\{F_r \wedge \Lambda^k[\Gamma]\},$$

and define $\bar{H}^k[r](\Gamma)$ and $\bar{\beta}^k[r](\Gamma)$ in terms of $\bar{B}^k[r](\Gamma)$ and $\bar{Z}^k[r](\Gamma)$ as above. We show below, following Corollary 4.3, that the two iterated cohomology definitions are different.

Definition Let F be a set of positive integers. Define

$$\begin{aligned}\text{init}(F) &= \min\{r > 0 : r \notin F\} - 1\\ &= \max\{r \ge 0 : [r] \subseteq F\}.\end{aligned}$$

In other words, $\text{init}(F)$ measures the largest "initial segment" in F, and is 0 if there is no initial segment (i.e., $1 \notin F$).

Theorem 4.1 *Let Γ be a simplicial complex, and let $\Delta(\Gamma)$ denote the result of applying algebraic shifting to Γ. Then*

$$\beta^k[r](\Gamma) = \#\{\text{facets } F \in \Delta(\Gamma) : |F| = k+1,\ \text{init}(F) = r\}.$$

Proof: (Very similar to the proof of the $r = 0$ case by Björner and Kalai, Claim 2 in [4, Theorem 3.1].) Let $\Delta = \Delta(\Gamma)$ and let

$$\Delta_k[r] = \{S \in \Delta : |S| = k+1,\ [r] \subseteq S\}.$$

We claim that

$$\Lambda^k[r](\Gamma) = \text{span}\{\tilde{f}_S : S \in \Delta_k[r]\}. \tag{7}$$

First, let $\mathcal{A} = \{S \in \binom{[n]}{k+1} : [r] \subseteq S\}$; since $\mathcal{A}$ is initial with respect to lexicographic ordering, $\{\tilde{f}_S : S \in \Delta_k[r]\}$ is a basis for $\text{span}\{\tilde{f}_S : S \in \mathcal{A}\}$. Now, if $y \in \Lambda^k[r](\Gamma)$, then $y = \tilde{f}_{[r]} \wedge x$ for some $x \in \Lambda^{k-r}[\Gamma]$. Say $x = \sum \gamma_R \tilde{f}_R$; then

$$y = \tilde{f}_{[r]} \wedge x = \sum_{R \cap [r] = \emptyset} \pm \gamma_R \tilde{f}_{R \cup [r]} \in \text{span}\{\tilde{f}_S : S \in \mathcal{A}\} = \text{span}\{\tilde{f}_S : S \in \Delta_k[r]\}.$$

Conversely, if $S \in \Delta_k[r]$, then $\tilde{f}_S = \tilde{f}_{[r]} \wedge \tilde{f}_{S-[r]} \in \Lambda^k[r](\Gamma)$, and Eq. (7) follows.

Now

$$\dim Z^k[r](\Gamma) = \dim \Lambda^k[r](\Gamma) - \dim B^{k+1}[r](\Gamma),$$

and, by definition,

$$B^k[r](\Gamma) = \Lambda^k[r+1](\Gamma);$$

therefore,

$$\begin{aligned}\beta^k[r](\Gamma) &= \dim Z^k[r](\Gamma) - \dim B^k[r](\Gamma)\\ &= (\dim \Lambda^k[r](\Gamma) - \dim B^{k+1}[r](\Gamma)) - \dim B^k[r](\Gamma)\\ &= (\dim \Lambda^k[r](\Gamma) - \dim \Lambda^{k+1}[r+1](\Gamma)) - \dim \Lambda^k[r+1](\Gamma)\\ &= \#\Delta_k[r] - \#\Delta_{k+1}[r+1] - \#\Delta_k[r+1].\end{aligned}$$

