Journal of Algebraic Combinatorics **13** (2001), 317–326

The Integral Tree Representation of the Symmetric Group

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Received January 11, 1999; Revised May 10, 2000

Abstract. Let T_n be the space of fully-grown n-trees and let V_n and V_n' be the representations of the symmetric groups Σ_n and Σ_{n+1} respectively on the unique non-vanishing reduced integral homology group of this space. Starting from combinatorial descriptions of V_n and V_n', we establish a short exact sequence of $\mathbb{Z}\Sigma_{n+1}$-modules, giving a description of V_n' in terms of V_n and V_{n+1}. This short exact sequence may also be deduced from work of Sundaram.

Modulo a twist by the sign representation, V_n is shown to be dual to the Lie representation of Σ_n, Lie_n. Therefore we have an explicit combinatorial description of the integral representation of Σ_{n+1} on Lie_n and this representation fits into a short exact sequence involving Lie_n and Lie_{n+1}.

Keywords: symmetric group representation, free Lie algebra

Introduction

The space of fully-grown n-trees T_n was described in [7]. This is a topological space with an action of the symmetric group Σ_{n+1}. It has the homotopy type of a bouquet of spheres and its unique non-vanishing reduced homology group affords a representation of Σ_{n+1}, the tree representation.

The space T_n arises naturally in joint work of the author with Alan Robinson on 'gamma homology', a homology theory for coherently homotopy commutative algebras (E_∞-algebras) [8]. This is motivated by the problem in stable homotopy theory of developing an obstruction theory for E_∞ structures on ring spectra. The representation of Σ_{n+1} on T_n plays an important role. Character calculations were carried out in [7] and related representations were studied in [11].

The same representation has arisen in several other contexts. As will be shown below, it is closely related to the Σ_{n+1} action on Lie_n, the multilinear component of the free Lie algebra on n generators. This action is due originally to Kontsevich and has been studied by Getzler and Kapranov [4].

In [6] Mathieu studies the same action on the top component of the rational cohomology of the complement of the braid arrangement. By looking at integral cohomology, he shows

*I acknowledge the support of a TMR grant from the European Union, held at the Laboratoire d'Analyse, Géométrie et Applications (UMR 7539 au CNRS), Université Paris-Nord.

that the action does not come from an action of a transformation group on the complement of the braid arrangement. It is possible that the results of this paper may shed some further light on this.

This and related representations have been extensively studied by Sundaram via subposets of the partition lattice [10]; see also [9]. Related work in the rational situation has also been carried out by Hanlon [5]. Finally, the same representation arises in the homology of the complex of 'not 2-connected' graphs [1].

The aim of the present paper is to study the *integral* representation of Σ_{n+1} on T_n. Let V_n and V_n' denote the representations of Σ_n and Σ_{n+1} respectively on the unique non-vanishing reduced integral homology group of the tree space T_n. We start from explicit combinatorial descriptions of these representations, involving shuffles. In the first section, these are used to deduce various properties of the representations. In particular, we prove (1.8) that there is a short exact sequence of $\mathbb{Z}\Sigma_{n+1}$-modules

$$0 \to V_{n+1} \to \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n \to V_n' \to 0\,.$$

This generalizes the *rational* direct sum decomposition

$$\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n \cong V_n' \oplus V_{n+1}\,,$$

of $\mathbb{Q}\Sigma_{n+1}$-modules of [11, 3.4]. We note that the short exact sequence of $\mathbb{Z}\Sigma_{n+1}$-modules may also be deduced from the work of Sundaram [10] which uses entirely different methods. We end the first section with a simple example where the sequence does not split over the integers.

The second section establishes the relationship between the integral tree representation V_n of Σ_n and Lie_n. We use a direct combinatorial argument to prove that there is an isomorphism of $\mathbb{Z}\Sigma_n$-modules

$$\mathrm{Lie}_n \cong \mathrm{Hom}(V_n, \mathbb{Z}[-1]),$$

where $\mathbb{Z}[-1]$ denotes the sign representation. Related results were proved by Barcelo [2]. From this result and those of the first section, we recover the fact that Lie_n carries an action of Σ_{n+1} and we have an explicit combinatorial description for this $\mathbb{Z}\Sigma_{n+1}$-module. This integral representation of Σ_{n+1} fits into a short exact sequence involving Lie_n and Lie_{n+1}.

1. The symmetric group representations, V_n and V_n'

The symmetric group Σ_n is the group of permutations of the set $\{1, \ldots, n\}$. Permutations are written in cycle notation and multiplied from right to left, so for example $(1\,2)(2\,3) = (1\,2\,3)$. The identity permutation is written 1. We regard Σ_n as contained in Σ_{n+1} as the subgroup of permutations fixing $n+1$, and similarly the group ring $\mathbb{Z}\Sigma_n$ is a subring of $\mathbb{Z}\Sigma_{n+1}$. We denote the sign of a permutation π by $\epsilon(\pi) \in \{-1,\ 1\}$.

