Journal of Algebraic Combinatorics **14** (2001), 133–144

On the Various Realizations of the Basic Representation of $\mathbf{A}_{n-1}^{(1)}$ and the Combinatorics of Partitions

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Received December 1, 2001; Revised April 2, 2001

Abstract. The infinite dimensional Lie algebra $\hat{sl}_n = A_{n-1}^{(1)}$ can be realized in several ways as an algebra of differential operators. The aim of this note is to show that the intertwining operators between the realizations of $\hat{sl}_n$ corresponding to all partitions of n can be described very simply by using combinatorial constructions.

Keywords: Lie algebras, representation theory, combinatorics of partitions, symmetric functions

1. Introduction

The infinite dimensional Lie algebra $\hat{sl}_n = A_{n-1}^{(1)}$ can be realized in several ways as an algebra of differential operators. This is done by identifying its simplest irreducible highest weight representation (the so-called basic representation) with the tensor product of a polynomial ring by the group algebra of a lattice. The two famous such realizations are the principal one, due to Kac, Kazhdan, Lepowsky and Wilson [6] and the homogeneous one, due to Frenkel and Kac [1]. But there are many other realizations. Kac and Peterson [5] have classified them and showed that they are parametrized by the partitions $\nu = (n_1, n_2, \dots, n_s)$ of n. The principal and homogeneous realizations can be regarded as the two extreme cases corresponding respectively to $\nu = (n)$ and $\nu = (1, \dots, 1)$. Ten Kroode and Van de Leur [4] have calculated the vertex operators for all partitions ν. In [8], we have shown that the intertwining operator between the principal and homogeneous realizations can be described very simply by using a combinatorial construction coming from the representation theory of symmetric groups: the n-quotient, n-core and n-sign of a partition. The aim of this note is to generalize this result to the realizations of $\hat{sl}_n$ corresponding to all partitions ν of n. To do this we introduce the notions of ν-quotient, ν-core and ν-sign of a partition, which reduce to the classical ones when $\nu = (1, \dots, 1)$.

It would be interesting to investigate if these notions have also a meaning in the representations of symmetric groups.

2. Several realizations of the basic representation of $A_{n-1}^{(1)}$

In this section we recall following [4] how to construct the different realizations of the basic representation of $A_{n-1}^{(1)}$. We put $\nu = (n_1, n_2, \dots, n_s)$, with $|\nu| = n$.

We follow the notation of [3]. Let $\bar{A}_{n-1}^{(1)}$ be the subalgebra of $\bar{a}_\infty$ consisting of the matrices $A = (a_{ij})_{i,j\in\mathbf{Z}}$ such that

$$a_{i+n,j+n} = a_{ij}, \quad \sum_{k=1}^{n} a_{ni+k,nj+k} = 0, \quad (i, j \in \mathbf{Z}). \tag{1}$$

The subalgebra $\bar{A}_{n-1}^{(1)} \oplus \mathbf{C}c$ of a_∞ is isomorphic to $A_{n-1}^{(1)'}$, and the affine Lie algebra $A_{n-1}^{(1)}$ is obtained by adjoining to $A_{n-1}^{(1)'}$ the degree operator

$$D = -\sum_{i\in\mathbf{Z}} \left\lfloor \frac{i}{n} \right\rfloor E_{ii}, \tag{2}$$

where the E_{ij} are the units matrix. The Chevalley generators e_i, f_i, h_i for $i \in \{0, \ldots, n-1\}$ are

$$e_i = \sum_{k\equiv i \bmod n} E_{i,i+1}, \quad f_i = \sum_{k\equiv i \bmod n} E_{i+1,i}, \quad h_i = [e_i, f_i]\,. \tag{3}$$

We denote by $\mathfrak{h}$ the Cartan subalgebra generated by $h_0, \ldots, h_{n-1}, D$.

To describe how to associate a Heisenberg subalgebra denoted by $\mathcal{H}_\nu$, to a partition $\nu = (n_1, n_2, \ldots, n_s)$ of n we need the construction of $A_{n-1}^{(1)}$ as a loop algebra.