Further,

$$\#\Delta_k[r+1] = \#\{S \in \Delta_k[r] : r+1 \in S\},$$

and, via the bijection $S \leftrightarrow S' = S \cup \{r+1\}$,

$$\begin{aligned}\#\Delta_{k+1}[r+1] &= \#\{S' \in \Delta_{k+1}[r] : r+1 \in S'\}\\ &= \#\{S \in \Delta_k[r] : r+1 \notin S,\ S \dot\cup \{r+1\} \in \Delta\},\end{aligned}$$

so

$$\begin{aligned}\beta^k[r](\Gamma) &= \#\Delta_k[r] - \#\{S \in \Delta_k[r] : r+1 \notin S,\ S \dot\cup \{r+1\} \in \Delta\}\\ &\quad - \#\{S \in \Delta_k[r] : r+1 \in S\}\\ &= \#\{S \in \Delta_k[r] : S \cup \{r+1\} \notin \Delta\}.\end{aligned}$$

Finally note that if $[r] \subseteq S$ and $S \cup \{r+1\} \notin \Delta$, then $\text{init}(S) = r$ and, by Lemma 2.1, S must be maximal, completing the proof. □

Corollary 4.2 *Let Γ be a simplicial complex. Then*

$$\beta^k[r](\Gamma) = \beta^k[r](\Delta(\Gamma)).$$

Proof: Using Theorem 4.1 twice and the stability of algebraic shifting (Corollary 3.3),

$$\begin{aligned}\beta^k[r](\Gamma) &= \#\{\text{facets } F \in \Delta(\Gamma) : |F| = k+1,\ \text{init}(F) = r\}\\ &= \#\{\text{facets } F \in \Delta(\Delta(\Gamma)) : |F| = k+1,\ \text{init}(F) = r\}\\ &= \beta^k[r](\Delta(\Gamma)).\end{aligned}$$ □

Corollary 4.3 *Let Γ be a simplicial complex, let $\Delta = \Delta(\Gamma)$ be the result of applying algebraic shifting to Γ, and define $B_1, \ldots, B_d$ as in Corollary 2.4. Then*

$$\beta^{k+r}[r](\Gamma) = f_k(B_r) = \beta^k\big(\Delta^{(r)}\big).$$

Proof: It is easy to see by Eq. (5) that

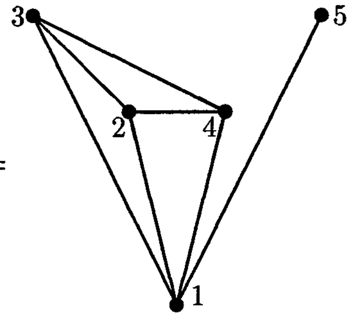
$$f_k(B_r) = \#\{\text{facets } F \in \Delta(\Gamma) : |F| = k+1+r,\ \text{init}(F) = r\}.$$

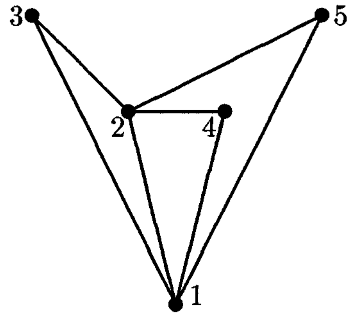
Then apply Theorem 4.1 and Eq. (6). □

Remark We can now show that Kalai's iterated cohomology is different from the one presented here. In [12], Kalai gives the formula

$$\bar{\beta}^k[r](\Gamma) = \#\{F \in \Delta(\Gamma) : |F| = k+1,\ F \cap [r] = \emptyset,\ F \cup [r] \notin \Delta(\Gamma)\}. \tag{8}$$

To see that the definitions are essentially different, consider the following 1-dimensional shifted simplicial complexes:

$\Delta_1 =$ 

$\Delta_2 =$ 

It is easy to check, using Eq. (8), that $\bar{\beta}^1[2](\Delta_1) = 1$ but $\bar{\beta}^1[2](\Delta_2) = 0$; it is also easy to check, using Theorem 4.1, that $\beta^k[r](\Delta_1) = \beta^k[r](\Delta_2)$ for all k, r. These complexes are built by taking a cone over four vertices and adjoining three of the six possible remaining edges in the only two ways to make shifted complexes.