Our starting point is the following description of representations V_n and V_n' of Σ_n and Σ_{n+1} respectively.

Definition 1.1 Recall that $\pi \in \Sigma_n$ is an $(i, n-i)$-shuffle if $\pi(1) < \pi(2) < \cdots < \pi(i)$ and $\pi(i+1) < \pi(i+2) < \cdots < \pi(n)$. For $i = 1, \ldots, n-1$, define

$$s_{i,n-i} = \sum \epsilon(\pi)\pi \in \mathbb{Z}\Sigma_n \quad \text{and} \quad \bar{s}_{i,n-i} = \sum \epsilon(\pi)\pi^{-1} \in \mathbb{Z}\Sigma_n\,,$$

where each sum is over all $(i, n-i)$-shuffles π in Σ_n. We set $s_{0,n} = s_{n,0} = \bar{s}_{0,n} = \bar{s}_{n,0} = 1$.

Now we define the integral representations V_n and V'_n of Σ_n and Σ_{n+1} respectively as follows. As a left $\mathbb{Z}\Sigma_n$-module V_n has a single generator, denoted c_n, subject to the relations

$$\bar{s}_{i,n-i}c_n = 0, \quad \text{for } i = 1, \ldots, n-1.$$

Let $t_{n+1} = (-1)^n(1\ 2\ \ldots\ n+1) \in \mathbb{Z}\Sigma_{n+1}$. As a $\mathbb{Z}\Sigma_{n+1}$-module V'_n also has a single generator, which we denote again by c_n, subject to the relations

$$\begin{aligned} \bar{s}_{i,n-i}c_n &= 0, \quad \text{for } i = 1, \ldots, n-1; \\ (1 - t_{n+1})c_n &= 0. \end{aligned}$$

It is proved in [12] that V_n and V'_n are the representations of Σ_n and Σ_{n+1} respectively given by the unique non-vanishing reduced homology group of the space of fully grown n-trees, T_n. In particular, the restriction of V'_n to Σ_n is V_n.

Notation 1.2 Let $p_{n,i} = (n\ i)(n-1\ i+1)(n-2\ i+2)\cdots \in \Sigma_n$, for $i = 1, \ldots, n$, and let $\rho_{n,i} = (-1)^{n-i}\epsilon(p_{n,i})p_{n,i} \in \mathbb{Z}\Sigma_n$.

Lemma 1.3 ([12, III.3.1]) *For $i = 1, \ldots, n$, in V_n we have the relations*

$$\rho_{n,i}c_n = \bar{s}_{i-1,n-i}c_n\,.$$

Proof: We use downward induction on i. When $i = n$ the result is trivial, since both sides of the equation above are equal to c_n. Now suppose the result holds for $i+1$. It is clear that if π is an $(i, n-i)$-shuffle, then either $\pi(i) = n$ or $\pi(n) = n$. Let $\bar{s}^{(i)}_{i,n-i} = \sum \epsilon(\pi)\pi^{-1}$, where the sum is over $(i, n-i)$-shuffles such that $\pi(i) = n$. Then $\bar{s}_{i,n-i} = \bar{s}^{(i)}_{i,n-i} + \bar{s}_{i,n-i-1}$ for $i = 1, \ldots, n-1$. Hence

$$\begin{aligned} 0 = \bar{s}_{i,n-i}c_n &= \bar{s}^{(i)}_{i,n-i}c_n + \bar{s}_{i,n-i-1}c_n \\ &= \bar{s}^{(i)}_{i,n-i}c_n + \rho_{n,i+1}c_n \quad \text{by induction hypothesis.} \end{aligned}$$

Now π is an $(i, n-i)$-shuffle such that $\pi(i) = n$ if and only if $\pi(i\ i+1 \ldots n)$ is an $(i-1, n-i)$-shuffle. Thus $(i\ i+1 \ldots n)^{-1}\bar{s}^{(i)}_{i,n-i} = (-1)^{n-i}\bar{s}_{i-1,n-i}$. Furthermore, $(i\ i+1 \ldots n)^{-1}\rho_{n,i+1} = (-1)^{n-i+1}\rho_{n,i}$. Composing each side of the above with the permutation $(i\ i+1 \ldots n)^{-1}$ gives the result. □

Next we give some identities in the group ring $\mathbb{Z}\Sigma_n$, which will be useful in studying our representations.

