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On a Conjectured Formula for Quiver Varieties

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Abstract. In A.S. Buch and W. Fulton [*Invent. Math.* **135** (1999), 665–687] a formula for the cohomology class of a quiver variety is proved. This formula writes the cohomology class of a quiver variety as a linear combination of products of Schur polynomials. In the same paper it is conjectured that all of the coefficients in this linear combination are non-negative, and given by a generalized Littlewood-Richardson rule, which states that the coefficients count certain sequences of tableaux called factor sequences. In this paper I prove some special cases of this conjecture. I also prove that the general conjecture follows from a stronger but simpler statement, for which substantial computer evidence has been obtained. Finally I will prove a useful criterion for recognizing factor sequences.

Keywords: quiver varieties, Littlewood-Richardson rule, Schur functions, Young tableaux

1. Introduction

The goal of this paper is to prove some combinatorial results about a formula for quiver varieties given in [4].

Let X be a non-singular complex variety and $E_0 \to E_1 \to E_2 \to \cdots \to E_n$ a sequence of vector bundles and bundle maps over X. A set of *rank conditions* for this sequence is a collection of non-negative integers $r = (r_{ij})$ for $0 \leq i < j \leq n$. This data defines a degeneracy locus in X,

$$\Omega_r(E_\bullet) = \{x \in X \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij}\ \forall i < j\}.$$

Let r_{ii} denote the rank of the bundle E_i. We will demand that the rank conditions can *occur*, i.e. that there exists a sequence of vector spaces and linear maps $V_0 \to V_1 \to \cdots \to V_n$ so that $\dim(V_i) = r_{ii}$ and $\operatorname{rank}(V_i \to V_j) = r_{ij}$. This is equivalent to the conditions $r_{ij} \leq \min(r_{i,j-1}, r_{i+1,j})$ for $i < j$, and $r_{ij} - r_{i,j-1} - r_{i+1,j} + r_{i+1,j-1} \geq 0$ for $j - i \geq 2$.

Given two vector bundles E and F on X and a partition λ, we let $s_\lambda(F - E)$ denote the super-symmetric Schur polynomial in the Chern roots of these bundles. By definition this is the determinant of the matrix whose (i, j)th entry is the coefficient of the term of degree $\lambda_i + j - i$ in the formal power series expansion of the quotient of total Chern polynomials $c_t(E^\vee)/c_t(F^\vee)$.

The expected (and maximal) codimension for the locus $\Omega_r(E_\bullet)$ in X is

$$d(r) = \sum_{i<j} (r_{i,j-1} - r_{ij}) \cdot (r_{i+1,j} - r_{ij}).$$

The main result of [4] gives a formula for the cohomology class of $\Omega_r(E_\bullet)$ when it has this codimension:

$$[\Omega_r(E_\bullet)] = \sum_{\mu} c_\mu(r)\, s_{\mu_1}(E_1 - E_0) \cdots s_{\mu_n}(E_n - E_{n-1}).$$

Here the sum is over sequences of partitions $\mu = (\mu_1, \ldots, \mu_n)$; the coefficients $c_\mu(r)$ are certain integers given by an explicit combinatorial algorithm which is described in Section 2. These coefficients are known to generalize Littlewood-Richardson coefficients as well as the coefficients appearing in Stanley symmetric functions [3, 4]. The formula specializes to give new expressions for all known types of Schubert polynomials related to type A geometry [7].

There is no immediate geometric reason for the products of Schur polynomials appearing in the formula. However, it is even more surprising that the coefficients $c_\mu(r)$ all seem to be non-negative. Attempts to prove this has led to a conjecture saying that these coefficients count the number of different sequences of tableaux satisfying certain conditions [4]. These sequences are called *factor sequences* and are defined in Section 2.

The main result in this paper is a proof of this conjecture in some special cases which include all situations where the sequence $E_\bullet$ has up to four bundles. We will also show that the conjecture follows from a stronger but simpler conjecture, for which substantial computational verification has been obtained. For both of these results, a sign-reversing involution on pairs of tableaux constructed by S. Fomin plays a fundamental role.

In Section 2 we will explain the algorithm for computing the coefficients $c_\mu(r)$, as well as the conjectured formula for these coefficients. In Section 3 we will prove a useful criterion for recognizing factor sequences. Section 4 gives an account of Fomin's involution, which in Section 5 is used to formulate the stronger conjecture mentioned above. Finally, Section 6 contains a proof of this stronger conjecture in special cases.

The work described in this paper, some of which was announced in [1], can be viewed as a continuation of a joint geometric project with W. Fulton, which resulted in the quiver formula described in [4]. We would like to thank him for introducing us to the subject of degeneracy loci during this very pleasant collaboration, and also for numerous suggestions, ideas, comments, etc. during the work on this paper. We are also extremely grateful to S. Fomin who provided the vital involution mentioned above, and who also collaborated with us in the attempts to prove the conjecture. Finally we thank Frank Sottile for many useful suggestions for improving our exposition.

2. Description of the algorithm

This section explains the algorithm for computing the coefficients $c_\mu(r)$ as well as the conjecture for these coefficients. We will first explain this in the ordinary case described

in the introduction. Then we will extend the notions to a more general situation, which for many purposes is easier to work with.

We will need some notation. Let $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ be the ring of symmetric functions. The variable h_i may be identified with the complete symmetric function of degree i. If $I = (a_1, a_2, \ldots, a_p)$ is a sequence of integers, define the Schur function $s_I \in \Lambda$ to be the determinant of the $p \times p$ matrix whose (i, j)th entry is h_{a_i+j-i}:

$$s_I = \det(h_{a_i+j-i})_{1\le i,j\le p}.$$

(Here one sets $h_0 = 1$ and $h_{-q} = 0$ for $q > 0$.) A Schur function is always equal to either zero or plus or minus a Schur function s_λ for a partition λ. This follows from interchanging the rows of the matrix defining s_I. Furthermore, the Schur functions given by partitions form a basis for the ring of symmetric functions [6, 10].

We will give the algorithm for computing the coefficients $c_\mu(r)$ by constructing an element P_r in the nth tensor power of the ring of symmetric functions $\Lambda^{\otimes n}$, such that

$$P_r = \sum_\mu c_\mu(r) s_{\mu_1} \otimes \cdots \otimes s_{\mu_n}.$$

It is convenient to arrange the rank conditions in a *rank diagram*:

$$\begin{array}{ccccccccc}
E_0 & \to & E_1 & \to & E_2 & \to & \cdots & \to & E_n \\
r_{00} & & r_{11} & & r_{22} & & \cdots & & r_{nn} \\
& r_{01} & & r_{12} & & \cdots & & r_{n-1,n} & \\
& & r_{02} & & \cdots & & r_{n-2,n} & & \\
& & & \ddots & & \iddots & & & \\
& & & & r_{0n} & & & &
\end{array}$$

In this diagram we replace each small triangle of numbers

$$\begin{array}{ccc}
r_{i,j-1} & & r_{i+1,j} \\
& r_{ij} &
\end{array}$$

by a rectangle R_{ij} with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns.

$$R_{ij} = \begin{array}{c} \boxed{} \\ r_{i,j-1} - r_{ij} \end{array} \quad r_{i+1,j} - r_{ij}$$

These rectangles are then arranged in a *rectangle diagram*:

$$\begin{array}{ccccccc}
R_{01} & & R_{12} & & \cdots & & R_{n-1,n} \\
& R_{02} & & \cdots & & R_{n-2,n} & \\
& & \ddots & & \iddots & & \\
& & & R_{0n} & & &
\end{array}$$

It turns out that the information carried by the rank conditions is very well represented in this diagram. First, the expected codimension $d(r)$ for the locus $\Omega_r(E_\bullet)$ is equal to the total number of boxes in the rectangle diagram. Furthermore, the condition that the rank conditions can occur is equivalent to saying that the rectangles get narrower when one travels south-west, while they get shorter when one travels south-east. Finally, the element P_r depends only on the rectangle diagram.