On the other hand, it is not hard to verify that if a simplicial complex is "s-fold acyclic" (i.e., all the rth iterated homology groups vanish for $r = 0, \ldots, s$) under either definition, then it is "s-fold acyclic" under the other definition (both conditions correspond to the algebraically shifted complex Δ being an "s-fold cone", i.e., $\Delta = [s] * \Delta'$ for some Δ').

Remark It is easy to see now that iterated homology is not topological, i.e., that two simplicial complexes whose realizations are homeomorphic need not have the same iterated Betti numbers. Simply take two triangulations of the same space that use different numbers of facets; the sum of the iterated Betti numbers is equal to the number of facets, by Theorem 4.1, so the two triangulations will have different sets of iterated Betti numbers.

5. Iterated homology and non-pure shelling

A simplicial complex is shellable [5, 6] if it can be constructed by adding one facet at a time, so that as each facet F is added, a *unique* new minimal face, called the restriction face $R(F)$, is added. Equivalently, as each facet is added, it intersects the existing complex (previous facets) in a union of codimension 1 faces. We take the following as the formal definition.

Definition (Björner-Wachs [5]) A simplicial complex Γ is **shellable** if there is a map

$R : \{\text{facets of } \Gamma\} \to \Gamma$

called the **restriction map** and an ordering of the facets $F_1, \dots, F_t$ of Γ such that:

$$\Gamma = \bigcup_{1 \le i \le t} [R(F_i), F_i]; \quad \text{and} \tag{9}$$

$$R(F_a) \subseteq F_b \Rightarrow a \le b. \tag{10}$$

Note that condition (10) implies that the union in Eq. (9) is disjoint. The **restriction numbers** are defined by

$$h_{k,j}(\Gamma) = \#\{\text{facets } F : |F| = k,\ |R(F)| = j\}$$

and are independent of the shelling order.

In [5], the numbers $h_{k,j}$ are defined differently, and for all complexes (not just shellable ones). But the $h_{k,j}$'s equal the restriction numbers for shellable complexes [5, Theorem 3.4], and since we are only interested in the $h_{k,j}$'s for shellable complexes, we will use $h_{k,j}$ to denote shelling restriction numbers.

The original definition of shellability also required Γ to be pure; we will refer to this property as **pure shellability**. In [5, 6], Björner and Wachs dropped the assumption of purity, and proved basic results about general shellability.

The restriction numbers of pure shellability are $h_j(\Gamma) = \#\{\text{facets } F : |R(F)| = j\}$, so $h_j(\Gamma) = h_{d,j}(\Gamma)$ for a pure $(d-1)$-dimensional shellable complex. It is well-known that a pure $(d-1)$-dimensional shellable complex has homology only in top dimension (i.e., $\beta_k(\Gamma) = 0$ for $k < d-1$) and that $\beta_{d-1}(\Gamma) = h_d(\Gamma) = h_{d,d}(\Gamma)$. Björner and Wachs extended this to (generalized) shelling, with the following theorem.

Proposition 5.1 (Björner-Wachs [5, Theorem 4.1]) *If Γ is shellable, then*

$$\beta_{k-1}(\Gamma) = h_{k,k}(\Gamma)$$

for any k.

Iterated homology provides an algebraic interpretation of the non-diagonal restriction numbers (i.e., $h_{k,j}(\Gamma)$, where $k \neq j$), generalizing Proposition 5.1. (See Corollary 5.8.)

We collect here other useful facts about shelling.

Proposition 5.2 (Björner-Wachs [5, Theorem 2.6]) *If Γ is a shellable, then there is a shelling $F_1, \dots, F_t$ of Γ such that*

$$a \le b \Rightarrow |F_a| \ge |F_b|.$$

This means that we can always construct a shellable complex using higher-dimensional facets first and lower-dimensional facets last. Recall from §4 that $\text{init}(F)$ measures the largest "initial segment" of a set F.

