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A Characterisation of the Generalized Quadrangle $Q(5, q)$ Using Cohomology

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Abstract. If a GQ $\mathcal{S}'$ of order (s, s) is contained in a GQ $\mathcal{S}$ of order (s, s^2) as a subquadrangle, then for each point X of $\mathcal{S}\backslash\mathcal{S}'$ the set of points $\mathcal{O}_X$ of $\mathcal{S}'$ collinear with X form an ovoid of $\mathcal{S}'$. Thas and Payne proved that if $\mathcal{S}' = Q(4, q)$, q even, and $\mathcal{O}_X$ is an elliptic quadric for each $X \in \mathcal{S}\backslash\mathcal{S}'$, then $\mathcal{S} \cong Q(5, q)$. In this paper we provide a single proof for the q odd and q even cases by establishing a link between the geometry involved and the first cohomology group of a related simplicial complex.

Keywords: generalized quadrangle, subquadrangle, cohomology, ovoid

1. Introduction and definitions

In this paper we apply the theory of cohomology and homology over $\mathbb{Z}_2$ to characterise the GQ $Q(5, q)$ by the embedding of the subquadrangle $Q(4, q)$. We shall see that considering the embedding as a covering problem that the cohomology may be naturally introduced and a relatively straightforward calculation provides the characterisation result.

If $Q(4, q)$ is contained as a subquadrangle in a GQ $\mathcal{S}$ of order (q, q^2), then for each point of $\mathcal{S}\backslash Q(4, q)$ the set of points of $Q(4, q)$ collinear with this point is an ovoid of $Q(4, q)$. In the classical case where $Q(4, q)$ is a subquadrangle of $Q(5, q)$ each such ovoid is an elliptic quadric. In [19] Thas and Payne gave the following characterisation of $Q(5, q)$.

Theorem 1.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a generalized quadrangle of order (q, q^2), q even, having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, q)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ consisting of all of the points collinear with a given point X of $\mathcal{S}\backslash\mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

They provided an elegant geometrical proof which made use of the nucleus of $Q(4, q)$ and the characterisation of $Q(5, q)$ as a GQ of order (s, s^2) with a 3-regular point and a non-incident regular line. In this paper we give a proof that applies in both the q odd and q even cases. Calculating a particular cohomology group (which will be reduced to a problem of quadric geometry) we show that given the classical case $\mathcal{S} = Q(5, q)$ there are no other examples of GQ of order (q, q^2) containing $Q(4, q)$ as a subquadrangle with all associated ovoids being elliptic quadrics.

Theorem 1.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a generalized quadrangle of order (q, q^2) having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, q)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ consisting of all of the points collinear with a given point X of $\mathcal{S}\backslash\mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

In this section we will deal with geometric definitions and preliminaries, including definitions of terms mentioned above. In Section 2 we give a brief introduction to algebraic topology on a simplicial complex over $\mathbb{Z}_2$, the connection between homology on a simplicial complex and properties of the associated graph, and finally covers of a graph and covers of a geometry. In Section 3 we show that Theorem 1.2 may be proved if it can be shown that a particular homology group of a simplicial complex related to the elliptic quadric ovoids of $Q(4, q)$ is trivial. In Section 4 we perform the calculation to show that the homology group in question is trivial. In Section 5 we discuss the possible generalisation of the results presented in this paper.

We now give some formal definitions that put the characterisation problem of Theorem 1.2 in context and lay the groundwork for the rest of the paper. We begin with the definition of a finite generalized quadrangle. (For more details on generalized quadrangles see [13]). A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
(ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{I} m \mathrm{I} Y \mathrm{I} \ell$.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s, t), then it follows that $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$ ([13, 1.2.1]).

The examples that we have already mentioned are the GQs $Q(5, q)$ and $Q(4, q)$. The GQ $Q(5, q)$ has order (q, q^2) and arises as the geometry of points and lines of a non-singular elliptic quadric in PG(5, q) with canonical form given by the equation $f(x_0, x_1) + x_2x_5 + x_3x_4 = 0$ where f is an irreducible quadratic binary form. The GQ $Q(4, q)$ has order q and arises as the geometry of points and lines of a non-singular (parabolic) quadric in PG(4, q) with canonical form given by the equation $x_0^2 + x_1x_4 + x_2x_3 = 0$. We note that $Q(5, q)$ contains subquadrangles isomorphic to $Q(4, q)$ in the form of non-singular hyperplane intersections with $Q(5, q)$. For more information on quadrics and their properties see [9].

An *ovoid* of a GQ $\mathcal{S}$ of order (s, t) is a set $\mathcal{O}$ of points such that each line of $\mathcal{S}$ is incident with precisely one point of $\mathcal{O}$. It follows that $\mathcal{O}$ has $st + 1$ points. In this paper we will be interested in the GQ $Q(4, q)$ and its classical ovoids the elliptic quadric ovoids. If a three-dimensional subspace of PG$(4, q)$ intersects $Q(4, q)$ in a non-singular elliptic quadric $\mathcal{E}$, then the points of $\mathcal{E}$ form an *elliptic quadric ovoid* of $Q(4, q)$.