$A_{n-1}^{(1)'}$ is the central extension $(\mathbf{C}[t, t^{-1}] \otimes_{\mathbf{C}} A_{n-1}) \oplus \mathbf{C}c$ of the loop algebra of A_{n-1}, the bracket being given by

$$[t^k \otimes x, t^l \otimes y] = t^{k+l} \otimes (xy - yx) + k\delta_{k,-l}\,\mathrm{tr}\,(xy)\,c, \qquad (x, y \in A_{n-1},\ k, l \in \mathbf{Z}), \tag{4}$$

(see [3], 7.1). The isomorphism ι between the two realizations is

$$\iota(t^j \otimes E_{r,s}) = \sum_{i\in\mathbf{Z}} E_{r+ni,\,s+ni+nj}, \quad (1 \le r, s \le n-1). \tag{5}$$

In this construction the degree operator acts by

$$[D, \iota(t^k \otimes x)] = k\,\iota(t^k \otimes x)\,, \quad (x \in A_{n-1},\ k \in \mathbf{Z}). \tag{6}$$

Then, we need to change the usual basis of $gl(n)$, $\{E_{i,j}\}_{1\le i,j\le n}$. We will work with partitions in blocks of an $n \times n$-matrix.

To ν we associate the matrix

$$\begin{pmatrix} B_{11} & B_{12} & \dots & B_{1s} \\ B_{21} & B_{22} & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ B_{s1} & \dots & \dots & B_{ss} \end{pmatrix},$$

where B_{ij} is a block of size $n_i \times n_j$. The standard basis for the (i, j)th-block is the set of matrices $\{E^{i,j}_{p,q}\}_{1\le p\le n_i, 1\le q\le n_j}$ defined by

$$E^{i,j}_{p,q} := E_{n_1+\dots+n_{i-1}+p, n_1+\dots+n_{j-1}+q}.$$

The commutation relations become

$$\left[E^{i,j}_{p,q}, E^{k,l}_{r,s}\right] = \delta_{j,k}\delta_{q,r} E^{i,l}_{p,s} - \delta_{i,l}\delta_{p,s} E^{k,j}_{r,q}.$$

We can now return to $A^{(1)}_{n-1}$. In its realization as a loop algebra we put for $1 \le l \le n_i$ and $1 \le i \le s$

$$\alpha^{(k)}_{il} = t^k \otimes A_{i,l}$$

where $A_{i,l}$ corresponds to a matrix with zero entries, except on its (i, i)th diagonal block. This block is made of 4 blocks of size $l \times (n_i - l)$, $l \times l$, $(n_i - l) \times (n_i - l)$ and $(n_i - l) \times l$:

$$\begin{pmatrix} 0 & \dots & 0 & 1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ 0 & \dots & 0 & & & 1 \\ t & & & 0 & \dots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & t & 0 & \dots & 0 \end{pmatrix}.$$

We then define for $1 \le i \le s - 1$

$$\beta^{(k)}_i = \frac{1}{n_i}\alpha^{(k)}_{i,n_i} - \frac{1}{n_{i+1}}\alpha^{(k)}_{i+1,n_{i+1}}.$$

Theorem 1 [4, 5]

(*i*) *For $\nu = (n_1, \dots, n_s)$ the subalgebra generated by the elements $\alpha^{(k)}_{il}, \beta^{(k)}_j$ for $k \in \mathbf{Z}$, $1 \le i \le s$, $1 \le l < n_i$, $1 \le j \le s - 1$ is an Heisenberg subalgebra $\mathcal{H}_\nu$.*

(*ii*) *The Heisenberg subalgebras $\mathcal{H}_\nu$ form a complete non-redundant list of Heisenberg subalgebras of $A_{n-1}^{(1)}$ up to conjugacy.*

In particular, the principal Heisenberg subalgebra is associated to $\nu = (n)$ and the homogeneous one to $\nu = (1, \ldots, 1)$.

There is a realization of the basic representation of $A_{n-1}^{(1)}$ associated to each Heisenberg subalgebra.

We then want to construct an intertwiner between all this realizations, and to do this we introduce combinatorial notions.