We will define $P_r \in \Lambda^{\otimes n}$ by induction on n. When $n = 1$ (corresponding to a sequence of two vector bundles), the rectangle diagram has only one rectangle $R = R_{01}$. In this case we set

$$P_r = s_R \in \Lambda^{\otimes 1}$$

where R is identified with the partition for which it is the Young diagram. This case recovers the Giambelli-Thom-Porteous formula.

If $n \geq 2$ we let $\bar{r}$ denote the bottom n rows of the rank diagram. Then $\bar{r}$ is a valid set of rank conditions, so by induction we can assume that

$$P_{\bar{r}} = \sum_{\mu} c_\mu(\bar{r}) s_{\mu_1} \otimes \cdots \otimes s_{\mu_{n-1}} \tag{1}$$

is a well defined element of $\Lambda^{\otimes n-1}$. Now P_r is obtained from $P_{\bar{r}}$ by replacing each basis element $s_{\mu_1} \otimes \cdots \otimes s_{\mu_{n-1}}$ in (1) with the sum

$$\sum_{\substack{\sigma_1,\ldots,\sigma_{n-1}\\ \tau_1,\ldots,\tau_{n-1}}} \left(\prod_{i=1}^{n-1} c_{\sigma_i\tau_i}^{\mu_i} \right) s_{\boxed{R_{01}\,\sigma_1}} \otimes \cdots \otimes s_{\begin{array}{|l|l|}\hline R_{i-1,i} & \sigma_i \\ \hline \multicolumn{2}{|l|}{\tau_{i-1}} \\ \hline \end{array}} \otimes \cdots \otimes s_{\begin{array}{|l|}\hline R_{n-1,n} \\ \hline \tau_{n-1} \\ \hline \end{array}}.$$

This sum is over all partitions $\sigma_1, \ldots, \sigma_{n-1}$ and $\tau_1, \ldots, \tau_{n-1}$ such that σ_i has fewer rows than $R_{i-1,i}$ and each Littlewood-Richardson coefficient $c_{\sigma_i\tau_i}^{\mu_i}$ is non-zero. A diagram consisting of a rectangle $R_{i-1,i}$ with (the Young diagram of) a partition σ_i attached to its right side, and τ_{i-1} attached beneath should be interpreted as the sequence of integers giving the number of boxes in each row of this diagram.

It can happen that the rectangle $R_{i-1,i}$ is empty, since the number of rows or columns can be zero. If the number of rows is zero, then σ_i is required to be empty, and the diagram is the Young diagram of τ_{i-1}. If the number of columns is zero, then the algorithm requires that the length of σ_i is at most equal to the number of rows $r_{ii} - r_{i-1,i}$ of $R_{i-1,i}$, and the diagram consists of σ_i in the top $r_{ii} - r_{i-1,i}$ rows and τ_{i-1} below this, possibly with some zero-length rows in between.

Next we will describe the conjectured formula for the coefficients $c_\mu(r)$. We will need the notions of (semistandard) Young tableaux and multiplication of tableaux. In particular we shall make use of the row and column bumping algorithms for Schensted insertion. See for example [11, 13], or [6].

A *tableau diagram* for a set of rank conditions is a filling of all the boxes in the corresponding rectangle diagram with integers, such that each rectangle R_{ij} becomes a tableau T_{ij}. Furthermore, it is required that the entries of each tableau T_{ij} are strictly larger than the

entries in tableaux above T_{ij} in the diagram, within 45 degree angles. These are the tableaux T_{kl} with $i \leq k < l \leq j$ and $(k, l) \neq (i, j)$.

A *factor sequence* for a tableau diagram with n rows is a sequence of tableaux $(W_1, \ldots, W_n)$, which is obtained as follows: If $n = 1$ then the only factor sequence is the sequence (T_{01}) containing the only tableau in the diagram. When $n \geq 2$, a factor sequence is obtained by first constructing a factor sequence $(U_1, \ldots, U_{n-1})$ for the bottom $n-1$ rows of the tableau diagram, and choosing arbitrary factorizations of the tableaux in this sequence:

$$U_i = P_i \cdot Q_i.$$

In other words we must choose tableaux P_i and Q_i such that U_i is their product in the plactic monoid [6, 9]. Then the sequence

$$(W_1, \ldots, W_n) = (T_{01} \cdot P_1, Q_1 \cdot T_{12} \cdot P_2, \ldots, Q_{n-1} \cdot T_{n-1,n})$$

is the factor sequence for the whole tableau diagram. The conjecture from [4], which is the theme of this paper, can now be stated as follows:

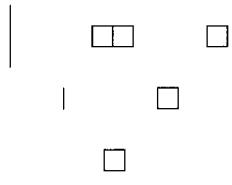
Conjecture 1 The coefficient $c_\mu(r)$ is equal to the number of different factor sequences $(W_1, \ldots, W_n)$ for any fixed tableau diagram for the rank conditions r, such that W_i has shape μ_i for each i.

This conjecture first of all implies that the coefficients $c_\mu(r)$ are non-negative and that they are independent of the side lengths of empty rectangles in the rectangle diagram. In addition it implies that the number of factor sequences only depends on the rectangle diagram and not on the choice of a filling of its boxes with integers.

Example 1 Suppose we are given a sequence of four vector bundles and the following rank conditions:

$$\begin{array}{ccccccc} E_0 & \to & E_1 & \to & E_2 & \to & E_3 \\ 1 & & 4 & & 3 & & 3 \\ & 1 & & 2 & & 2 & \\ & & 1 & & 1 & & \\ & & & 0 & & & \end{array}$$

These rank conditions then give the following rectangle diagram:



From the bottom row of this diagram we get

$$P_{\tilde{r}} = s_{\square}.$$

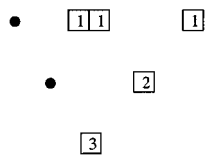
Then using the algorithm we obtain

$$P_{\tilde{r}} = s_{\square} \otimes s_{\square} + 1 \otimes s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

and

$$P_r = s_{\square} \otimes s_{\square\square\square} \otimes s_{\square} + s_{\square} \otimes s_{\square\square} \otimes s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + 1 \otimes s_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} \otimes s_{\square} + 1 \otimes s_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} \otimes s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + 1 \otimes s_{\square\square\square} \otimes s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + 1 \otimes s_{\square\square} \otimes s_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}.$$

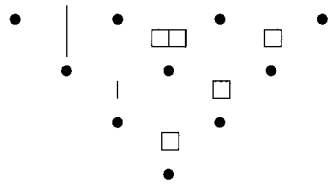
Thus the formula for the cohomology class of $\Omega_r(E_\bullet)$ has six terms. Now, one possible tableau diagram for the given rank conditions is the following:



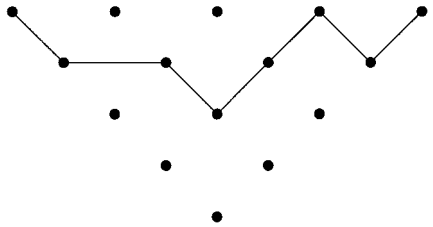
This diagram has the following six factor sequences:

$$\left(\boxed{3}, \boxed{1}\boxed{1}\boxed{2}, \boxed{1}\right),\ \left(\boxed{3}, \boxed{1}\boxed{1}, \begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}\right),\ \left(\emptyset, \begin{smallmatrix}\boxed{1}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix}, \boxed{1}\right),$$
$$\left(\emptyset, \begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{3}\end{smallmatrix}, \begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}\right),\ \left(\emptyset, \boxed{1}\boxed{1}\boxed{3}, \begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}\right),\ \left(\emptyset, \boxed{1}\boxed{1}, \begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}\right).$$

Since only the rectangle diagram matters for the formula, we will often depict a rank diagram simply as a triangle of dots in place of a triangle of numbers. This is especially convenient when working with paths through the rank diagram, which we shall do shortly. Such a diagram will often be decorated with the rectangles from the rectangle diagram, or by the tableaux from a tableau diagram. When this is done, each rectangle or tableau is put in the middle of the triangle of dots representing the numbers that produced the rectangle. In this way the rank conditions used in the above example would be represented by the diagram:

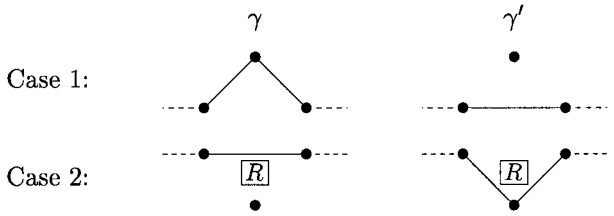


We will now introduce a generalization of the formula for P_r. Define a *path* through the rank diagram to be a union of line segments between neighboring rank conditions, which form a continuous path from r_{00} to r_{nn} such that any vertical line intersects this path at most once.



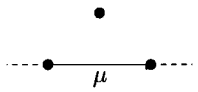
The length of a path is the number of line segments it contains (which is between n and $2n$). Given a path γ of length ℓ, we will define an element $P_\gamma \in \Lambda^{\otimes \ell}$. It is convenient to identify the basis elements of $\Lambda^{\otimes \ell}$ with labelings of the line segments of γ by partitions. More generally, if $I_1, \ldots, I_\ell$ are sequences of integers, then we will identify the labeling of the line segments in γ by these sequences, left to right, with the element $s_{I_1} \otimes \cdots \otimes s_{I_\ell} \in \Lambda^{\otimes \ell}$. All basis elements occurring in P_γ with non-zero coefficient will assign the empty partition to line segments on the left and right sides of the rank diagram. If γ is the highest path, going horizontally from r_{00} to r_{nn}, then P_γ is equal to P_r.

We define P_γ inductively as follows. If γ is the lowest possible path, going from r_{00} to r_{0n} to r_{nn}, then we set $P_\gamma = 1 \otimes 1 \otimes \cdots \otimes 1 \in \Lambda^{\otimes 2n}$. In other words P_γ is equal to the single basis element which assigns the empty partition to each line segment. If γ is any other path, then we can find a path γ' which is equal to γ, except it goes lower at one place, in one of the following ways:



By induction we may assume that $P_{\gamma'}$ is well defined.

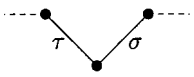
If we are in Case 1 we now obtain P_γ from $P_{\gamma'}$ by replacing each basis element



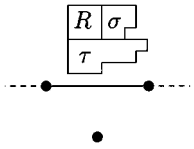
occurring in $P_{\gamma'}$ with the sum

$$\sum_{\sigma,\tau} c^{\mu}_{\sigma\tau} \left(\sigma \quad \tau \right) .$$

For Case 2, let R be the rectangle associated to the triangle where γ and γ' differ. Then P_γ is obtained from $P_{\gamma'}$ by replacing each basis element



occurring in $P_{\gamma'}$ with zero if σ has more rows than R, and otherwise with the element:



An easy induction shows that this definition is independent of the choice of the path γ'. The element P_γ has geometric meaning similar to that of P_r. It describes the cohomology class of a degeneracy locus $\Omega_r(\gamma)$ defined in [4].

If we are given a tableau diagram, the notion of a factor sequence can also be extended to paths. Any factor sequence for a path γ will contain one tableau for each line segment in γ. As with elements of $\Lambda^{\otimes \ell}$, we will often regard such a sequence as a labeling of the line segments in γ with tableaux.

If γ is the lowest path from r_{00} to r_{0n} to r_{nn} then the only factor sequence is the sequence $(\emptyset, \ldots, \emptyset)$ which assigns the empty tableau to each line segment. Otherwise we can find a lower path γ' as in Case 1 or Case 2 above. In order to obtain a factor sequence for γ we must first construct one for γ'.

If we are in Case 1, let $(\ldots, W, \ldots)$ be a factor sequence for γ' such that W is the label of the displayed line segment, and let $W = P \cdot Q$ be an arbitrary factorization of W. Then the sequence $(\ldots, P, Q, \ldots)$ is a factor sequence for γ. For Case 2, let T be the tableau corresponding to the rectangle R. If $(\ldots, Q, P, \ldots)$ is a factor sequence for γ' with Q and P the tableaux assigned to the displayed line segments, then $(\ldots, Q \cdot T \cdot P, \ldots)$ is a factor sequence for γ.

Finally we define coefficients $c_\mu(\gamma) \in \mathbb{Z}$ by the expression

$$P_\gamma = \sum_\mu c_\mu(\gamma) s_{\mu_1} \otimes \cdots \otimes s_{\mu_\ell} \in \Lambda^{\otimes \ell}$$

where ℓ is the length of γ. Conjecture 1 then has the following generalization:

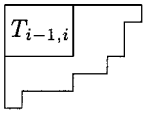
Conjecture 1A The coefficient $c_\mu(\gamma)$ is equal to the number of different factor sequences $(W_1, \ldots, W_\ell)$ for the path γ, such that W_i has shape μ_i for each i.

3. A criterion for factor sequences

In this section we will prove a simple criterion for recognizing factor sequences. As in the previous section we will start by discussing ordinary factor sequences.

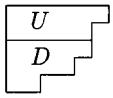
Let $\{T_{ij}\}$ be a tableau diagram and let $(W_1, \ldots, W_n)$ be a sequence of tableaux. At first glance it would appear that to check if this sequence is a factor sequence, we would have to find all factor sequences $(U_1, \ldots, U_{n-1})$ for the bottom $n-1$ rows of the tableau diagram, as well as all factorizations $U_i = P_i \cdot Q_i$, to see if our sequence $(W_1, \ldots, W_n)$ is obtained from any of these, i.e. $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ for all i. Equivalently we could find all factorizations of each W_i into three factors $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ (with $Q_0 = P_n = \emptyset$), and check if $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for any of these choices. The criterion for factor sequences allows us to check this for just one factorization of each W_i.

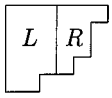
Notice that if the sequence $(W_1, \ldots, W_n)$ is a factor sequence, obtained from an inductive factor sequence $(U_1, \ldots, U_{n-1})$ as above, then the conditions on the filling of a tableau diagram imply that the entries of each tableau $T_{i-1,i}$ are strictly smaller than the entries of Q_{i-1} and P_i. This implies that $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ contains the rectangular tableau $T_{i-1,i}$ in its upper-left corner.