Lemma 1.3 ([17], [12], see [13], 2.2.1) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a* GQ *of order (s, s^2) with a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ of order s and let $P \in \mathcal{P} \setminus \mathcal{P}'$. Then the set of points of $\mathcal{S}'$ which are collinear with P form an ovoid of $\mathcal{S}'$.*

An ovoid defined as in Lemma 1.3 is said to be *subtended* by P, or just *subtended* if P is understood. The ovoids of $\mathcal{S}'$ subtended by the points in $\mathcal{P}\backslash\mathcal{P}'$ are said to be the ovoids *subtended by $\mathcal{S}$* or just the *subtended* ovoids.

A *rosette* based at a point X of a GQ $\mathcal{S}$ of order (s, t) is a set $\mathcal{R}$ of ovoids with pairwise intersection $\{X\}$ and such that $\{\mathcal{O}\backslash\{X\} : \mathcal{O} \in \mathcal{R}\}$ is a partition of the points of $\mathcal{S}$ not collinear with X. The point X is called the *base point* of $\mathcal{R}$. It follows that a rosette $\mathcal{R}$ contains s ovoids.

If $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (s, s^2) with a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ of order s, then every line of $\mathcal{S}$ is either a line of $\mathcal{S}'$ or is incident with exactly one point of $\mathcal{S}'$ (by [13, 2.1] and a count). A line of $\mathcal{S}$ meeting $\mathcal{S}'$ in exactly one point is called a *tangent*. Given a tangent line ℓ to $\mathcal{S}'$, the set of s ovoids subtended by points of ℓ not in $\mathcal{S}'$ form a rosette of $\mathcal{S}'$. We say that this rosette is the rosette *subtended* by the line ℓ or that the rosette is *subtended* if ℓ is understood.

The GQ $Q(5, q)$ has a single orbit of subquadrangles isomorphic to $Q(4, q)$ under the group of $Q(5, q)$ each member of which arises as a hyperplane section of the quadric defining $Q(5, q)$. If P is a point of $Q(5, q)\backslash Q(4, q)$, then the ovoid of $Q(4, q)$ subtended by P is precisely the set of points in the intersection of $Q(4, q)$ with $P^{\perp}$, where $\perp$ denotes the polarity of $Q(5, q)$. Consequently, each subtended ovoid of $Q(4, q)$ is an elliptic quadric ovoid. Further, if $\mathcal{E}$ is an elliptic quadric ovoid of $Q(4, q)$ contained in the three-dimensional space Σ, then the points of $Q(5, q)$ collinear in $Q(5, q)$ with each point of $\mathcal{E}$ are the points of $\Sigma^{\perp} \cap Q(5, q)$ and none of these points is contained in $Q(4, q)$. Since $\Sigma^{\perp}$ is a secant to $Q(5, q)$ it follows that $\mathcal{E}$ is subtended by precisely two points of $Q(5, q)\backslash Q(4, q)$. If $\mathcal{S}$ is a GQ of order (q, q^2) containing a subquadrangle $Q(4, q)$ such that each subtended ovoid of $Q(4, q)$ is an elliptic quadric ovoid (and hence each elliptic quadric ovoid of $Q(4, q)$ is subtended by precisely two points of $\mathcal{S}\backslash Q(4, q)$), then $Q(4, q)$ is said to be *classically embedded* in $\mathcal{S}$.

A (finite) *semipartial geometry* (SPG) is an incidence structure $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
(ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
(iii) If X is a point and ℓ is a line not incident with X, then the number of pairs $(Y, m) \in \mathcal{P}\times\mathcal{B}$ for which $X \mathrm{~I~} m \mathrm{~I~} Y \mathrm{~I~} \ell$ is either a constant α ($\alpha > 0$), or 0;

(iv) For any pair of non-collinear points (X, Y) there are μ $(\mu > 0)$ points Z such that Z is collinear with both X and Y.

The integers s, t, α, μ are the *parameters* of $\mathcal{T}$. For more information on SPGs see [5].