3. Combinatorial notions

A partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is a weakly decreasing sequence of nonnegative integers. We denote by $\mathcal{P}$ the set of partitions, by $|\lambda|$ the weight of λ and by $l(\lambda) = t$ the length of λ.

3.1. *ν-core, ν-quotient, ν-sign*

Fix $\nu = (n_1, \ldots, n_s)$ a partition of n. To any partition λ, we associate the ν-core $\lambda_{(\nu)}$, the ν-quotient $\lambda^{(\nu)}$ and the ν-sign $\epsilon_\nu(\lambda)$ of λ. These notions generalize the classical notions of n-core, n-quotient and n-sign of a partition λ (see [2, 11]) which correspond to the case $\nu = (1, \ldots, 1)$.

To λ we associate an infinite decreasing sequence

$$\theta(\lambda) = (\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \ldots) = (\theta^1, \theta^2, \ldots).$$

Such a decreasing sequence can be encoded by a configuration of beads on an abacus with s infinite runners numbered $0, \ldots, s-1$ from left to right. The possible positions of a bead on the abacus are indicated by integer labels, the ith-runner being labelled by all integers congruent modulo n to the numbers $-(n_s + \cdots + n_{s-i+1})$, $-(n_s + \cdots + n_{s-i+1} + 1)$, $\ldots$, $-(n_s + \cdots + n_{s-i+1} + n_{s-i} - 1)$.

Then one encodes θ by putting a bead in position θ^i for each i. Note that all but a finite number of negative positions are occupied by a bead. Thus, for $\lambda = (6, 5, 5, 3, 3, 2, 1)$, $n = 3$ and $\nu = (2, 1)$, we have

$$\theta(\lambda) = (6, 4, 3, 0, -1, -3, -5, -7, -8, \ldots)$$

and the corresponding (2, 1)-abacus is shown in the left-part of figure 1.

It is convenient to call 'holes' the unoccupied positions with a label $i \leq 0$, and 'particles' the occupied positions with a label $i > 0$ (see the infinite wedge construction in [7]). By definition, there are as many holes as particles.

One can now read the ν-core of λ on the corresponding abacus. This is the partition $\lambda_{(\nu)}$ corresponding to the bead configuration obtained by sliding the beads up as high as possible on their respective runners. Thus, continuing our previous example, the bead configuration

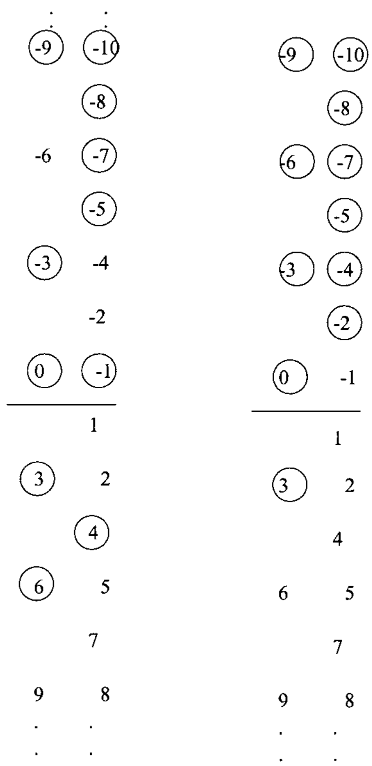

Figure 1. The (2, 1)-abacus of $\lambda = (6, 5, 5, 3, 3, 2, 1)$ and of its (2, 1)-core $\lambda_{(2,1)} = (3, 1)$.

of the (2, 1)-core of $\lambda = (6, 5, 5, 3, 3, 2, 1)$ is shown in the right-part of figure 1, so that $\lambda_{(2,1)} = (3, 1)$.

This description shows that there is a bijection between the set $\mathcal{C}_\nu$ of ν-core partitions (that is, partitions μ such that $\mu = \mu_{(\nu)}$) and the set of s-tuples $(a_0, a_1, \dots, a_{s-1})$ of integers such that $\sum_i a_i = 0$. It is obtained by defining a_i to be either minus the number of holes or plus the number of particles on runner $s - 1 - i$ of the bead configuration corresponding to $\mu \in \mathcal{C}_\nu$. Hence, one can associate to each $\mu \in \mathcal{C}_\nu$ a monomial $q_0^{a_0} \cdots q_{s-1}^{a_{s-1}}$ in $\mathbf{C}[q_0^{\pm 1}, \dots, q_{s-1}^{\pm 1}]$.