$W_i =$ 

We shall therefore investigate ways to factor a tableau into three pieces, one of which is a contained rectangular tableau.

A trivial way to factor any tableau is by cutting it along a horizontal or vertical line. Let T be a tableau and $a \geq 0$ an integer. Let U the top a rows of T, and D the rest of T. Then $T = D \cdot U$. We will call this factorization the *horizontal cut* through T after the ath row. Vertical cuts are defined similarly.

$T =$ 

$= D \cdot U$ $\qquad$ $T =$ 

$= L \cdot R$

Lemma 1 *Let $T = P \cdot Q$ be any factorization of T and let a be the number of rows in Q. The following are equivalent*:

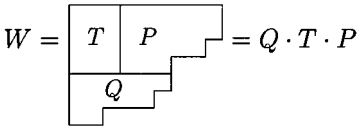
(i) *$T = P \cdot Q$ is a horizontal cut.*
(ii) *The ith row of T has the same number of boxes as the ith row of Q for $1 \leq i \leq a$.*
(iii) *Whenever the top row of P has a box in column $j \geq 1$, the ath row of Q has a strictly smaller box in this column (unless $a = 0$).*

Similarly, if P has b columns, then $T = P \cdot Q$ is a vertical cut iff the first b columns of T and P have the same heights, iff the boxes in the last column of P are smaller than or equal to the boxes in similar positions in the first column of Q.

Proof: It is clear that (i) implies (ii) and (iii). If (iii) is true then P and Q fit together to form a tableau with Q in the top a rows and P below. By taking a horizontal cut through this tableau, we see that it must be the product of P and Q. But then it is equal to T and (i) follows. Finally, suppose (ii) is true. When the boxes of P are column bumped into Q to form the product T, all of these boxes must then stay below the ath row. This process

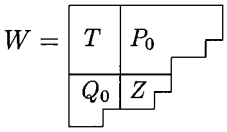
therefore reconstructs P below Q and (i) follows. The statements about vertical cuts are proved similarly. □

Now let W be any tableau whose shape contains a rectangle $(b)^a$ with a rows and b columns. We define the *canonical factorization* of W with respect to the rectangle $(b)^a$ to be the one obtained by first taking a horizontal cut through W after the ath row, and then a vertical cut through the top part of W after the bth column.



Note that this definition depends on a, even when b is zero and the rectangle $(b)^a$ is empty. When the product of three tableau Q, T, and P looks like in this picture, we shall say that the pair of tableaux (Q, P) *fits around* the rectangular tableau T.

More generally, let Q_0 be the part of W below T, P_0 the part of W to the right of T, and let Z be the remaining part between Q_0 and P_0.



We define a *simple factorization* of W with respect to the rectangle $(b)^a$ to be any factorization $W = Q \cdot T \cdot P$, such that $Q = Q_0 \cdot \tilde{Q}$ and $P = \tilde{P} \cdot P_0$ for some factorization $Z = \tilde{Q} \cdot \tilde{P}$.

Note that if $Z = \tilde{Q} \cdot \tilde{P}$ is any factorization of Z and if we put $Q = Q_0 \cdot \tilde{Q}$ and $P = \tilde{P} \cdot P_0$, then $Q \cdot T \cdot P = W$. This follows because $P = \tilde{P} \cdot P_0$ must be a horizontal cut through P, and therefore $T \cdot P = \tilde{P} \cdot T \cdot P_0$. In fact, given arbitrary tableaux $\tilde{Q}$ and $\tilde{P}$ one can show that $\tilde{Q} \cdot \tilde{P} = Z$ if and only if $Q_0 \cdot \tilde{Q} \cdot T \cdot \tilde{P} \cdot P_0 = W$, but we shall not need this here.

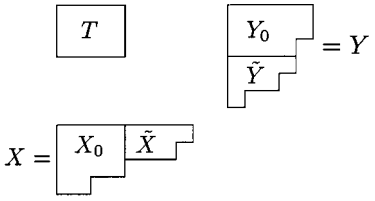
We are now ready to formulate the criterion for factor sequences. Let $\{R_{ij}\}$ be the set of rectangles corresponding to the tableau diagram $\{T_{ij}\}$. If $(W_1, \ldots, W_n)$ is a factor sequence, a simple factorization of any W_i will always be with respect to the relevant rectangle $R_{i-1,i}$ from the rectangle diagram.

Theorem 1 *Let $(W_1, \ldots, W_n)$ be a sequence of tableaux such that each W_i contains $T_{i-1,i}$ in its upper-left corner. Let $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ be any simple factorization of W_i with respect to the rectangle $R_{i-1,i}$. Then $(W_1, \ldots, W_n)$ is a factor sequence if and only if Q_0 and P_n are empty tableaux and $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for the bottom $n-1$ rows of the tableau diagram $\{T_{ij}\}$.*

We shall derive this result from Proposition 1 below. Since this criterion can be applied recursively to the sequence $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$, it gives an easy algorithm to determine if a sequence $(W_1, \ldots, W_n)$ is a factor sequence. An easy way to produce the

simple factorizations required by the theorem is to take the canonical factorization of each W_i. When this choice is made, the work required in the algorithm essentially consists of $n(n-1)/2$ tableau multiplications. Note also that this method makes use of the height of any empty rectangles in the rectangle diagram.

To prove this criterion we need some definitions. Let T be a tableau whose shape is the rectangle $(b)^a$ with a rows and b columns. We will consider pairs of tableaux (X, Y) such that all entries in X and Y are strictly larger than the entries of T. For such a pair, let $X = X_0 \cdot \tilde{X}$ be the vertical cut through X after the bth column, and let $Y = \tilde{Y} \cdot Y_0$ be the horizontal cut after row a.



If (X', Y') is another pair of tableaux, we will write $(X, Y) \models (X', Y')$ if either

1. for some factorization $\tilde{X} = M \cdot N$ we have $X' = X_0 \cdot M$ and $Y' = N \cdot Y$, or
2. for some factorization $\tilde{Y} = M \cdot N$ we have $X' = X \cdot M$ and $Y' = N \cdot Y_0$.

Note that this implies that $X' \cdot T \cdot Y' = X \cdot T \cdot Y$. In the first case this follows because $X \cdot T = X_0 \cdot T \cdot \tilde{X}$ and $X' \cdot T = X_0 \cdot T \cdot M$, and the second case is similar. We will let $\rightarrow$ denote the transitive closure of the relation $\models$. This notation depends on the choice of T, as well as the numbers a and b if T is empty.

Lemma 2 *Let W be a tableau containing T in its upper-left corner. Suppose that the entries of T are smaller than all other entries in W. If $W = Q \cdot T \cdot P$ is a simple factorization of W with respect to the rectangle $(b)^a$, and $W = X \cdot T \cdot Y$ is any factorization, then $(X, Y) \rightarrow (Q, P)$.*

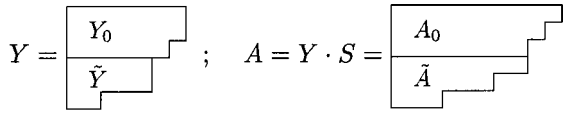
Proof: Let $X = X_0 \cdot \tilde{X}$ be the vertical cut through X after column b, and put $Y' = \tilde{X} \cdot Y$. Then let $Y' = \tilde{Y}' \cdot Y_0'$ be the horizontal cut through Y' after row a, and put $X'' = X_0 \cdot \tilde{Y}'$.