Lemma 1.4 *Let* $\nu_n = (1-\mu_{n-1,n})(1-\mu_{n-2,n})\cdots(1-\mu_{1,n}) \in \mathbb{Z}\Sigma_n$, *where* $\mu_{i,n} = (-1)^{n-i}(i\ i+1\ \ldots\ n)$. *Then, for* $i = 1, \ldots, n-1$

$$\nu_n s_{i,n-i} = 0\,.$$

Proof: This is a straightforward deduction from relations in the group ring $\mathbb{Z}\Sigma_n$ given by Garsia in [3]. A special case of [3, 2.2] (modulo an introduction of signs) is $p_{n,1}\nu_n p_{n,1}s_{i,n-i} = 0$, for $i = 1, \ldots, n-1$. Since $p_{n,1}s_{i,n-i} = s_{n-i,i}p_{n,1}$, for $i = 1, \ldots, n-1$, the required result follows. □

Proposition 1.5 *As a* $\mathbb{Z}$*-module*, V_n *is free of rank* $(n-1)!$. *A basis is given by* $\{\pi c_n \mid \pi \in \Sigma_{n-1}\}$.

Proof: Note that $\{p_{n,i} \mid i = 1, \ldots, n\}$ is a set of coset representatives for Σ_{n-1} in Σ_n. The equations of 1.3 express the action of each of these on the generator c_n in terms of only πc_n's where $\pi \in \Sigma_{n-1}$. Therefore, $\{\pi c_n \mid \pi \in \Sigma_{n-1}\}$ is a generating set for V_n.

Now we claim that in V_n, $\{\pi c_n \mid \pi \in \Sigma_{n-1}\}$ is linearly independent. Equivalently, we show that, for $z_\pi \in \mathbb{Z}$ and $a_i \in \mathbb{Z}\Sigma_n$,

$$\sum_{\pi\in\Sigma_{n-1}} z_\pi \pi = \sum_{i=1}^{n-1} a_i \bar{s}_{i,n-i} \quad\Rightarrow\quad z_\pi = 0 \quad \text{for all } \pi \in \Sigma_{n-1}.$$

For suppose some $z_\pi \neq 0$; without loss of generality, we may assume $z_1 \neq 0$ (since otherwise we multiply by π^{-1}). For $\sigma \in \Sigma_n$, let $C(\sigma) \in \mathbb{Z}$ denote the coefficient of σ in $\sum_{i=1}^{n-1} a_i \bar{s}_{i,n-i}$. Then if $x = \sum_{\sigma\in\Sigma_n} x_\sigma \sigma$, set $C(x) = \sum_{\sigma\in\Sigma_n} x_\sigma C(\sigma)$. Consider ν_n as in 1.4. Let $a_i = \sum_{\pi\in\Sigma_n}(a_i)_\pi \pi$. Then $C(\nu_n)$ is the sum over i and over π in Σ_n of $(a_i)_\pi$ multiplied by the coefficient of π in $\nu_n s_{i,n-i}$, which is zero by 1.4. Thus $C(1) = C(1-\nu_n)$. Now note that $1-\nu_n$ is a $\mathbb{Z}$-linear combination of permutations in $\Sigma_n\backslash\Sigma_{n-1}$. But in $\sum_{\pi\in\Sigma_{n-1}} z_\pi\pi$ we have $z_1 \neq 0$ and the coefficient of each $\sigma \in \Sigma_n\backslash\Sigma_{n-1}$ is zero, giving a contradiction. □

Now we prove certain identities in the integral group ring $\mathbb{Z}\Sigma_{n+1}$, which will be used to establish the module structure of V'_n.

Proposition 1.6 *For* $k = 1, \ldots, n$, $\bar{s}_{k,n+1-k}(1-t_{n+1})$ *is an element of the left ideal* $\sum_{i=1}^{n-1}\mathbb{Z}\Sigma_{n+1}\bar{s}_{i,n-i}$ *of* $\mathbb{Z}\Sigma_{n+1}$.

Proof: Let $\mu_{i,n} \in \mathbb{Z}\Sigma_n$ be as in 1.4. We show that, for $k = 1, \ldots, n$,

$$\bar{s}_{k,n+1-k}(1-t_{n+1}) = (1-\mu_{k+1,n+1})\bar{s}_{k,n-k} + (\mu_{k,n+1}-t_{n+1})\bar{s}_{k-1,n+1-k}\,.$$

As in the proof of 1.3, we have

$$(*)\qquad \bar{s}_{k,n+1-k} = \mu_{k,n+1}\bar{s}_{k-1,n+1-k} + \bar{s}_{k,n-k}, \quad \text{for } k = 1, \ldots, n.$$