The example of an SPG of interest to us in this paper is that of R. Metz, which we shall refer to as $\mathcal{T}$ from this point. We shall present two models of the SPG. The first model, that of Metz, takes as points the elliptic quadrics of $Q(4, q)$ and as lines the sets of q elliptic quadrics meeting pairwise in a common point (and sharing a common tangent plane at that common point). This incidence structure is an SPG with parameters $s = q - 1,\ t = q^2$, $\alpha = 2,\ \mu = 2q(q - 1)$. Since we will be often working with $\mathcal{T}$ and its properties we now give a proof that $\mathcal{T}$ is indeed an SPG. If $\mathcal{E}$ is an elliptic quadric ovoid of $Q(4, q)$, P a point of $\mathcal{E}$ and π_P the tangent plane to $\mathcal{E}$ at P, then π_P is contained in $q + 1$ three-dimensional subspaces of PG$(4, q)$. Of these one is the tangent space of $Q(4, q)$ at P and q intersect $Q(4, q)$ in the elliptic quadric ovoids of a rosette. Since π_P is the unique tangent plane to $\mathcal{E}$ at P we have that $s = q - 1, t = q^2$ and two points of $\mathcal{T}$ are collinear if and only if they are elliptic quadric ovoids intersecting in exactly one point. If $\mathcal{E}$ and $\mathcal{E}'$ are two distinct elliptic quadric ovoids in three-dimensional spaces Σ and Σ', respectively, then $\Sigma \cap \Sigma'$ is some plane π. If π is a tangent plane to $Q(4, q)$, then it is also a tangent plane to both $\mathcal{E}$ and $\mathcal{E}'$, and so they are collinear in $\mathcal{T}$. Otherwise, π meets $Q(4, q)$ in a conic $\mathcal{C}$ which is also the intersection of $\mathcal{E}$ and $\mathcal{E}'$. Now suppose that $\mathcal{R} = \{\mathcal{E}_1, \ldots, \mathcal{E}_q\}$ is rosette of elliptic quadric ovoids with base point P and common tangent plane π_P such that $\mathcal{E} \notin \mathcal{R}$. If $P \in \mathcal{E}$, then since the tangent plane to $\mathcal{E}$ at P is not π_P it must be that $\mathcal{E}$ intersects each element of $\mathcal{R}$ in a conic and so in $\mathcal{T}$ there are no lines on $\mathcal{E}$ intersecting $\mathcal{R}$. If, on the other hand, $P \notin \mathcal{E}$, then π_P meets Σ in a line ℓ which is external to $\mathcal{E}$. Of the $q + 1$ planes of Σ on ℓ one intersects $\mathcal{E}$ in a conic which is the intersection of $\mathcal{E}$ and the tangent space to $Q(4, q)$ at P, and the q other planes intersect $\mathcal{E}$ in $\mathcal{E} \cap \mathcal{E}_i$, $i = 1, \ldots, q$, respectively. Of these intersections $q - 2$ are conics and two are single points. Thus $\mathcal{E}$ is collinear, in $\mathcal{T}$, with precisely two of the points of $\mathcal{R}$ and so $\alpha = 2$. Finally, suppose that $\mathcal{E}$ and $\mathcal{E}'$ are two non-collinear points of $\mathcal{T}$, that is two elliptic quadric ovoids meeting in a conic $\mathcal{C}$. Let $P \in \mathcal{E} \backslash \mathcal{C}$ and π_P be the tangent plane to $\mathcal{E}$ at P. If Σ' is the three-space of $\mathcal{E}'$, then $\Sigma' \cap \pi_P$ is a line external to $\mathcal{E}'$. The line $\Sigma' \cap \pi_P$ is contained in two tangent planes to $\mathcal{E}'$, π_Q and π_R, say, tangent at the points Q and R, respectively. Now both $\langle \pi_P, \pi_Q \rangle$ and $\langle \pi_P, \pi_R \rangle$ intersect $Q(4, q)$ in elliptic quadric ovoids meeting both $\mathcal{E}$ and $\mathcal{E}'$ in a single point and hence are collinear with them in $\mathcal{T}$. Repeating this argument for all points of $\mathcal{E} \backslash \mathcal{C}$ yields $\mu = 2q(q - 1)$. Note that this SPG is the subtended ovoid/rosette structure of $Q(4, q)$ in $Q(5, q)$.

The second model is due to Hirschfeld and Thas in [8, page 268]. Let $\mathcal{Q}$ be the non-singular elliptic quadric in PG$(5, q)$ with polarity $\perp$. Let P be a point of PG$(5, q) \backslash \mathcal{Q}$ and Π a hyperplane of PG$(5, q)$ not incident with P. Then the SPG has point set the projection of $\mathcal{Q} \backslash P^{\perp}$ onto Π and line set the set of lines of Π containing q points of the SPG.

2. Preliminaries

In this section we introduce a number of algebraic and geometrical tools that will be used in Section 3 and Section 4.

2.1. *Algebraic topology of a simplicial complex over* $\mathbb{Z}_2$

In this section we give a brief introduction to concepts of a simplicial complex and the homology and cohomology groups of a simplicial complex over $\mathbb{Z}_2$. Useful introductory texts to algebraic topology are [6], [15] and [16].

A *simplicial complex* $\Gamma = (V, S)$ consists of a set V of *vertices* and a set S of finite non-empty subsets of V called *simplexes* such that

(a) Any set consisting of exactly one vertex is a simplex.
(b) Any non-empty subset of a simplex is a simplex.

A simplex containing exactly $q + 1$ vertices is called a *q-simplex*. If s is a q-simplex of Γ comprising the vertices $P_0, P_1, \ldots, P_q$, then we represent s by the ordered list $(P_0 P_1 \ldots P_q)$. Any permutation of the list represents the same q-simplex s.

We define $C_q(\Gamma, \mathbb{Z}_2)$ to be the group of formal linear combinations $\sum_\sigma r_\sigma \sigma$, where σ runs through the q-simplexes of Γ and $r_\sigma \in \mathbb{Z}_2$. An element of $C_q(\Gamma, \mathbb{Z}_2)$ is called a *q-chain* of Γ over $\mathbb{Z}_2$. If Γ is non-empty, then we define $C_{-1}(\Gamma, \mathbb{Z}_2) = 0$ and we define $C_q(\Gamma, \mathbb{Z}_2) = 0$ for $q > n$ if $n < \infty$ and n is the size of the largest simplex of Γ.