We claim that the pair (X'', Y_0') fits around T. Using Lemma 1 and that the entries of T are smaller than all other entries, it is enough to prove that the $b + j$th entry in the top row of X'' is strictly larger than the jth entry in the bottom row of Y_0'. This will follow if the $b + j$th entry in the top row of X'' is larger than or equal to the jth entry in the top row of $\tilde{Y}'$. Since $X'' = X_0 \cdot \tilde{Y}'$ and X_0 has at most b columns, this follows from an easy induction on the number of rows of $\tilde{Y}'$.

It follows from the claim that $W = X'' \cdot T \cdot Y_0'$ is the canonical factorization of W, and therefore we have $(X, Y) \models (X_0, Y') \models (X'', Y_0') \models (Q, P)$ as required. □

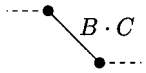
Notice that if $W = X \cdot T \cdot Y$ is a simple factorization and $(X, Y) \models (X', Y')$, then $W = X' \cdot T \cdot Y'$ must also be a simple factorization. It follows that Lemma 2 would be false without the requirement that $W = Q \cdot T \cdot P$ is simple.

Lemma 3 *Let $a \geq 0$ be an integer, and let Y and S be tableaux with product $A = Y \cdot S$. Let $A = \tilde{A} \cdot A_0$ and $Y = \tilde{Y} \cdot Y_0$ be the horizontal cuts through A and Y after row a, and let $\tilde{Y} = M \cdot N$ be any factorization. Then $N \cdot Y_0 \cdot S = \tilde{A}' \cdot A_0$ for some tableau $\tilde{A}'$, and $M \cdot \tilde{A}' = \tilde{A}$.*

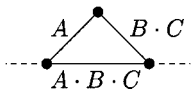


Proof: The first statement follows from the observation that the bottom rows of Y can't influence the top part of $Y \cdot S$, which is a consequence of the row bumping algorithm. Lemma 1 then shows that the factorization $A = (M \cdot \tilde{A}') \cdot A_0$ is a horizontal cut, so $M \cdot \tilde{A}' = \tilde{A}$ as required. □

Lemma 4 *Let γ be a path through the rank diagram, and let $(\ldots, A, B \cdot C, \ldots)$ be a factor sequence for γ such that the product $B \cdot C$ is the label of a down-going line segment. Then $(\ldots, A \cdot B, C, \ldots)$ is also a factor sequence for γ.*



Proof: We will first consider the case where the line segment corresponding to A goes up. Let γ' be the path under γ that cuts short this line segment and its successor.

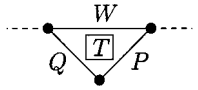


Then by definition $(\ldots, A \cdot B \cdot C, \ldots)$ is a factor sequence for γ', which means that $(\ldots, A \cdot B, C, \ldots)$ is a factor sequence for γ. In general γ lies over a path like the one above, and the general case follows from this. □

Similarly one can prove that if $(\ldots, A \cdot B, C, \ldots)$ is a factor sequence for a path, such that $A \cdot B$ is the label of an up-going line segment, then $(\ldots, A, B \cdot C, \ldots)$ is also a factor sequence for this path.

Proposition 1 *Let γ and γ' be paths related as in Case 2 of Section 2, and let $(\ldots, W, \ldots)$ be a factor sequence for γ such that W is the label of the displayed horizontal line*

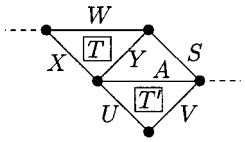
segment.



If $W = Q \cdot T \cdot P$ is any simple factorization of W, then $(\ldots, Q, P, \ldots)$ is a factor sequence for γ'.

Proof: Since $(\ldots, W, \ldots)$ is a factor sequence for γ, there exists a factorization $W = X \cdot T \cdot Y$ such that $(\ldots, X, Y, \ldots)$ is a factor sequence for γ'. By Lemma 2 we have $(X, Y) \to (Q, P)$. It is therefore enough to show that if $(X, Y) \models (X', Y')$ then $(\ldots, X', Y', \ldots)$ is a factor sequence for γ'.

Let a be the number of rows in (the rectangle corresponding to) T, and let $Y = \tilde{Y} \cdot Y_0$ be the horizontal cut through Y after the ath row. We will do the case where a factor of $\tilde{Y}$ is moved to X, the other case is proved using a symmetric argument. We then have a factorization $\tilde{Y} = M \cdot N$ such that $X' = X \cdot M$ and $Y' = N \cdot Y_0$. We can assume that the paths γ and γ' go down after they meet, and that the original factor sequence for γ is $(\ldots, W, S, \ldots)$.

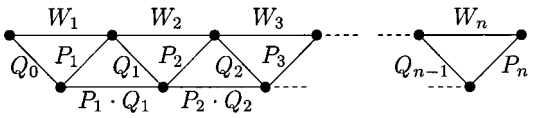


Put $A = Y \cdot S$. Then $(\ldots, X, A, \ldots)$ is a factor sequence for the path with these labels in the picture. Now let T' be the rectangular tableau associated to the lower triangle, and let $A = U \cdot T' \cdot V$ be the canonical factorization of A. Since this is a simple factorization we may assume by induction that $(\ldots, X, U, V, \ldots)$ is a factor sequence. Using Lemma 3 we deduce that $N \cdot Y_0 \cdot S = U' \cdot T' \cdot V$ for some tableau U', such that $M \cdot U' = U$. Since $(\ldots, X, M \cdot U', V, \ldots)$ is a factor sequence, so is $(\ldots, X \cdot M, U', V, \ldots)$ by Lemma 4. This means that $(\ldots, X \cdot M, U' \cdot T' \cdot V, \ldots) = (\ldots, X', Y' \cdot S, \ldots)$ is a factor sequence, which in turn implies that $(\ldots, X', Y', S, \ldots)$ is a factor sequence for γ' as required. □

The proof of Proposition 1 also gives the following:

Corollary 1 *Let $(\ldots, X, Y, \ldots)$ be a factor sequence for the path γ' in the proposition. If $(X, Y) \to (X', Y')$ then $(\ldots, X', Y', \ldots)$ is also a factor sequence for γ'.*

Proof of Theorem 1: The "if" implication follows from the definition. If the sequence $(W_1, \ldots, W_n)$ is a factor sequence, then n applications of Proposition 1 shows that $(Q_0, P_1, Q_1, P_2, \ldots, Q_{n-1}, P_n)$ is a factor sequence for the path with these labels.



It follows that Q_0 and P_n are empty, and $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for the bottom $n-1$ rows. This proves "only if". □

4. An involution of Fomin

In this section we will describe a sign-reversing involution on pairs of tableaux constructed by Sergey Fomin. The purpose of this involution is to cancel out the difference between the coefficients $c_\mu(r)$ produced by the algorithm in Section 2, and their conjectured values.