We can then define the ν-quotient of λ. It is a $(s - 1)$-tuple of partitions denoted by $\lambda^{(\nu)} = (\lambda^1, \dots, \lambda^{s-1})$. We are going to read the ith part of the ν-quotient on the $s - 1 - i$th runner. For each particle, in decreasing order, we count all the holes above it on the same runner. This gives us a partition. For our example we have $\lambda^0 = (4, 2)$ and $\lambda^1 = (1, 1, 1, 1)$.

Proposition 1 *The map $\lambda \to (\lambda_{(\nu)}, \lambda^{(\nu)})$ is a bijection from $\mathcal{P}$ to $\mathcal{C}_\nu \times \mathcal{P}^s$.*

Proof: It is obvious by the construction of the ν-core, the ν-quotient. □

To finish we now define the ν-sign of λ by using two different orderings of the beads of its bead configuration. Since in our setting the number of these beads is infinite, we have

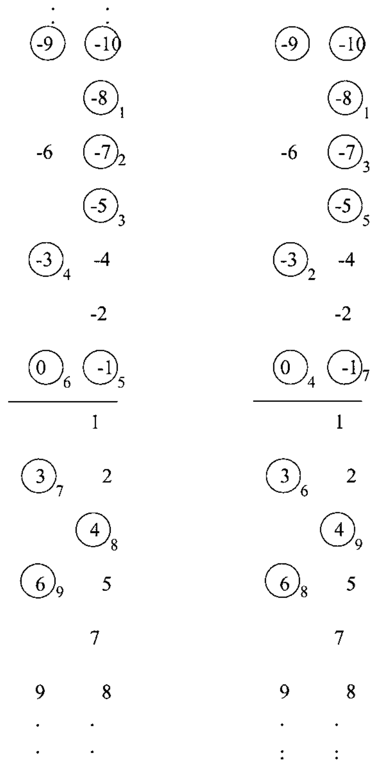

Figure 2. The two numberings of the finite bead configuration of $\lambda = (6, 5, 5, 3, 3, 2, 1)$.

to restrict to a finite subset. An incomplete row with beads on all remaining integers is also without holes. To obtain the finite part of the abacus we discard all the negatives rows without holes such that the first row discarded is a complete row. The finite part gives us the finite bead configuration of λ. The first way of numbering the beads is given by the natural ordering of their labels. For the second one, we sort the beads into different layers, the jth layer ($j \geq 1$) consisting of those beads which have $j-1$ beads above them on their respective runners in the finite bead configuration. In this numbering, called π-numbering, a bead on runner i and layer j is numbered before a bead on runner i_1 and layer j_1 if and only if $j < j_1$, or $j = j_1$ and $i > i_1$. When we compare these two numberings, we get a permutation π_ν whose sign is denoted by $\epsilon_\nu(\lambda)$. This is the ν-sign of λ. In our running example, figure 2 shows the natural numbering (left) and the π-numbering (right). Thus

$$\pi_{(2,1)}(\lambda) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 5 & 2 & 7 & 4 & 6 & 9 & 8 \end{pmatrix},$$

so that $\epsilon_{(2,1)}(\lambda) = 1$.

3.2. Symmetric functions

We review the necessary background in the theory of symmetric functions (see [9]). We denote by $\mathbf{Sym}(X)$ the algebra of symmetric functions in an infinite set of variables $X = \{x_1, x_2, \ldots\}$ with coefficients in $\mathbf{C}$. When there is no danger of confusion we shall omit X and simply write $\mathbf{Sym}$. The algebra $\mathbf{Sym}$ can also be regarded as the polynomial ring $\mathbf{Sym} = \mathbf{C}[p_k; k \in \mathbf{N}^*]$ where $p_k = \sum_i x_i^k$ is the power sum symmetric function. An important linear basis of $\mathbf{Sym}$ is the basis of Schur's S-functions

$$s_\lambda = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu$$

where for λ and $\mu = (1^{m_1} \cdots r^{m_r})$ two partitions of m, we put

$$p_\mu = p_1^{m_1} \cdots p_r^{m_r}, \quad z_\mu = 1^{m_1} m_1! \cdots r^{m_r} m_r!,$$

and $\chi_\lambda(\mu)$ is the irreducible character χ_λ of $\boldsymbol{S}_m$ evaluated on the conjugacy class of cycle-type μ.