Hence, the required result is equivalent to

$$\bar{s}_{k,n+1-k}t_{n+1} = \mu_{k+1,n+1}\bar{s}_{k,n-k} + t_{n+1}\bar{s}_{k-1,n+1-k}\,,$$

that is

$$\bar{s}_{k,n+1-k} = \mu_{k+1,n+1}\bar{s}_{k,n-k}t_{n+1}^{-1} + t_{n+1}\bar{s}_{k-1,n+1-k}t_{n+1}^{-1}\,.$$

This identity may be derived in a similar manner to $(*)$. A $(k, n+1-k)$-shuffle π must have either $\pi(1) = 1$ or $\pi(k+1) = 1$. Now note that α is a $(k, n-k)$-shuffle if and only if $(1\ 2 \ldots n+1)\alpha(k+1\ k+2 \ldots n+1)^{-1}$ is a $(k, n+1-k)$-shuffle taking $k+1$ to 1. Furthermore, σ is a $(k-1, n+1-k)$-shuffle if and only if $(1\,2 \ldots n+1)\sigma(1\,2 \ldots n+1)^{-1}$ is a $(k, n+1-k)$-shuffle fixing 1. The result follows. □

Corollary 1.7 *The following relations hold in* $\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n = \mathbb{Z}\Sigma_{n+1} \otimes_{\mathbb{Z}\Sigma_n} V_n$,

$$\bar{s}_{k,n+1-k}((1 - t_{n+1}) \otimes c_n) = 0 \quad \text{for } k = 1, \ldots, n.$$

Proof: By 1.6, $\bar{s}_{k,n+1-k}(1 - t_{n+1}) = \sum_{i=1}^{n-1} x_i \bar{s}_{i,n-i}$, for some x_i in $\mathbb{Z}\Sigma_{n+1}$. So

$$\bar{s}_{k,n+1-k}((1 - t_{n+1}) \otimes c_n) = \sum_{i=1}^{n-1} x_i \bar{s}_{i,n-i} \otimes c_n = \sum_{i=1}^{n-1} x_i \otimes \bar{s}_{i,n-i} c_n = 0.$$ □

Theorem 1.8 *There is a short exact sequence of* $\mathbb{Z}\Sigma_{n+1}$*-modules*

$$0 \to V_{n+1} \to \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n \to V_n' \to 0\,.$$

Proof: The $\mathbb{Z}\Sigma_{n+1}$-module V_{n+1} is determined by the relations $\bar{s}_{i,n+1-i} c_{n+1} = 0$ for $i = 1, \ldots, n$. Let $\theta : V_{n+1} \to \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n$ be defined by

$$\theta(x c_{n+1}) = x(1 - t_{n+1}) \otimes c_n, \quad \text{for } x \in \mathbb{Z}\Sigma_{n+1}.$$

By 1.7, θ respects the relations, so θ is a well-defined $\mathbb{Z}\Sigma_{n+1}$-module homomorphism.

Next we claim θ is injective. From 1.5, V_{n+1} has a basis $\{\pi c_{n+1} \mid \pi \in \Sigma_n\}$. For $\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n = \mathbb{Z}\Sigma_{n+1} \otimes_{\mathbb{Z}\Sigma_n} V_n$ we have a basis $\{t_{n+1}^i \otimes \sigma c_n \mid \sigma \in \Sigma_{n-1},\ i = 0, \ldots, n\}$. Let $x = \sum_{\pi \in \Sigma_n} x_\pi \pi$, for $x_\pi \in \mathbb{Z}$. Then

$$\theta(x c_{n+1}) = \sum_{\pi \in \Sigma_n} x_\pi \pi(1 - t_{n+1}) \otimes c_n = \sum_{\pi \in \Sigma_n} 1 \otimes x_\pi \pi c_n \pm t_{n+1}^{\pi(1)} \otimes x_\pi \bar{\pi} c_n\,,$$

where $\bar{\pi} = t_{n+1}^{-\pi(1)} \pi t_{n+1} \in \Sigma_n$. Then $\theta(x c_{n+1}) = 0$ implies $\sum_{\pi \in \Sigma_n, \pi(1)=k} x_\pi \bar{\pi} c_n = 0$ for $k = 1, \ldots, n$, which in turn implies $\sum_{\pi \in \Sigma_n, \pi(1)=k} x_\pi (n\ n+1-k)\bar{\pi} c_n = 0$ for $k = 1, \ldots, n$. But

$$\begin{aligned} &\{(n\ n+1-k)\bar{\pi} c_n \mid \pi \in \Sigma_n,\ \pi(1) = k\} \\ &\quad = \{(n\ n+1-k) t_{n+1}^{-k} \pi t_{n+1} c_n \mid \pi \in \Sigma_n,\ \pi(1) = k\} \\ &\quad = \{\sigma c_n \mid \sigma \in \Sigma_{n-1}\}\,, \end{aligned}$$

which is linearly independent by 1.5. Hence, $x_\pi = 0$ for all $\pi \in \Sigma_n$ and θ is injective.