Fix an integer $a > 0$. If P and Q are tableaux of shapes σ and τ such that P has at most a rows, we let $S(\frac{P}{Q})$ denote the symmetric function $s_I \in \Lambda$ where I is the sequence of integers $I = (\sigma_1, \ldots, \sigma_a, \tau_1, \tau_2, \ldots)$. Let $\mathcal{P}_a$ be the set of all pairs (Q, P) such that $S(\frac{P}{Q}) \neq 0$ and such that P and Q do not fit together as a tableau with P in the top a rows and Q below. This means that the ath row of P must be shorter than the top row of Q, or some box in the top row of Q must be smaller than or equal to the box in the same position of the ath row of P. For example, if $a = 2$ the following pairs are in P_a:

$$\left(\begin{array}{|c|c|c|c|}\hline 3&5&6&7\\\hline 4\\\cline{1-1}\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline 1&3&7&8\\\hline 2&4\\\cline{1-2}\end{array}\right) \text{ and } \left(\begin{array}{|c|c|c|}\hline 4&4&7\\\hline 5&6\\\cline{1-2}8\\\cline{1-1}\end{array}\;,\;\begin{array}{|c|c|c|c|c|}\hline 2&3&3&4&6\\\hline 3&4&5&5\\\cline{1-4}\end{array}\right).$$

Lemma 5 (*Fomin's involution*) *There exists an involution of $\mathcal{P}_a$ with the property that if (Q, P) is mapped to (Q', P') then*

(i) $Q' \cdot P' = Q \cdot P$,
(ii) $S(\frac{P'}{Q'}) = -S(\frac{P}{Q})$, *and*
(iii) *the first column of Q' is equal to the first column of Q.*

Fomin supplied the proof of this lemma in the form of the beautiful algorithm described below. While Fomin's original description uses path representations of tableaux, we have translated the algorithm into notation that is closer to the rest of this paper.

We will work with *diagrams with weakly increasing rows*. These will be "Young diagrams" for finite sequences of non-negative integers, where all boxes are filled with integers so that the rows are weakly increasing. Empty rows are allowed as in the following example:

$$\begin{array}{|c|c|c|c|}\hline 2&3&6&6\\\hline 3&4&5&6&7&7\\\hline\end{array}$$
$$\begin{array}{|c|c|}\hline 4&6\\\hline\end{array}$$

A *violation* for such a diagram to be a tableau is a box in the second row or below, such that there is no box directly above it, or the box directly above it is not strictly smaller. The above diagram has 4 violations in its second row and 2 in row four.

If D is a diagram with weakly increasing rows, and if I is the sequence of row lengths, we put $S(D) = s_I \in \Lambda$. Let rect(D) denote the tableau obtained by multiplying the rows of D together, from bottom to top. We will identify a pair $(Q, P) \in \mathcal{P}_a$ with the diagram D consisting of P in the top a rows and Q below. For this diagram we then have $Q \cdot P =$ rect(D) and $S(\frac{P}{Q}) = S(D)$.

We will start by taking care of the special case where $a = 1$ and both P and Q have at most one row. In this case Lemma 5 without property (iii) is equivalent to the identity $s_{\ell,k} = h_\ell h_k - h_{\ell+1} h_{k-1}$ in the plactic monoid, which is a special case of a result by Lascoux and Schützenberger [9, 12]. The simple proof of this result given in [5] develops techniques which Fomin used to establish Lemma 5 in full generality.

Lemma 6 *Let D be a diagram with two rows and at least one violation in the second row. Then there exists a unique diagram D' such that* rect(D') = rect(D) *and $S(D') = -S(D)$. Furthermore, D' also has two rows and at least one violation in the second row. The leftmost violations of D and D' appear in the same column and contain the same number. The parts of D and D' to the left of this column agree.*

Proof: Let p and q be the lengths of the top and bottom rows of D. The requirement $S(D') = -S(D)$ then implies that D' must have two rows with $q-1$ boxes in the top row and $p+1$ in the bottom row. Now it follows from the Pieri formula [6, section 2.2] that the product rect(D) of the rows in D has at most two rows. Furthermore, since D contains a violation, the second row of rect(D) has at most $q-1$ boxes. Using the Pieri formula again, this implies that there is exactly one way to factor rect(D) into a row of length $p+1$ times another of length $q-1$. This establishes the existence and uniqueness of D'.

Explicitly, one may use the inverse row bumping algorithm to obtain this factorization of rect(D). This is done by bumping out a horizontal strip of $q-1$ boxes which includes all boxes in the second row, working from right to left.

Let x be the leftmost violation of D, where D has the form:

$$D = \begin{array}{|c|c|}\hline A & E \\ \hline B & x \\ \cline{1-2} \end{array} F .$$

Suppose the parts A and B each contain t boxes. Now form the product $F \cdot E$ and let c_j and d_j be the boxes of this product as in the picture:

$$F \cdot E = \begin{array}{|c|c|c|c|c|c|}\hline c_1 & c_2 & c_3 & c_4 & \cdots & c_k \\ \hline d_1 & d_2 & \cdots & d_l \\ \cline{1-4} \end{array} .$$

Since x is a violation in D, it must be smaller than all boxes in E and F. Therefore we have

$$x \cdot F \cdot E = \begin{array}{|c|c|c|c|c|c|c|}\hline x & c_1 & c_2 & c_3 & c_4 & \cdots & c_k \\ \hline d_1 & d_2 & \cdots & d_l \\ \cline{1-4} \end{array} .$$

Now since each $d_j > c_j$ it follows that if a horizontal strip of length $q-t-1$ is bumped off this tableau, x will remain where it is. In other words we can factor $x \cdot F \cdot E$ into $x \cdot F' \cdot E'$ such that $x \cdot F'$ and E' are rows of lengths $p-t+1$ and $q-t-1$ respectively. Since the entries of A and B are no larger than x, the products $B \cdot x \cdot F'$ and $A \cdot E'$ are rows of lengths $p+1$ and $q-1$. But the product of these rows is rect(D), so they must be the rows of D' by the uniqueness. This proves that D' has the stated properties. □

Notice that the uniqueness also implies that the transformation of diagrams described in the lemma is inverse to itself, i.e. an involution.

Now suppose D is any diagram with weakly increasing rows. Then Lemma 6 can be applied to any subdiagram of two consecutive rows, such that the second of these rows contains a violation. If this subdiagram is replaced by the new two-row diagram given by the lemma, we arrive at a diagram D' satisfying $S(D') = -S(D)$ and rect(D') = rect(D). We will call this an *exchange operation* between the two rows of D.

We shall need an ordering on the violations in a diagram. Here the smallest of two violations is the south-west most one. If the two violations are equally far south-west, then the north-west most one is smaller. In other words, a violation in row i and column j is smaller than another in row i' and column j' iff $j-i < j'-i'$, or $j-i = j'-i'$ and $i < i'$.

Notice that when an exchange operation between two rows is carried out, violations may appear or disappear in these two rows as well as in the row below them. However, the properties given in Lemma 6 imply that all of the changed violations will be larger than the left-most violation in the second of the rows exchanged. It follows that the minimal violation in a diagram will remain constant if any (sequence of) exchange operations is carried out. Similarly, all boxes south-west of the minimal violation will remain fixed.

Proof of Lemma 5: Given a pair $(Q, P) \in \mathcal{P}_a$, let $\mathcal{D}_{Q,P}$ be the finite set of all non-tableau diagrams D with weakly increasing rows, such that rect$(D) = Q \cdot P$ and $S(D) = \pm S(\frac{P}{Q})$, and so that the minimal violation in D is in row $a+1$. The pair (Q, P) is then identified with one of the diagrams in this set. We will describe an involution of the set $\mathcal{D}_{Q,P}$ and another of the complement of $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ in $\mathcal{D}_{Q,P}$. The restriction of Fomin's involution to $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ is then obtained by applying the involution principle of Garsia and Milne [8] to these involutions.