Proposition 5.3 (Björner-Wachs [6, Corollary 11.4]) *If Δ is shifted, then it is shellable with restriction numbers given by*

$$h_{k,j}(\Delta) = \#\{\text{facets } F \in \Delta : |F| = k,\ \text{init}(F) = k - j\}.$$

Theorem 5.4 *Let Γ be simplicial complex, and let $\Delta(\Gamma)$ denote the result of applying algebraic shifting to Γ. Then*

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Delta(\Gamma)).$$

Proof: Apply Theorem 4.1 and Proposition 5.3. □

Example We illustrate Proposition 5.3 for the shifted complex Δ in figure 1. The shelling order is 123, 124, 15, 16, 34, 7. The restriction faces are given by the following table.

F	123	124	15	16	34	7
$R(F)$	$\emptyset$	4	5	6	34	7

We tabulate the data $h_{k,j}(\Delta)$, indexing rows by k and columns by j.

k, j	0	1	2	3
0	0			
1	0	1		
2	0	2	1	
3	1	1	0	0

The table of $h_{k,j}(\Delta)$ data differs from the table of $f_{k-1}(B_r)$ data at the end of §2 only in whether each column starts in the top row or ends in the bottom row. This is a consequence of Corollary 4.3 and Theorem 5.4, since Δ is shifted:

$$h_{k,k-r}(\Delta) = \beta^{k-1}[r](\Delta) = f_{k-r-1}(B_r).$$

Collapsing is a different kind of decomposition and is closely related to shelling.

Definition (Kalai [12, §4]) A face R of a simplicial complex Γ is called **free** if it is included in a unique facet F. The empty set is a free face of Γ if and only if Γ is a simplex. (This definition is slightly nonstandard in that facets are themselves free.) If $|R| = p$ and $|F| = q$, then we say R is of **type** (p, q). A (p, q)**-collapse step** is the deletion from Γ of a free face of type (p, q) and all faces containing it (i.e., the deletion of the interval $[R, F]$). Performing a collapse step may create new free faces. A **collapsing sequence** is a sequence of collapse steps that reduce Γ to the empty simplicial complex.

The following lemma is implicit in [12, §4]. It is also a special case of [7, Proposition 2.2], where "S-partitions" are used intead of collapsing sequences, but it is easy to see the two concepts are equivalent.

Lemma 5.5 *If a collapsing sequence of a simplicial complex Γ consists of the intervals $[R_t, F_t], \ldots, [R_1, F_1]$, and each F_i is a facet in the original complex Γ, then $F_1, \ldots, F_t$ is a shelling order of Γ. Furthermore, the restriction map of this shelling is given by setting $R(F_i) = R_i$.*

Conversely, if $G_1, \ldots, G_t$ is a shelling order of a simplicial complex Γ, then

$$[R(G_t), G_t], \ldots, [R(G_1), G_1]$$

is a collapsing sequence of Γ.

Proof: The collapsing sequence and definition of $R(F_i)$ give the decomposition of Γ, Eq. (9); each $R(F_i)$ being a free face at the ith collapse establishes condition (10). The important assumption here is that each F_i is a facet; *every* collapsing sequence gives a decomposition that satisfies (9) and (10), but the tops of the intervals are not necessarily facets.