Let $\langle \cdot\,,\, \cdot \rangle$ be the scalar product of $\mathbf{Sym}$ defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ is Kronecker's symbol. We denote by D_f the adjoint of the multiplication by f with respect to $\langle \cdot\,,\, \cdot \rangle$, that is,

$$\langle D_f g, h \rangle = \langle g, fh \rangle, \quad (f, g, h \in \mathbf{Sym}).$$

D_f is in fact a differential operator since we have for $f = f(p_1, p_2, \ldots)$

$$D_f = f\left(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \ldots, n\frac{\partial}{\partial p_n}, \ldots\right).$$

The multiplication of an S-function by a power sum p_k is conveniently described as follows. It is a particular case of a formula of Muir for multiplying an S-function by a monomial symmetric function (see [10]).

Let $\tau = (\tau_i)_{i \geq 1}$ be an infinite sequence of pairwise distinct integers such that $\tau_i = -i + 1$ for i large enough. Then there is a finite permutation w such that $w(\tau) = \theta(\lambda)$ for some partition λ. It is convenient to introduce the notation

$$S_\tau = \operatorname{sgn} w s_\lambda, \tag{7}$$

and to extend this notation to sequences τ such that $\tau_i = \tau_j$ for some pair i, j, by putting $S_\tau = 0$ in this case. Then (see e.g. [9] I 3, Ex. 11)

$$p_k S_\tau = \sum_{i \geq 1} S_{\tau + k\epsilon_i} \tag{8}$$

where $\epsilon_i = (\delta_{ij})_{j \geq 1}$. Clearly, only a finite number of summands have all their parts pairwise distinct and the above sum is finite.

The partitions μ such that s_μ occurs in $p_k s_\lambda$ are those obtained from λ by addition of a ribbon (or rim-hook) of length k. The sign of s_μ is then equal to the sign of the permutation w such that $s_{\tau+k\epsilon_i} = \operatorname{sgn} w s_\lambda$. Similarly, we have

$$D_{p_k} S_\tau = \sum_{i\geq 1} S_{\tau-k\epsilon_i}. \tag{9}$$

Finally, we denote by $\mathbf{Sym}(X_1, \ldots, X_j)$ the $\mathbf{C}$-algebra of functions of j sets of variables $X_1, \ldots, X_j$, symmetric in each set separately. In other words, $\mathbf{Sym}(X_1, \ldots, X_j) = \mathbf{C}[p_k(X_i);\, k \in \mathbf{N}^*,\, 1 \leq i \leq n]$. A linear basis of this algebra is given by the products $s_{\alpha^1}(X_1) \cdots s_{\alpha^j}(X_j)$ where $\alpha^1, \ldots, \alpha^j$ are arbitrary partitions.

We denote respectively by $\hat{p}_k(X_i)$ and $D_{p_k(X_i)}$ the endomorphisms

$$\hat{p}_k(X_i)(f) = p_k(X_i) f, \quad D_{p_k(X_i)}(f) = k \frac{\partial}{\partial p_k(X_i)} f, \quad (f \in \mathbf{Sym}(X_1, \ldots, X_j)).$$

More generally, for $g \in \mathbf{Sym}(X_1, \ldots, X_j)$, we define $\hat{g}$ and D_g by expanding g as a polynomial in the variables $p_k(X_i)$ and taking the corresponding polynomials in the $\hat{p}_k(X_i)$ or $D_{p_k(X_i)}$.

4. Construction of the intertwiners

We generalize to all the realizations of the basic representation of $A_{n-1}^{(1)}$ constructed in [4] the results obtained for the homogeneous one in [8]. Thoughout this section we put $\nu = (n_1, n_2, \ldots, n_s)$, with $|\nu| = n$.