Let $\Psi : \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n \to V_n'$ be defined by

$$\Psi(x \otimes c_n) = x c_n , \quad \text{for } x \in \mathbb{Z}\Sigma_{n+1}.$$

This is clearly a well-defined surjective $\mathbb{Z}\Sigma_{n+1}$-module homomorphism. Furthermore,

$$\Psi\theta(x c_{n+1}) = \Psi(x(1 - t_{n+1}) \otimes c_n) = x(1 - t_{n+1})c_n = 0 ,$$

so $\Psi\theta = 0$.

Now let $\Phi : V_n' \to (\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n)/\theta(V_{n+1})$ be the homomorphism of $\mathbb{Z}\Sigma_{n+1}$-modules determined by $\Phi(c_n) = [1 \otimes c_n]$. The relations determining the module V_n' are $\bar{s}_{i,n-i} c_n = 0$ for $i = 1, \ldots, n$ and $(1 - t_{n+1})c_n = 0$. We have $\Phi(\bar{s}_{i,n-i} c_n) = [\bar{s}_{i,n-i} \otimes c_n] = [1 \otimes \bar{s}_{i,n-i} c_n] = [0]$. Also $\Phi((1 - t_{n+1})c_n) = [(1 - t_{n+1}) \otimes c_n] = [0]$. So Φ is a well-defined $\mathbb{Z}\Sigma_{n+1}$-module homomorphism. It is clear that Φ is surjective.

The set $\{\sigma c_n \mid \sigma \in \Sigma_{n-1}\}$ is a basis for V_n'. Consider the elements $\Phi(\sigma c_n) = [1 \otimes \sigma c_n]$ for $\sigma \in \Sigma_{n-1}$. The elements $1 \otimes \sigma c_n$, for $\sigma \in \Sigma_{n-1}$, are independent in $\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n$ and clearly remain independent on passing to the quotient by $\theta(V_{n+1})$. Thus Φ is injective.

So $V_n' \cong \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} V_n/\theta(V_{n+1})$ and the sequence is exact. □

The short exact sequence of $\mathbb{Z}\Sigma_{n+1}$-modules of 1.8 may also be deduced from the work of Sundaram; see particularly 1.1 and the proof of 3.5 in [10].

Proposition 1.9 *The short exact sequence of* 1.8 *splits if $n + 1$ is inverted.*

Proof: If $n + 1$, the index of Σ_n in Σ_{n+1}, is invertible in the ring R then the restriction map

$$\mathrm{res}_{\Sigma_{n+1},\Sigma_n} : \mathrm{Ext}^1_{R\Sigma_{n+1}}(V_n', V_{n+1}) \to \mathrm{Ext}^1_{R\Sigma_n}(V_n', V_{n+1})$$

is injective. But, from 1.5, V_{n+1} is free over Σ_n, so $\mathrm{Ext}^1_{R\Sigma_n}(V_n', V_{n+1}) = 0$. Thus $\mathrm{Ext}^1_{R\Sigma_{n+1}}(V_n', V_{n+1}) = 0$ when $n + 1$ is invertible in R. □

Example 1.10 We consider the case $n = 2$. Here V_2 is the trivial representation of Σ_2, with basis c_2; V_2' is the trivial representation of Σ_3 with basis c_2; $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} V_2$ is a 3-dimensional module, with basis $\alpha = 1 \otimes c_2$, $\beta = t_3 \otimes c_2$, $\gamma = t_3^2 \otimes c_2$, and V_3 is a 2-dimensional module with basis c_3, $(1\ 2)c_3$. By 1.8, $\theta(V_3)$ is a submodule of $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} V_2$, with basis $\theta(c_3) = \alpha - \beta$, $\theta((1\ 2)c_3) = \alpha - \gamma$. The quotient $(\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} V_2)/\theta(V_3)$ is the trivial representation of Σ_3, V_2'. Now any trivial submodule of $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} V_2$ has basis a multiple of $\alpha + \beta + \gamma$. The matrix changing the basis α, β, γ to $\alpha - \beta, \alpha - \gamma, \alpha + \beta + \gamma$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

This has determinant 3. So if we invert 3, the trivial module V_2' is a complementary submodule to $\theta(V_3)$ in $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} V_2$. However, if 3 is not inverted, $\theta(V_3)$ is not a direct summand of $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} V_2$.