The *q-th boundary operator* is a homomorphism $\partial_q : C_q(\Gamma, \mathbb{Z}_2) \to C_{q-1}(\Gamma, \mathbb{Z}_2)$. Let s be a q-simplex of Γ (and so also a q-chain) represented by $(P_0 P_1 \ldots P_q)$. Then the action of ∂_q on s is

$$\partial_q(s) = \sum_{i=0}^{q} (P_0 P_1 \ldots P_{i-1} P_{i+1} \ldots P_q).$$

Note that this definition is independent of the representation of s that we use. A *face* of s is a $(q-1)$-simplex with vertices in s and the $(q-1)$-simplex $(P_0 P_1 \ldots P_{i-1} P_{i+1} \ldots P_q)$ is the *face* of s *opposite* P_i. We extend ∂_q to act on q-chains of Γ by letting it act linearly. The $(q-1)$-chain $\partial_q(s)$ is called the *boundary* of s.

The following lemma is an elementary result in algebraic topology.

Lemma 2.1 $\partial_q \circ \partial_{q+1} = 0$.

A q-chain s such that $\partial_q(s) = 0$ is called a *q-cycle* and if $s = \partial_{q+1}(s')$ for some $(q+1)$-chain s', then s is called a *q-boundary*. Two q-chains that differ by a boundary are called *homologous* and a q-cycle that is homologous to the zero q-chain is called *null homologous*. The set of q-cycles form a group (the kernel of ∂_q) denoted by $Z_q(\Gamma, \mathbb{Z}_2)$ and the set of q-boundaries also form a group (the image of ∂_{q+1}) which is denoted by $B_q(\Gamma, \mathbb{Z}_2)$. If s is a q-boundary with $s = \partial_{q+1}(s')$, then $\partial_q(s) = \partial_q \circ \partial_{q+1}(s') = 0$ by Lemma 2.1 and so $B_q(\Gamma, \mathbb{Z}_2)$ is a subgroup of $Z_q(\Gamma, \mathbb{Z}_2)$. The quotient group $Z_q(\Gamma, \mathbb{Z}_2)/B_q(\Gamma, \mathbb{Z}_2)$ is called the *q-th homology group* of Γ over $\mathbb{Z}_2$, and is denoted $H_q(\Gamma, \mathbb{Z}_2)$.

We denote by $C^q(\Gamma, \mathbb{Z}_2)$ the group of homomorphisms from $C_q(\Gamma, \mathbb{Z}_2)$ to $\mathbb{Z}_2$. Any element of $C^q(\Gamma, \mathbb{Z}_2)$ is called a *q-cochain* of Γ into $\mathbb{Z}_2$. The *q-th coboundary operator* is a group homomorphism $\delta^q : C^q(\Gamma, \mathbb{Z}_2) \to C^{q+1}(\Gamma, \mathbb{Z}_2)$. Let c be a q-cochain and s a $q + 1$-simplex of Γ represented by $(P_0 P_1 \ldots P_{q+1})$. Then the action of $\delta^q c$ on s is defined to be

$$\delta^q c(s) = \sum_{i=0}^{q+1} c(P_0 P_1 \dots P_{i-1} P_{i+1} \dots P_{q+1});$$

Which is independent of the representation of s used. By linearity this determines the action of $\delta^q c$ on all $q+1$-chains. The $q+1$-cochain $\delta^q c$ is called the *coboundary* of c.

We have an analogous result to Lemma 2.1 for δ^q:

Lemma 2.2 $\delta^{q+1} \circ \delta^q = 0$.

The following lemma links the boundary and coboundary operators:

Lemma 2.3 *Let s be a $(q+1)$-chain and c a q-cochain. Then*

$$\delta^q c(s) = c(\partial_{q+1} s).$$

A q-cochain c such that $\delta^q c = 0$ is called a *q-cocycle* and if $c = \delta^{q-1} c'$ for some $(q-1)$-cochain c', then c is called a *q-coboundary*. Two q-cochains that differ by a boundary are called *cohomologous*. $Z^q(\Gamma, \mathbb{Z}_2)$ is the group of q-cocycles (the kernel of δ^q) and $B^q(\Gamma, \mathbb{Z}_2)$ is the group of q-coboundaries (the image of δ^{q-1}). Lemma 2.2 tells us that $B^q(\Gamma, \mathbb{Z}_2)$ is a subgroup of $Z^q(\Gamma, \mathbb{Z}_2)$. The quotient group $Z^q(\Gamma, \mathbb{Z}_2)/B^q(\Gamma, \mathbb{Z}_2)$ is the *q-th cohomology group* of Γ over $\mathbb{Z}_2$, and is denoted $H^q(\Gamma, \mathbb{Z}_2)$.

The following theorem states the precise connection between homology groups and cohomology groups.