In fact, the aim of this section is to show that the notions of ν-sign, ν-quotient and ν-core introduced before allow to give a simple combinatorial description of the isomorphism between constructions of [6] (for the principal realization) and [4] (for the others) of the basic representation of $A_{n-1}^{(1)}$.

Let $(\Lambda_0, \ldots, \Lambda_{n-1}, \delta)$ denote the basis of $\mathfrak{h}^*$ dual to $(h_0, \ldots, h_{n-1}, D)$. We are going to describe the several explicit realizations of the basic representation of $A_{n-1}^{(1)}$ linked to the several Heisenberg subalgebras introduced in Section 2. This representation is the irreducible representation $L(\Lambda_0)$ with highest weight Λ_0 (see [3] 9.3, 14).

We consider the Fock representation κ_{P} of $A_{n-1}^{(1)}$, obtained by restricting to $A_{n-1}^{(1)}$ the action of a_∞ on **Sym** (see [3] or [8]). We have the explicit description of the Chevalley generators in this representation:

$$\kappa_{\mathrm{P}}(e_i) s_\lambda = \sum_\mu s_\mu, \quad \kappa_{\mathrm{P}}(f_i) s_\lambda = \sum_\nu s_\nu, \tag{10}$$

where μ (*resp.* ν) runs through the partitions obtained from λ by removing (*resp.* adding) a node with content $d \equiv i \bmod n$.

We know (see [8]) that $L(\Lambda_0)$ in its principal picture is isomorphic to the subalgebra of **Sym** $\mathcal{T}^{(n)} = \mathbf{C}[p_i,\, i \not\equiv 0 \bmod n]$.

We shall now transport this Fock representation into the picture corresponding to the Heisenberg subalgebra $\mathcal{H}_\nu$, and describe its unique irreducible component of type $L(\Lambda_0)$. Let us recall that $L(\Lambda_0)$ does not remain irreducible by restriction to $\mathcal{H}_\nu$, we have

$$L(\Lambda_0)\downarrow_{\mathcal{H}_\nu} \simeq \Omega(\Lambda_0)\otimes_{\mathbf{C}} \mathcal{S}(\mathcal{H}_\nu^-) \tag{11}$$

where the space $\Omega(\Lambda_0)$ is an irreducible module over the Heisenberg subalgebra consisting of all vectors in $L(\Lambda_0)$ which are killed by the $\alpha_{il}^{(k)}$ and $\beta_i^{(k)}$ with $k > 0$.

We define the $\mathbf{C}$-vector space $\mathcal{B}_\nu = M \otimes_{\mathbf{C}} \mathbf{Sym}(X_0, X_1, \ldots, X_{s-1})$ where M is the subspace of $\mathbf{C}[q_0^{\pm 1}, \ldots, q_{s-1}^{\pm 1}]$ with basis $\{q_0^{a_0}\cdots q_{s-1}^{a_{s-1}} \mid a_0+\cdots+a_{s-1}=0\}$.

We can now introduce the main tool of the construction which is the generalization of the intertwiner Φ_n of [8].

Definition 1 Denote by Π_ν the isomorphism of $\mathbf{C}$-vector spaces given by

$$\begin{aligned}\Pi_\nu : \mathbf{Sym}(X) &\to \mathcal{B}_\nu \\ s_\lambda(X) &\to \epsilon_\nu(\lambda) q_0^{a_0}\cdots q_{s-1}^{a_{s-1}} \otimes s_{\lambda^0}(X_0)\cdots s_{\lambda^{s-1}}(X_{s-1})\end{aligned} \tag{12}$$

where $a_0, \ldots, a_{s-1}$ are related to λ_ν as in Section 2.1.

The fact that this is indeed an isomorphism of vector spaces results from Proposition 1.