2. The relationship with Lie$_n$

The free Lie algebra on n generators is the subalgebra generated by brackets $[a, b] = ab - ba$ of the algebra of all $\mathbb{Z}$-linear combinations of words in an alphabet with n letters. Then Lie_n is that part of the free Lie algebra consisting of $\mathbb{Z}$-linear combinations of brackets which contain each generator exactly once. There is a natural left action of Σ_n on Lie_n by permuting the generators. The main result of this section is Theorem 2.6, which establishes an isomorphism of $\mathbb{Z}\Sigma_n$-modules

$$\mathrm{Lie}_n \cong \mathrm{Hom}(V_n, \mathbb{Z}[-1]),$$

where V_n is the representation defined in §1 and $\mathbb{Z}[-1]$ denotes the sign representation.

We begin with the following result, which is well-known. For a proof see [3], for example.

Proposition 2.1 *As a* $\mathbb{Z}$*-module*, Lie_n *is free of rank* $(n-1)!$. *Let*

$$A_n = [...[[a_n, a_{n-1}], a_{n-2}] \dots a_1] \in \mathrm{Lie}_n .$$

Then the elements πA_n *for* $\pi \in \Sigma_{n-1}$ *are generators for* Lie_n *as a free abelian group.*

We need the following notation, in addition to that of §1.

Notation 2.2 *Let* τ_n *denote the* (*unsigned*) *n-cycle* $(1\,2\,\dots\,n)$ *in* Σ_n. *We define inductively elements* $\theta_{n,k} \in \mathbb{Z}\Sigma_{n-1}$, *for each* $n \geq 2$ *and for* $k = 1, \dots, n$ *by*

$$\theta_{2,1} = -1, \quad \theta_{n,1} = \theta_{n-1,1}\big(\tau_{n-1}^{-1} - 1\big), \quad \textit{for } n \geq 3,$$
$$\theta_{2,2} = 1, \quad \theta_{n,k} = \tau_{n-1}\theta_{n-1,k-1}\tau_{n-1}^{-1}, \quad \textit{for } k = 2, \dots, n.$$

It follows easily that $\theta_{n,n} = 1$.

In 1.3, we gave the action of the permutation $p_{n,k}$ on the generator c_n of V_n. The following result describes the action of this element on the generator A_n of Lie_n.

Proposition 2.3 *For all $n \geq 2$ and for $k = 1, \ldots, n$,*

$$p_{n,k} A_n = \theta_{n,k} A_n \, .$$

Proof: The proof is by a double induction. First we fix $k = 1$ and use induction on n to prove that $p_{n,1} A_n = \theta_{n,1} A_n$ for all n. This is evident for $n = 2$. Assume the result for $j < n$. Let $B_{n-1} = [\ldots[[b_{n-1}, b_{n-2}], b_{n-3}] \ldots b_1]$, where $b_i = a_i$, for $i = 1, \ldots, n-2$, and $b_{n-1} = [a_{n-1}, a_n]$. Then

$$\begin{aligned} p_{n,1} A_n &= [...[[a_1, a_2], a_3] \ldots a_n] \\ &= [[\,...\,[[a_1, a_2], \ldots a_{n-2}], a_n], a_{n-1}] + [...[[a_1, a_2], \ldots a_{n-2}], [a_{n-1}, a_n]] \\ &= \theta_{n-1,1}[[\,...\,[[a_n, a_{n-2}], \ldots, a_2], a_1], a_{n-1}] + p_{n-1,1} B_{n-1} \quad \text{by hypothesis} \\ &= \theta_{n-1,1}(n-1\ n-2\ \ldots\ 1) A_n + \theta_{n-1,1} B_{n-1} \quad \text{by induction hypothesis} \\ &= \theta_{n-1,1}(n-1\ n-2\ \ldots\ 1) A_n - \theta_{n-1,1} A_n \quad \text{since } \theta_{n-1,1} \in \mathbb{Z}\Sigma_{n-2} \\ &= \theta_{n-1,1}\big(\tau_{n-1}^{-1} - 1\big) A_n = \theta_{n,1} A_n \, . \end{aligned}$$

Now suppose that $p_{n-1,k-1} A_{n-1} = \theta_{n-1,k-1} A_{n-1}$ for given n and k. This time set $b_i = a_{i+1}$ for $i = 1, \ldots, n-1$. Then, for $k > 1$,

$$\begin{aligned} p_{n,k} A_n &= p_{n,k}[\,...\,[[a_n, a_{n-1}], a_{n-2}] \ldots a_1] \\ &= p_{n-1,k-1}[[\,...\,[[b_{n-1}, b_{n-2}], b_{n-3}] \ldots b_1], a_1] \\ &= \theta_{n-1,k-1}[[\,...\,[[b_{n-1}, b_{n-2}], b_{n-3}] \ldots b_1], a_1] \quad \text{by hypothesis} \\ &= \tau_{n-1} \theta_{n-1,k-1} \tau_{n-1}^{-1} A_n \\ &= \theta_{n,k} A_n \, . \end{aligned}$$

The general result now follows by induction. □

The following lemma gives an alternative description of the elements $\theta_{n,k}$.