Theorem 2.4 *Let $\Gamma = (V, S)$ be a simplicial complex with V non-empty and finite. Then*

$$H_q(\Gamma, \mathbb{Z}_2) \cong H^q(\Gamma, \mathbb{Z}_2) \quad \textit{for all } q \geq -1.$$

This result will prove extremely useful to us in both Section 3 and Section 4. In Section 3 we will reformulate our desired characterisation of $Q(5, q)$ in terms of the order of a particular cohomology group, however it is the corresponding homology group that we actually calculate, which is done in Section 4.

2.2. *Graphs and homology of a simplicial complex over $\mathbb{Z}_2$*

In this section we first define the graph associated with a simplicial complex and the simplicial complex associated with a graph. We then proceed to describe the connections between algebraic topology on a simplicial complex and properties of the graph associated with the simplicial complex. In particular we establish necessary and sufficient conditions on a simplicial complex Γ for the group $H_1(\Gamma, \mathbb{Z}_2)$ to be trivial and then reformulate these conditions on the graph associated with Γ.

If Γ is a simplicial complex, then the *q-skeleton* Γ^q of Γ is the simplicial complex consisting of all p-simplexes of Γ for $p \leq q$. The 1-skeleton of Γ is the simplicial complex of 0-simplexes and 1-simplexes of Γ, which we will consider as a graph.

Let G be a graph with vertex set V and edge set E. We define Γ_G to be the simplicial complex with 0-simplexes the vertices of G, 1-simplexes the edges of G, 2-simplexes the complete subgraphs on 3 vertices and in general, with the set of q-simplexes the set of complete subgraphs of G on $q + 1$ vertices.

An alternative representation of $C_q(\Gamma, \mathbb{Z}_2)$ is the following. The elements of $C_q(\Gamma, \mathbb{Z}_2)$ are the subsets of the set of q-simplexes and the group operation on two elements of $C_q(\Gamma, \mathbb{Z}_2)$ is symmetric difference. Given an element s of $C_q(\Gamma, \mathbb{Z}_2)$, as a sum of q-simplexes, the subset of q-simplexes corresponding to s contains those q-simplexes whose coefficient in s is 1. The boundary of a q-simplex s is the set of $(q - 1)$-faces of s and the boundary of a q-chain σ is the symmetric difference of the sets of $(q - 1)$-faces of the q-simplexes of σ. Note that for a q-simplex s we will often use s to mean the q-chain $\{s\}$.

We now consider the group $H_1(\Gamma, \mathbb{Z}_2)$ and the graph Γ^1, for some finite simplicial complex Γ. Let σ be a 1-cycle of Γ, then $\partial_1(\sigma) = 0$ and so each 0-simplex of Γ is contained in an even number of 1-simplexes of σ (the number may be 0). Let σ' be a 1-boundary of Γ with $\sigma' = \partial_2(\sigma'')$, then σ' is the symmetric difference of the sets of 1-faces of the 2-simplexes of σ''.

We now make some important observations, which will be assumed for the rest of the paper regarding the correspondence between 1-cycles of a simplicial complex and circuits of its 1-skeleton. Let $\sigma = \{s_1, s_2, \ldots, s_n\}$ be a 1-cycle of Γ. We say that σ is an *elementary* 1-cycle of Γ, if each vertex of Γ appears in exactly none or two 1-simplexes of σ. We say that σ is *induced* if for any 1-simplex s such that $\partial_1(s) = \{P, Q\}$ with $P \in \partial_1(s_i)$, $Q \in \partial_1(s_j)$, for $i, j \in \{1, \ldots, n\}$, $i \neq j$, then $s \in \sigma$. If σ is an induced 1-cycle that is not the boundary of a 2-simplex, we say that σ is a *proper* induced 1-cycle. The above definitions make more intuitive sense in the graph Γ^1, the 1-skeleton of Γ. Recall that a 2-simplex of Γ is a triangle in Γ^1, a 1-simplex an edge and a 0-simplex a vertex of Γ^1. In Γ^1 a 1-cycle of Γ is a set of edges such that each vertex appears in an even number of edges, that is, the set of edges of a *circuit* of Γ^1. An elementary 1-cycle is the set of edges of an *elementary circuit* of Γ^1 and an induced 1-cycle is the set of edges of an *induced circuit* of Γ^1. Note that in Γ^1 a 1-cycle does not correspond to a single circuit, since the 1-simplexes of a 1-cycle aren't ordered, but rather to a set of circuits. In the work that follows we will often abuse notation and refer to a circuit as both a set of vertices and a set of edges.

We now state some elementary results which will allow us to establish when $H_1(\Gamma, \mathbb{Z}_2)$ is trivial.

Lemma 2.5 *A* 1*-cycle of* Γ *may be written as the sum of elementary* 1*-cycles, with no common* 1*-simplexes.*

Lemma 2.6 *An elementary* 1*-cycle of* Γ *containing* n 1*-simplexes can be written as the sum of induced* 1*-cycles of* Γ *each of which contains at most* n 1*-simplexes.*

Combining Lemma 2.5 and Lemma 2.6 gives us the following result.