The involution of $\mathcal{D}_{Q,P}$ simply consists of doing an exchange operation between the rows a and $a+1$ of a diagram. This is possible because all diagrams are required to have a violation in row $a+1$.

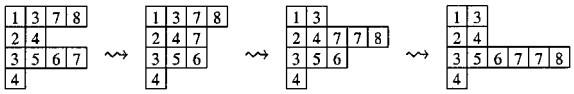
Now note that a diagram $D \in \mathcal{D}_{Q,P}$ is in the complement of $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ if and only if D has a violation outside the $a+1$st row. We take the involution of $\mathcal{D}_{Q,P} \backslash \mathcal{P}_a$ to be an exchange operation between the row of the minimal violation outside row $a+1$, and the row above this violation. This is indeed an involution since the minimal violation outside row $a+1$ stays the same.

These involutions now combine to give an involution of $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ by the involution principle. To carry it out, start by forming the diagram with P in the top a rows and Q below it. Then do an exchange operation between row a and row $a+1$. If all violations in the resulting diagram are in row $a+1$ we are done. P' is then the top a rows of this diagram and

Q' is the rest. Otherwise we continue by doing an exchange operation between the row of the minimal violation outside row $a+1$ and the row above it, followed by another exchange operation between row a and row $a+1$. We continue in this way until all violations are in row $a+1$.

Finally, the properties of P' and Q' follow from the properties of exchange operations. In particular, the requirement $S(\frac{P'}{Q'}) = -S(\frac{P}{Q})$ follows because we always carry out an odd number of exchange operations. □

Example 2 The pair $(P, Q) = (\begin{smallmatrix}3&5&6&7\\4\end{smallmatrix}, \begin{smallmatrix}1&3&7&8\\2&4\end{smallmatrix})$ in $\mathcal{P}_2$ gives the following sequence of exchange operations:

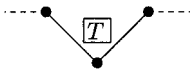


This pair therefore corresponds to $(P', Q') = (\begin{smallmatrix}3&5&6&7&7&8\\4\end{smallmatrix}, \begin{smallmatrix}1&3\\2&4\end{smallmatrix})$ by Fomin's involution.

There are examples of pairs (Q, P) for which the set $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ has more than two diagrams, all with the same first column. This means that the involution constructed above is not the only one that satisfies the conditions of Lemma 5. One way to produce different involutions is to use another ordering among violations. The only property of the order that we have used is that when an exchange operation is carried out, any violations that are created or removed by this operation must be larger than the leftmost violation in the second of the rows being exchanged. For example, given any irrational parameter $\xi \in (0, 1)$, we obtain a new order by letting a violation in position (i, j) be smaller than another in position (i', j') if and only if $j - \xi i < j' - \xi i'$.

5. The stronger conjecture

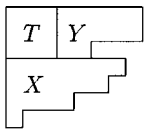
In this section we present a simple conjecture which implies Conjecture 1A. Let γ be a path through the rank diagram which at some triangle has an angle pointing down:



Let T be the rectangular tableau associated to this triangle, and suppose the corresponding rectangle has a rows and b columns.

If X and Y are tableaux whose entries are strictly larger than the entries of T, and if Y has at most a rows, we will let

$$\frac{T|Y}{X} =$$



denote the diagram consisting of $T \cdot Y$ in the top a rows and X below. The sequence of row lengths of this diagram then gives an element $S(\frac{T|Y}{X})$ in the ring of symmetric functions Λ. Note that (X, Y) fits around T if and only if the diagram $\frac{T|Y}{X}$ is a tableau.

Suppose that (X, Y) does not fit around T and $S(\frac{T|Y}{X})$ is non-zero. Let $X = X_0 \cdot \tilde{X}$ be the vertical cut through X after the bth column. Then $(\tilde{X}, Y)$ is an element of the set $\mathcal{P}_a$ defined in the previous section. Let $(\tilde{X}', Y')$ be the result of applying Fomin's involution to this pair, and set $X' = X_0 \cdot \tilde{X}'$. Since the first columns of $\tilde{X}$ and $\tilde{X}'$ agree, X' consists of X_0 with $\tilde{X}'$ attached to its right side by Lemma 1. It follows that $S(\frac{T|Y'}{X'}) = -S(\frac{T|Y}{X})$. (Note that one could also get from (X, Y) to (X', Y') by applying Fomin's involution to the pair $(X, T \cdot Y)$.)

Conjecture 2 Let $(\ldots, X, Y, \ldots)$ be a factor sequence for γ with X and Y the labels of the displayed line segments, such that Y has at most a rows. Suppose (X, Y) does not fit around T and $S(\frac{T|Y}{X}) \neq 0$. If X' and Y' are obtained from X and Y by applying Fomin's involution as described above, then $(\ldots, X', Y', \ldots)$ is also a factor sequence for γ.

If we fix the location of the down-pointing angle of γ (i.e. the location of T in the tableau diagram), then the strongest case of this conjecture is when the rest of γ goes as low as possible. If Conjecture 2 is true for all locations of the down-pointing angle, then the conjectured formula for the coefficients $c_\mu(\gamma)$ is correct.

Theorem 2 *Conjecture 1A follows from Conjecture* 2.

Proof: If $W_1, \ldots, W_\ell$ are diagrams with weakly increasing rows, e.g. tableaux, we will write $S(W_1, \ldots, W_\ell) = S(W_1) \otimes \cdots \otimes S(W_\ell) \in \Lambda^{\otimes \ell}$. With this notation we must prove that if γ is a path of length ℓ, then

$$P_\gamma = \sum_{(W_i)} S(W_1, \ldots, W_\ell) \tag{2}$$

where the sum is over all factor sequences (W_i) for γ.

Let γ' be a path under γ as in Case 1 or Case 2 of Section 2. By induction we can assume that Conjecture 1A is true for γ', i.e.

$$P_{\gamma'} = \sum_{(U_i)} S(U_1, \ldots, U_{\ell'}) \tag{3}$$

where this sum is over the factor sequences for γ'. We must prove that the right hand side of (2) is obtained by replacing each basis element of (3) in the way prescribed by the definition of P_γ. If we are in Case 1 then this follows from the Littlewood-Richardson rule [6, section 5.1]: If U is a tableau of shape μ and σ and τ are partitions, then there are $c^\mu_{\sigma\tau}$ ways to factor U into a product $U = P \cdot Q$ such that P has shape σ and Q has shape τ.

Assume we are in Case 2. By induction we then know that $P_{\gamma'} = \sum S(\ldots, X, Y, \ldots)$ where the sum is over all factor sequences $(\ldots, X, Y, \ldots)$ for γ'; X and Y are the labels of the two line segments where γ' is lower than γ. Let T be the rectangular tableau of the corresponding triangle, and let a be the number of rows in its rectangle. Then by definition

we get

$$P_\gamma = \sum_{(\ldots,X,Y,\ldots)} S\Big(\ldots, \frac{T \mid Y}{X}, \ldots\Big) \tag{4}$$

where the sum is over all factor sequences $(\ldots, X, Y, \ldots)$ for γ' such that Y has at most a rows.