Lemma 2.4 *For $n \geq 2$ and $k = 1, \ldots, n$,*

$$\theta_{n,k} = \sum (-1)^{(n-\pi^{-1}(k))} \pi,$$

where the sum is over all permutations π in Σ_{n-1} such that $p_{n,k} \pi p_{n,\pi^{-1}(k)}$ is a $(\pi^{-1}(k) - 1, n - \pi^{-1}(k))$-shuffle.

Proof: Let $X_{n,k} = \sum (-1)^{(n-\pi^{-1}(k))} \pi$, where the sum is over all permutations π in Σ_{n-1} such that $p_{n,k} \pi p_{n,\pi^{-1}(k)}$ is a $(\pi^{-1}(k) - 1, n - \pi^{-1}(k))$-shuffle. First we consider the case $k = 1$. Since $X_{2,1} = -1 = \theta_{2,1}$, it is sufficient to show that $X_{n,1} = X_{n-1,1}(\tau_{n-1}^{-1} - 1)$. Let $\pi \in \Sigma_{n-1}$ be a summand of $X_{n,1}$, that is $\sigma = p_{n,1} \pi p_{n,\pi^{-1}(1)}$ is a $(\pi^{-1}(1) - 1, n - \pi^{-1}(1))$-shuffle. Note that $\sigma(\pi^{-1}(1)) = 1$. It follows that either $\sigma(\pi^{-1}(1) + 1) = 2$, in which case $\pi(n-1) = n-1$; or $\sigma(1) = 2$, in which case $\pi(1) = n-1$.

In the first case, it is straightforward to check that π also appears in $X_{n-1,1}$ but with opposite sign. In the second case, we may check that $\pi\tau_{n-1}$ is a summand of $X_{n-1,1}$. The result for $k = 1$ follows.

To deduce the general case, first note that it follows immediately from the definition that $X_{n,n} = 1$. Then, for $k = 2, \ldots, n$, it is sufficient to show that $X_{n,k} = \tau_{n-1}X_{n-1,k-1}\tau_{n-1}^{-1}$. Suppose σ is a summand of $X_{n-1,k-1}$, and let $\pi = \tau_{n-1}\sigma\tau_{n-1}^{-1}$. So, setting $j = \sigma^{-1}(k-1)$, $\alpha = p_{n-1,k-1}\sigma p_{n-1,j}$ is a $(j-1, n-j-1)$-shuffle. Then $\pi^{-1}(k) = j+1$. Consider $\beta = p_{n,k}\pi p_{n,j+1}$. One can check that $\beta = x\alpha y$ where $x = (1\,2\ldots k-1)(k\,k+1\ldots n)$, and $y = (j\,j-1\ldots 2\,1)(n\,n-1\ldots j+1)$. Then a straightforward argument shows that β is a $(j, n-j-1)$-shuffle if and only if α is a $(j-1, n-j-1)$-shuffle. Hence σ is a summand of $X_{n-1,k-1}$ if and only if $\tau_{n-1}\sigma\tau_{n-1}^{-1}$ is a summand of $X_{n,k}$. □

Now we turn to consideration of $\mathrm{Hom}(V_n, \mathbb{Z}[-1])$. Let $f_n \in \mathrm{Hom}(V_n, \mathbb{Z}[-1])$ be the dual of c_n with respect to the basis of 1.5.

Proposition 2.5 *For $k = 1, \ldots, n$*

$$p_{n,k}f_n = \theta_{n,k}f_n\,.$$

Proof: For $\pi \in \Sigma_{n-1}$, we have

$$\begin{aligned}
(p_{n,k}f_n)(\pi c_n) &= \epsilon(p_{n,k})f_n(p_{n,k}\pi c_n)\\
&= \epsilon(p_{n,k})f_n\big(\sigma p_{n,\pi^{-1}(k)}c_n\big) \quad \text{where } \sigma = p_{n,k}\pi p_{n,\pi^{-1}(k)} \in \Sigma_{n-1}\\
&= \epsilon(p_{n,k})(-1)^{(n-\pi^{-1}(k))}\epsilon\big(p_{n,\pi^{-1}(k)}\big)f_n\big(\sigma\rho_{n,\pi^{-1}(k)}c_n\big)\\
&= \epsilon\big(p_{n,k}p_{n,\pi^{-1}(k)}\big)(-1)^{(n-\pi^{-1}(k))}f_n\big(\sigma\bar{s}_{\pi^{-1}(k)-1,n-\pi^{-1}(k)}c_n\big) \quad \text{by 1.3}\\
&= \begin{cases}(-1)^{(n-\pi^{-1}(k))}\epsilon(\pi), & \text{if } \sigma \text{ is a } (\pi^{-1}(k)-1, n-\pi^{-1}(k))\text{-shuffle;}\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