Corollary 2.7 *A* 1*-cycle of* Γ *can be expressed as the sum of induced* 1*-cycles of* Γ.

Recall that $H_1(\Gamma, \mathbb{Z}_2)$ is trivial if and only if each 1-cycle of Γ is a 1-boundary. Since each 1-cycle of Γ can be expressed as the sum of induced 1-cycles of Γ we have the following theorem.

Theorem 2.8 *$H_1(\Gamma, \mathbb{Z}_2)$ is trivial if and only if each induced 1-cycle of Γ is a 1-boundary.*

A circuit C of a graph G is said to be *decomposed* into circuits $C_1, C_2, \ldots, C_n$ if the edge set of C is the symmetric difference of the edge sets of the C_i. With this definition we can reformulate the above results in terms of the graph Γ^1. Lemma 2.5 says that every circuit may be decomposed into elementary circuits, while Lemma 2.6 says that any elementary circuit may be decomposed into induced circuits. Thus Theorem 2.8 becomes the following result.

Corollary 2.9 *The group $H_1(\Gamma, \mathbb{Z}_2)$ is trivial if and only if each induced circuit of the graph Γ^1 can be decomposed into triangles.*

2.3. Covers of a graph and covers of a geometry

In this section we introduce the cover of a graph and the cover of a geometry, as in [4] (see also [14]), which we will use in Section 3 to reformulate the characterisation of $Q(5, q)$ as a GQ $\mathcal{S}$ with a classically embedded subquadrangle isomorphic to $Q(4, q)$ in terms of the first homology group over $\mathbb{Z}_2$ of a particular simplicial complex.

Let G be a graph, then an *m-fold cover of G* is a pair $(\bar{G}, p)$ where $\bar{G}$ is a graph and p is a map from the vertex set of $\bar{G}$ to the vertex set of G satisfying:

(i) For any vertex $P \in G$, $p^{-1}(P)$ consists of m pairwise non-adjacent vertices
(ii) For any edge $e = \{P, Q\}$ of G, $p^{-1}(e)$ consists of m disjoint edges
(iii) For any non-edge $\{P, Q\}$ of G, $p^{-1}(\{P, Q\})$ is a graph with no edges.

The graph $\bar{G}$ is called the *covering graph*, the map p the *covering map*, m is called the *index* of the cover and any set of vertices of $\bar{G}$ of the form $p^{-1}(S)$ for some set S of vertices of G (possibly a single vertex) is called a *fibre* of S.

If G is the point graph of a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ and $(\bar{G}, p)$ satisfies

(iv) For any line ℓ of $\mathcal{S}$, if $\mathcal{P}_\ell = \{P \in \mathcal{P} : P \ \mathrm{I} \ \ell\}$, then $p^{-1}(\mathcal{P}_\ell)$ consists of m disjoint complete graphs,

then we can form a geometry $\bar{\mathcal{S}}$ with point set the vertices of $\bar{G}$ and lines (as sets of points) defined to be the complete graphs from (iv). The map p naturally induces a map from the point set of $\bar{\mathcal{S}}$ to the point set of $\mathcal{S}$, which, introducing an abuse of notation, we also call p. The pair $(\bar{\mathcal{S}}, p)$ is called an *m-fold cover of $\mathcal{S}$*. The geometry $\bar{\mathcal{S}}$ will be called the *covering geometry* and the terms *covering map*, *index* and *fibre* are defined as for the cover of a graph. We will often take the existence of the map p to be understood and call $\bar{\mathcal{S}}$ an m-fold cover of $\mathcal{S}$. Any element of the fibre $p^{-1}(P)$ will be called a *cover* of P. The map p induces a well-defined map from the line set of $\bar{\mathcal{S}}$ to the line set of $\mathcal{S}$ that, abusing notation once again, we shall denote by p.

Note that if (iv) is satisfied, and ℓ is a line of $\mathcal{S}$, then (i) and (ii) imply that each line in the set $p^{-1}(\ell)$ has the same size as ℓ (as a set of points). Also, if P I ℓ, then each point in the set $p^{-1}(P)$ is incident with exactly one element of $p^{-1}(\ell)$. This means that if ℓ (as a set of points) is $\{P_1, P_2, \dots, P_{s+1}\}$, then each line of the set $p^{-1}(\ell)$ has the form $\{P'_1, P'_2, \dots, P'_{s+1}\}$, where $P'_i \in p^{-1}(P_i)$, for $i = 1, 2, \dots, s+1$. That is, P' incident with ℓ' in $\bar{\mathcal{S}}$ implies that $p(P')$ is incident with $p(\ell')$ in $\mathcal{S}$.