Now suppose we have a factor sequence $(\ldots, X, Y, \ldots)$ such that the diagram $\frac{T|Y}{X}$ is a tableau. Then this tableau must be the product $X \cdot T \cdot Y$, and so $(\ldots, \frac{T|Y}{X}, \ldots)$ is a factor sequence for γ. Thus the term $S(\ldots, \frac{T|Y}{X}, \ldots)$ matches one of the terms of (2). On the other hand it follows from Proposition 1 that every term of (2) is matched in this way.

We conclude from this that the terms in (2) is the subset of the terms in (4) which come from factor sequences such that (X, Y) fits around T. We claim that the sum of the remaining terms in (4) is zero. In fact, if $(\ldots, X, Y, \ldots)$ is a factor sequence for γ' such that (X, Y) doesn't fit around T and $S(\frac{T|Y}{X}) \neq 0$, then we may apply Fomin's involution in the way described above to get tableaux X' and Y'. If Conjecture 2 is true, then the sequence $(\ldots, X', Y', \ldots)$ is also a factor sequence, and since $S(\frac{T|Y'}{X'}) = -S(\frac{T|Y}{X})$, the terms of (4) given by these two factor sequences cancel each other out. □

The number of factor sequences for a tableau diagram can be extremely large. For this reason it is almost impossible to verify Conjecture 1 or Conjecture 1A directly by counting factor sequences. In contrast, instances of Conjecture 2 can be tested easily even on large examples. Given a tableau diagram and a path, one can generate a factor sequence for this path by choosing factorizations of tableaux by random. Then one can apply Fomin's involution to the sequence, and use the criterion of Proposition 1 to check that the result is still a factor sequence. Such checks have been carried out repeatedly for each of 500,000 randomly chosen tableau diagrams with up to 10 rows of tableaux, without finding any violations of Conjecture 2. Together with the results in the next section, we consider this to be convincing evidence for the conjectures.

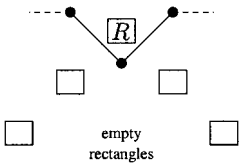
6. Proof in a special case

In this final section we will show that Conjecture 2 is true in certain special cases. These cases will be sufficient to prove the conjectured formula for $c_\mu(r)$ when all rectangles in and below the fourth row of the rectangle diagram are empty, and when no two non-empty rectangles in the third row are neighbors. This covers all situations with at most four vector bundles.

Let γ be a path through the rank diagram with a down-pointing angle as in the previous section. Let R be the rectangle of the corresponding triangle.

We will describe two cases where Conjecture 2 can be proved. Both cases require a special configuration of the rectangles surrounding R. Suppose R is the rectangle R_{ij} in the rectangle diagram. We will say that a different rectangle $R' = R_{kl}$ is *below* R if $k \leq i < j \leq l$. R' is *strictly below* R if $k < i < j < l$.

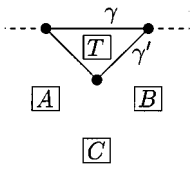
Proposition 2 *Conjecture* 2 *is true for* γ *if all rectangles strictly below* R *are empty.*



Note that this covers all rectangles on the left and right sides of the rectangle diagram.

Proof: Let T be the tableau corresponding to R, and let $(\ldots, X, Y, \ldots)$ be a factor sequence for γ. Since all tableau on the line going south-west from T in the tableau diagram are narrower than T, it follows that also X has fewer columns than T. Similarly Y has fewer rows than T. But this means that (X, Y) fits around T and the statement of Conjecture 2 is trivially true. □

In the other situation we shall describe, we allow three non-empty tableaux below T as shown in the picture.



All other tableaux below T are required to be empty. Let γ be the higher and γ' the lower of the two paths in the diagram.

Lemma 7 *Let* $(\ldots, X, Y, \ldots)$ *be a labeling of the line segments of* γ' *with tableaux. The following are equivalent*:

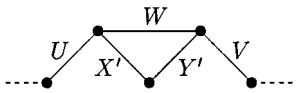
(1) $(\ldots, X, Y, \ldots)$ *is a factor sequence for* γ'.

(2) $(\ldots, X \cdot T \cdot Y, \ldots)$ *is a factor sequence for* γ *and the part of* X *that is wider than* T *and the part of* Y *that is taller than* T *have entries only from* C.

Proof: It is clear that (1) implies (2). For the other implication, put $W = X \cdot T \cdot Y$ and let $W = X' \cdot T \cdot Y'$ be the canonical factorization of W. Then it follows from Proposition 1 that $(\ldots, X', Y', \ldots)$ is a factor sequence for γ'. Since $(X, Y) \rightarrow (X', Y')$ by Lemma 2, we may assume that $(X, Y) \models (X', Y')$.

We will handle the case where a factor of the bottom part of Y is moved to X, the other case being symmetric. This means that for some tableau M we have $X' = X \cdot M$ and $Y = M \cdot Y'$. Since the bottom part of Y has entries only from C, this is also true for M.

We may assume that γ and γ' go down outside the displayed angle and that our factor sequence is $(\ldots, U, X', Y', V, \ldots)$.



Then by definition there exists a factorization $C = C'_1 \cdot C'_2$ such that $A \cdot C'_1 = U \cdot X'$ and $C'_2 \cdot B = Y' \cdot V$. Since $U \cdot X \cdot M$ consists of A with C'_1 attached on its right side, and since all entries of M are strictly larger than the entries of A, it follows that $U \cdot X$ consists of A with some tableau C_1 attached on the right side. Furthermore $C_1 \cdot M = C'_1$ by Lemma 1.

Put $C_2 = M \cdot C'_2$. Then we have $C_1 \cdot C_2 = C$, $A \cdot C_1 = U \cdot X$, and $C_2 \cdot B = Y \cdot V$. It follows that $(\ldots, U, X, Y, V, \ldots)$ is a factor sequence as required. □

Proposition 3 *Conjecture* 2 *is true for the path* γ' *in Lemma* 7.

Proof: Let $(\ldots, X, Y, \ldots)$ be a factor sequence for γ' which satisfies the conditions in Conjecture 2, and let X' and Y' be the tableaux obtained from X and Y using Fomin's involution. Since the part of X that is wider than T has entries only from C, the same will be true for X' by Lemma 5 (iii). Since Y' has fewer rows than T and since $(\ldots, X' \cdot T \cdot Y', \ldots) = (\ldots, X \cdot T \cdot Y, \ldots)$ is a factor sequence for γ, it follows from Lemma 7 that $(\ldots, X', Y', \ldots)$ is a factor sequence for γ'. □

Corollary 2 *Conjecture* 1 *is true if all rectangles in and below the fourth row of the rectangle diagram are empty, and if no two non-empty rectangles in the third row are neighbors.*

Proof: When the rectangle diagram satisfies these properties, then all instances of Conjecture 2 follow from either Proposition 2 or Proposition 3. The corollary therefore follows from Theorem 2. □

In Section 2 we defined a rectangle diagram to be something you get by replacing the small triangles of numbers in a rank diagram with rectangles. However, everything we have done is still true if one defines a rectangle diagram to be any diagram of rectangles, each given by a number of rows and columns, such that the number of rows decreases when one moves south-east while the number of columns decreases when one moves south-west. This definition is slightly more general because the side lengths of the rectangles in a rectangle diagram obtained from rank conditions satisfy certain relations. Although we don't know any geometric interpretation of these more general rectangle diagrams, they seem to be the natural definition for combinatorial purposes.

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