The result now follows from 2.4. □

Theorem 2.6 *There is an isomorphism of $\mathbb{Z}\Sigma_n$-modules*

$$\mathrm{Lie}_n \cong \mathrm{Hom}(V_n, \mathbb{Z}[-1]).$$

Proof: Define $\Phi : \mathrm{Lie}_n \to \mathrm{Hom}(V_n, \mathbb{Z}[-1])$ by $\Phi(\pi A_n) = \pi f_n$. By 2.1 and 1.5, this takes the basis $\{\pi A_n \mid \pi \in \Sigma_{n-1}\}$ of Lie_n to the basis $\{\pi f_n \mid \pi \in \Sigma_{n-1}\}$ of $\mathrm{Hom}(V_n, \mathbb{Z}[-1])$. By 2.3 and 2.5, Φ respects the $\mathbb{Z}\Sigma_n$-module structures. Thus Φ is a well-defined $\mathbb{Z}\Sigma_n$-module isomorphism. □

Let the dual of a module M be denoted M^*. Combining 2.6 with 1.8 allows us to recover the following result.

Corollary 2.7 *There is a* Σ_{n+1} *action on* Lie_n^* *and hence on* Lie_n. *Denoting these* $\mathbb{Z}\Sigma_{n+1}$*-modules by* $(\mathrm{Lie}_n^*)'$ *and* Lie_n', *we have short exact sequences of* $\mathbb{Z}\Sigma_{n+1}$*-modules*

$$0 \to \mathrm{Lie}_{n+1}^* \to \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} \mathrm{Lie}_n^* \to (\mathrm{Lie}_n^*)' \to 0,$$
$$0 \leftarrow \mathrm{Lie}_{n+1} \leftarrow \mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} \mathrm{Lie}_n \leftarrow \mathrm{Lie}_n' \leftarrow 0.$$

We remark that 2.6 and the results of the first section give an explicit combinatorial description of the Σ_{n+1} action on Lie_n.

References

1. E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker, "Complexes of not i-connected graphs," extended abstract, in Proceedings of the 9th Formal Power Series and Algebraic Combinatorics Conference, Vienna, July 1997.
2. H. Barcelo, "On the action of the symmetric group on the free Lie algebra and the partition lattice," *J. Combin. Theory (A)* **55** (1990), 93–129.
3. A.M. Garsia, "Combinatorics of the free Lie algebra and the symmetric group," in *Analysis*, research papers published in honour of J. Moser's 60th birthday, P.H. Rabinowitz and E. Zehnder (Eds.), Academic Press, San Diego, 1990.
4. E. Getzler and M.M. Kapranov, "Cyclic operads and cyclic homology," in *Conference Proceedings and Lecture Notes in Geometry and Topology*, Vol. VI, S.-T. Yau (Ed.), International Press, Cambridge, MA, 1995, pp. 167–201.
5. P. Hanlon, "Otter's method and the homology of homeomorphically irreducible k-trees," *J. Combin. Theory (A)* **74**(2) (1996), 301–320.
6. O. Mathieu, "Hidden Σ_{n+1}-actions," *Commun. Math. Phys.* **176** (1996), 467–474.
7. A. Robinson and S. Whitehouse, "The tree representation of Σ_{n+1}," *J. Pure Appl. Algebra* **111** (1996), 245–253.
8. A. Robinson and S. Whitehouse, "Operads and Γ-homology of commutative rings," *Math. Proc. Combin. Phil. Soc.* Preprint, Université Paris 13, 1998.
9. S. Sundaram, "Decompositions of S_n-submodules in the free Lie algebra," *J. Algebra* **154** (1993), 507–558.
10. S. Sundaram, "Homotopy of non-modular partitions and the Whitehouse module," *J. Alg. Combin.* **9** (1999), 251–269.
11. S. Whitehouse, "The Eulerian representations of Σ_n as restrictions of representations of Σ_{n+1}," *J. Pure Appl. Algebra* **115** (1997), 309–320.
12. S.A. Whitehouse, "Γ-(co)homology of commutative algebras and some related representations of the symmetric group," Ph.D. Thesis, University of Warwick, 1994.