Let G be a graph and Γ_G the simplicial complex associated with G. To simplify matters, we will identify a vertex of Γ_G with the corresponding vertex of G, and a 1-simplex of Γ_G with the corresponding pair of adjacent vertices of G. Recall from Section 2.1 that a 1-cochain c of Γ_G over $\mathbb{Z}_2$ is a homomorphism from the group $C_1(\Gamma_G, \mathbb{Z}_2)$ into $\mathbb{Z}_2$; which is completely determined by its action on the 1-simplexes of Γ_G. An *algebraic* 2-fold cover of G over $\mathbb{Z}_2$ is an 2-fold cover of G, $(\bar{G}, p)$ where $\bar{G}$ is the graph with

$$\begin{aligned}&\text{Vertex Set: } \{(P, \alpha) : P \in G,\ \alpha \in \mathbb{Z}_2\}\\&\text{Adjacency: } (P, \alpha) \sim (Q, \beta) \quad \text{if } P \sim Q \text{ and } c(P, Q) = \alpha + \beta\end{aligned}$$

and p is the map $p((P, \alpha)) = P$. It should be noted that $c(P, Q) = c(Q, P)$, since (P, Q) and (Q, P) represent the same 1-simplex.

Any 1-cochain c defines an 2-fold cover of G in the above way. If G is the point graph of a geometry $\mathcal{S}$ and $(\bar{\mathcal{S}}, p)$ is an algebraic 2-fold cover of $\mathcal{S}$, then we say that $(\bar{\mathcal{S}}, p)$ is an *algebraic* 2-fold cover of $\mathcal{S}$. It is relatively straight-forward to show that any 2-fold cover of a graph may be represented as a algebraic 2-fold cover of the graph.

Let G be the point graph of a geometry $\mathcal{S}$ and let $(\bar{G}, p)$ be an algebraic 2-fold cover of G. We investigate conditions under which condition (iv) above is satisfied.

Let ℓ be a line of $\mathcal{S}$ such that $\mathcal{P}_\ell = \{P_1, P_2, \dots, P_{s+1}\}$, then (P_1, α) is collinear to the set of points $\{(P_2, \alpha + c(P_1, P_2)), (P_3, \alpha + c(P_1, P_3)), \dots, (P_{s+1}, \alpha + c(P_1, P_{s+1}))\}$. Thus, $p^{-1}(\mathcal{P}_\ell)$ consists of m disjoint complete graphs if and only if the m complete graphs have vertex sets

$$\{(P_1, \alpha), (P_2, \alpha + c(P_1, P_2)), \dots, (P_{s+1}, \alpha + c(P_1, P_{s+1}))\} \quad \text{for } \alpha \in \mathbb{Z}_2.$$

This is true if and only if

$$\begin{aligned}&(P_i, \alpha + c(P_1, P_i)) \sim (P_j, \alpha + c(P_1, P_j)) \quad \text{for all } P_i, P_j\\&\quad \text{where } i \neq j,\ i, j \neq 1 \text{ and } \alpha \in \mathbb{Z}_2.\end{aligned}$$

Writing δ for the first coboundary operator δ^1, we see that this is true if and only if

$$\delta c(P_1, P_i, P_j) = 0 \quad \text{for all } P_i, P_j \text{ where } i \neq j \text{ and } i, j \neq 1.$$

This is the case if and only if $\delta c(P_i, P_j, P_k) = 0$ for all P_i, P_j where i, j, k are distinct and $i, j, k \neq 1$. Thus $(\bar{G}, p)$ gives rise to an algebraic 2-fold cover of $\mathcal{S}$ if and only if

$$\delta c(P, Q, R) = 0 \quad \text{for all distinct collinear points } P, Q, R.$$

We will call $(\bar{G}, p)$ and $(\bar{\mathcal{S}}, p)$ the algebraic 2-fold covers of G and $\mathcal{S}$ respectively, *defined* by c, or say that c *defines* $(\bar{G}, p)$ and $(\bar{\mathcal{S}}, p)$ respectively.

If $(\bar{\mathcal{S}}, p)$ and $(\bar{\mathcal{S}}', p')$ are two algebraic 2-fold covers of the geometry $\mathcal{S}$, defined by c and c' respectively, such that $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}'$ are isomorphic geometries, then we say that $(\bar{\mathcal{S}}, p)$ and $(\bar{\mathcal{S}}', p')$ are *equivalent*. Where $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}'$ are understood to be covers of $\mathcal{S}$, we say that c and c' are *equivalent*. Note that if c and c' are cohomologous, with $c' = c + \delta b$, then c and c' are equivalent, and the map $i : (P, \alpha) \mapsto (P, \alpha + b(P))$ is an isomorphism from $\bar{\mathcal{S}}$ to $\bar{\mathcal{S}}'$.

3. Introducing the cohomology

In this section we reformulate our characterisation problem of Section 1 using cohomology.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q, q^2) with a subquadrangle isomorphic to $Q(4, q)$, which we will denote by $Q(4, q) = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$. Let each ovoid of $Q(4, q)$ subtended by a point of $\mathcal{S} \backslash Q(4, q)$ be an elliptic quadric ovoid of $Q(4, q)$; that is, $Q(4, q)$ is a classically embedded subquadrangle of $\mathcal{S}$. Define $\mathcal{T}^* = (\mathcal{P} \backslash \mathcal{P}', \mathcal{B} \backslash \mathcal{B}', \mathrm{I}^*)$ where I^* is the natural restriction of I. Let $\mathcal{T}$ be the SPG of Metz that has point set the set of elliptic quadric ovoids of $Q(4, q)$ and line set the set of rosettes of $Q(4, q)$ subtended by $\mathcal{S}$. Note that each elliptic quadric of $Q(4, q)$ is subtended and each rosette of $Q(4, q)$ consisting entirely of elliptic quadrics is subtended. We will denote the point graph of $\mathcal{T}$ by G and, as in Section 2.2, the simplicial complex constructed from the graph G by Γ_G.