The map Π_ν allows to transport the representation κ_{P} of $A_{n-1}^{(1)}$ on $\mathbf{Sym}$ to a representation κ_ν on $\mathcal{B}_\nu$, defined by

$$\kappa_\nu(x) = \Pi_\nu \circ \kappa_{\mathrm{P}}(x)\circ \Pi_\nu^{-1}, \quad \left(x \in A_{n-1}^{(1)}\right).$$

For $g \in \mathbf{Sym}(X_0, \ldots, X_{s-1})$ we define endomorphisms of $\mathcal{B}_\nu$ by

$$\hat{g}(m\otimes f) = m\otimes \hat{g}(f), \quad D_g(m\otimes f) = m \otimes D_g(f),$$
$$(m \in M,\ f \in \mathbf{Sym}(X_0, \ldots, X_{s-1})).$$

Theorem 2 *For $k \in \mathbf{N}$, $i \in \{1, \ldots, s\}$, $l \in \{1, \ldots, n_i - 1\}$, we have*

$$\kappa_\nu\left(\alpha_{il}^{(k)}\right) = D_{p_{l+n_ik}(X_{s-i-1})}, \tag{13}$$
$$\kappa_\nu\left(\alpha_{il}^{(-k)}\right) = \hat{p}_{n_ik-l}(X_{s-i-1})\ k \neq 0, \tag{14}$$
$$\kappa_\nu\left(\beta_i^{(k)}\right) = \frac{1}{n_i} D_{p_{n_ik}(X_{s-i-1})} - \frac{1}{n_{i+1}} D_{p_{n_{i+1}k}(X_{s-i})}, \tag{15}$$
$$\kappa_\nu\left(\beta_i^{(-k)}\right) = \frac{1}{n_i}\hat{p}_{n_ik}(X_{s-i-1}) - \frac{1}{n_{i+1}}\hat{p}_{n_{i+1}k}(X_{s-i}). \tag{16}$$

Proof: To prove this result we are going to use the Boson-Fermion correspondence, i.e. the isomorphism between the realization of $L(\Lambda_0)_{a_\infty}$ in $\mathbf{Sym}$ and its realization as an infinite-wedge representation described below (see [3], 14.9). This is the alternative realization of $L(\Lambda_0)_{a_\infty}$ in a space $\mathcal{F}$ with basis vectors

$$u_I = u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k} \wedge \cdots$$

where $I = (i_k)_{k\geq 1}$ runs through all decreasing sequences of integers such that $i_k = -k+1$ for k sufficiently large. Here, u_i denotes the canonical basis of $\mathbf{C}^{\mathbf{Z}}$ on which the matrix units operate by $E_{ij}u_k = \delta_{jk}u_i$. The Lie algebra A_∞ of $\mathbf{Z}\times\mathbf{Z}$-matrices with a finite number of nonzero entries acts on $\mathcal{F}$ by derivation, that is,

$$E_{ij}(u_{i_1} \wedge u_{i_2} \wedge \cdots) = (E_{ij}u_{i_1}) \wedge u_{i_2} \wedge \cdots + u_{i_1} \wedge (E_{ij}u_{i_2}) \wedge \cdots + \cdots$$

and this extends uniquely to a projective representation of $\bar{a}_\infty$, hence to a linear representation of a_∞ hence by restriction to a linear representation of $A^{(1)}_{n-1}$.

In fact, here, we use the multicomponent fermionic version of [4]. Instead of the u_I we relabel the basis vector according to ν, we put:

$$u_j(l+n_jk) := u_{n_1+\cdots+n_{j-1}+l+nk}. \tag{18}$$

Then the basis vectors become $u_{i_1}(k_1) \wedge u_{i_2}(k_2) \wedge \cdots$.

We have then to introduce fermionic creation and annihilation operators for the basis vectors:

$$\begin{aligned}\psi_i(k)(u_{i_1}(k_1) \wedge u_{i_2}(k_2) \wedge \cdots) &:= u_i(k) \wedge u_{i_1}(k_1) \wedge u_{i_2}(k_2) \wedge \cdots,\\ \psi_i^*(k)(u_{i_1}(k_1) \wedge u_{i_2}(k_2) \wedge \cdots) &:= \sum_{j=1}^{\infty}(-1)^j \delta_{i,i_j}\delta_{k,k_j} u_{i_1}(k_1) \wedge u_{i_2}(k_2)\\ &\quad\ \wedge \cdots \wedge \widehat{u_{i_j}(k_j)} \wedge \cdots.\end{aligned}$$