The proof of the converse is similar: Condition (10) ensures that each $R(G_i)$ is free and Eq. (9) shows that the sequence of collapses reduces Γ to the empty complex. $\square$

Proposition 5.6 (Kalai [12, Theorem 4.2]) *If Γ' is obtained from Γ by a collapse step, then $\Delta(\Gamma')$ is obtained from $\Delta(\Gamma)$ by a collapse step of the same type.*

Theorem 5.7 *If Γ is a shellable simplicial complex, and $\Delta(\Gamma)$ denotes the result of applying algebraic shifting to Γ, then*

$$h_{k,j}(\Gamma) = h_{k,j}(\Delta(\Gamma)).$$

Proof: Let the shelling order of Γ be $G_1, \ldots, G_t$. By Proposition 5.5,

$$[R(G_t), G_t], \ldots, [R(G_1), G_1]$$

is a collapsing sequence of Γ, and Proposition 5.6 then implies that $\Delta(\Gamma)$ has a collapsing sequence

$$[R_t, F_t], \ldots, [R_1, F_1]$$

such that

$$|R_i| = |R(G_i)| \quad \text{and} \quad |F_i| = |G_i|$$

for all i. To apply Proposition 5.5 again to show that $\Delta(\Gamma)$ has the desired shelling, we must show that every F_i is a facet in $\Delta(\Gamma)$; it suffices to show that F_a is not contained in F_b for any $a \neq b$.

If $a > b$, then at the collapse step when F_a is removed as a maximal face of the remaining complex, F_b is still present, so $F_a \not\subset F_b$. On the other hand, by Proposition 5.2, we may assume that if $a < b$, then

$$|G_a| \geq |G_b|,$$

so

$$|F_a| = |G_a| \geq |G_b| = |F_b|,$$

and $F_a \not\subset F_b$.

Thus every F_i is a facet, and therefore, by Proposition 5.5, $F_1, \ldots, F_t$ is a shelling order of $\Delta(\Gamma)$ with

$$|R(F_i)| = |R_i| = |R(G_i)| \quad \text{and} \quad |F_i| = |G_i|$$

for all i, so $h_{k,j}(\Delta(\Gamma)) = h_{k,j}(\Gamma)$. □

We can now prove the desired generalization of Proposition 5.1.

Corollary 5.8 *If Γ is a shellable simplicial complex, then*

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Gamma).$$

Proof: By Theorems 5.4 and 5.7,

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Delta(\Gamma)) = h_{k,k-r}(\Gamma). \qquad \square$$

6. Depth

A sequence $(x_1, \ldots, x_k)$ of elements of a ring R is a **regular sequence** on R if each x_i is not a zero divisor on the quotient $R/(x_1, \ldots, x_{i-1})$. The **depth** of a ring is the length of the longest regular sequence on R, and the depth of a simplicial complex Γ is defined to be the depth of $K[\Gamma]$, the face ring of Γ over K (see [16] for more details). Smith [15] and Munkres [13] have described the depth of Γ in terms of combinatorial and topological properties of Γ. In [1] and [2, §§2, 3], Björner gives a description of the depth of a shellable complex Γ in terms of the shelling restriction numbers $h_{i,j}(\Gamma)$. Using Theorem 5.4 we describe depth in terms of iterated homology.

Theorem 6.1 *Let Γ be a simplicial complex*; *then* depth$(\Gamma) = k$ *if and only if*:
(a) $\beta^i[r](\Gamma) = 0$ *for* $i < k$; *and*
(b) $\beta^k[r](\Gamma) \neq 0$ *for some* r.

Proof: Using [15, Theorem 4.8] (see also Hibi [9]), we know that depth$(\Gamma) = k$ if and only if k is the largest integer such that the k-skeleton of Γ is Cohen-Macaulay. From [12, Theorem 5.3], this is equivalent to k being the largest integer such that the k-skeleton of the shifted complex $\Delta(\Gamma)$ is pure. This means that all facets of $\Delta(\Gamma)$ have dimension at least k, and there exists a facet of dimension exactly k. Thus, in any shelling of $\Delta(\Gamma)$, we have $h_{i,j}(\Delta(\Gamma)) = 0$ whenever $i \le k$, but $h_{k+1,j}(\Delta(\Gamma)) \neq 0$ for some j. By Theorem 5.4, this is equivalent to $\beta^i[r](\Gamma) = 0$ if $i < k$, for any r, but $\beta^k[r](\Gamma) \neq 0$, for some r. $\square$

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