One can easily check the anti-commutation relations of these multicomponent fermions. One has

$$:\psi_i(k)\psi_i^*(l) := \begin{cases}\psi_i(k)\psi_i^*(l) & \text{if } k>0\\ -\psi_i^*(l)\psi_i(k) & \text{otherwise,}\end{cases}$$

The unique isomorphism between the 'bosonic' realization **Sym** and the 'fermionic' realization $\mathcal{F}$ is the so-called boson-fermion correspondence $\sigma : \mathbf{Sym} \to \mathcal{F}$. It sends the S-function s_λ onto the infinite wedge

$$\sigma(s_\lambda) = u_{\theta(\lambda)}.$$

Using (18) we can easily compute $\theta(\lambda)$ from the multicomponent fermions. Now, following [4], we have

$$\alpha_{il}^{(k)} = \sum_{r\in\mathbf{Z}} :\psi_i(r)\psi_i^*(r+l+n_ik): . \tag{19}$$

Then

$$\alpha_{il}^{(k)}\big(u_{\nu(\theta(\lambda))}\big) = \sum_I u_I$$

where $u_{\nu(\theta(\lambda))}$ is obtained from $u_{\theta(\lambda)}$ by relabelling it following (18) and u_I is obtained from $u_{\nu(\theta(\lambda))}$ by changing the parts equal to $u_i(r+l+n_ik)$ into $u_i(r)$ by using (19). Then with this result, the definition of the ν-quotient, Eqs. (8), (9), and the definition of Π_ν we obtain the result. □

Theorem 2 shows that the representation κ_ν is a realization of the Fock representation of $A_{n-1}^{(1)}$ in which the operators of the subalgebra $\mathcal{H}_\nu$ act by multiplication and derivation with respect to the variables $p_{n_ik-l}(X_{s-i-1})$. Hence we have obtained the Fock representation in the picture corresponding to the Heisenberg subalgebra $\mathcal{H}_\nu$. It remains to single out the representation $L(\Lambda_0)$ inside $\mathcal{B}_\nu$.

Theorem 3 *The realization of $L(\Lambda_0)$ in B_ν is obtained by applying Π_ν to the realization of $L(\Lambda_0)$ in* **Sym**. *Let A be the subring of* $\mathbf{Sym}(X_0,\ldots,X_{s-1})$ *generated by the power sums $p_{n_ik-l}(X_{s-i-1})$ $k \geq 1,\ 1 < i \leq s$. We have*

$$\Pi_\nu\big(\mathcal{T}^{(n)}\big) = M \otimes A\,.$$

Proof: Because of the decomposition (11) we have to prove that M is equal to $\Omega(\Lambda_0)$. We use the operators $T_i = Q_iQ_{i+1}^{-1}$ introduced by [4]. The Q_i satisfy the relations:

$$\begin{aligned}
Q_i\psi_j(k) &= -\psi_j(k)Q_i \quad \text{if } i \neq j,\\
Q_i\psi_j^*(k) &= -\psi_j^*(k)Q_i \quad \text{if } i \neq j,\\
Q_i\psi_i(k) &= -\psi_i(k+1)Q_i,\\
Q_i\psi_i^*(k) &= \psi_i^*(k+1)Q_i,\\
Q_i^{-1}\psi_j(k) &= -\psi_j(k)Q_i^{(-1)} \quad \text{if } i \neq j,\\
Q_i^{(-1)}\psi_j^*(k) &= -\psi_j^*(k)Q_i^{(-1)} \quad \text{if } i \neq j,\\
Q_i^{(-1)}\psi_i(k) &= -\psi_i(k-1)Q_i^{(-1)},\\
Q_i^{(-1)}\psi_i^*(k) &= \psi_i^*(k-1)Q_i^{(-1)}.
\end{aligned}$$

Then by [4] we know that $\Omega(\Lambda_0)$ is isomorphic to the group generated by T_i for $1 \leq i \leq s-1$. With the definition of the action of T_i described above on the vacuum vector v_0 written in the multicomponent basis vector and the definition of the ν-core we obtain the desired result. The fact that $\mathcal{S}(\mathcal{H}_\nu^-)$ is isomorphic to A follows from Theorem 2. □

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