A number of the results presented in this section are special cases of results in [3]. We include proofs here for the sake of clarity. The following result is a special case of [3, Theorem 3.1].

Lemma 3.1 *Let Θ be the set of elliptic quadric ovoids of $Q(4, q)$ and represent $\mathcal{P} \backslash \mathcal{P}'$ as the set $\{(\mathcal{O}, 0), (\mathcal{O}, 1) : \mathcal{O} \in \Theta\}$. Let c be a 1-cochain of Γ_G over $\mathbb{Z}_2$ acting on the 1-simplex $(\mathcal{O}, \mathcal{O}')$ by*

$$c(\mathcal{O}, \mathcal{O}') = \begin{cases} 0 \text{ if } (\mathcal{O}, 0) \text{ and } (\mathcal{O}', 0) \text{ are collinear,} \\ 1 \text{ if } (\mathcal{O}, 0) \text{ and } (\mathcal{O}', 0) \text{ are not collinear.} \end{cases}$$

Then $\mathcal{T}^*$ is an algebraic 2-fold cover of $\mathcal{T}$ defined by c.

Further, c satisfies the coboundary condition

$$\delta c(\mathcal{O}, \mathcal{O}', \mathcal{O}'') = 0 \Longleftrightarrow \mathcal{O}, \mathcal{O}', \mathcal{O}'' \text{ are collinear in } \mathcal{T}. \tag{1}$$

Proof: Let $\mathcal{O}$ and $\mathcal{O}'$ be two collinear points of $\mathcal{T}$. So $\mathcal{O}$ and $\mathcal{O}'$ are two elliptic quadric ovoids of $Q(4, q)$ contained in a common rosette that is subtended by two distinct lines of $\mathcal{S}$. The point $(\mathcal{O}, 0)$ of $\mathcal{T}^*$ is incident with one of these lines and $(\mathcal{O}, 1)$ is incident with the other, and similarly for $(\mathcal{O}', 0)$ and $(\mathcal{O}', 1)$. Thus $(\mathcal{O}, \alpha)$ is collinear with $(\mathcal{O}', \beta)$ if and only if $c(\mathcal{O}, \mathcal{O}') = \alpha + \beta$, and so c defines an algebraic 2-fold cover of the point graph of $\mathcal{T}$.

To show that $\mathcal{T}^*$ is an algebraic 2-fold cover of $\mathcal{T}$, defined by c, we need to show that $\delta c(\mathcal{O}, \mathcal{O}', \mathcal{O}'') = 0$ whenever $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ are distinct collinear points of $\mathcal{T}$. So let

$\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ be distinct collinear points of $\mathcal{T}$ and $\mathcal{R}$ the subtended rosette of $Q(4, q)$ containing them. Now $(\mathcal{O}, 0)$ is collinear with $(\mathcal{O}', c(\mathcal{O}, \mathcal{O}'))$ and with $(\mathcal{O}'', c(\mathcal{O}, \mathcal{O}''))$. Since $\langle(\mathcal{O}, 0), (\mathcal{O}', c(\mathcal{O}, \mathcal{O}'))\rangle$ and $\langle(\mathcal{O}, 0), (\mathcal{O}'', c(\mathcal{O}, \mathcal{O}''))\rangle$ both subtend the rosette $\mathcal{R}$, it follows that $(\mathcal{O}', c(\mathcal{O}, \mathcal{O}'))$ and $(\mathcal{O}'', c(\mathcal{O}, \mathcal{O}''))$ are collinear and so $\delta c(\mathcal{O}, \mathcal{O}', \mathcal{O}'') = 0$. Thus c defines a cover of $\mathcal{T}$.

If $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ are pairwise collinear but not incident with a common line of $\mathcal{T}$, then it follows that they are not contained in a common subtended rosette of $Q(4, q)$. Thus $(\mathcal{O}, 0)$, $(\mathcal{O}', c(\mathcal{O}, \mathcal{O}'))$ and $(\mathcal{O}'', c(\mathcal{O}, \mathcal{O}''))$ are not incident with a common line of $\mathcal{T}^*$ and so $(\mathcal{O}', c(\mathcal{O}, \mathcal{O}'))$ and $(\mathcal{O}'', c(\mathcal{O}, \mathcal{O}''))$ are not collinear since this would be a triangle in $\mathcal{S}$. Hence $\delta c(\mathcal{O}, \mathcal{O}', \mathcal{O}'') = 1$ and c satisfies (1). □

We have shown that if $Q(4, q)$ is classically embedded in $\mathcal{S}$ then we may define a 1-cochain c of Γ_G over $\mathbb{Z}_2$ satisfiying (1). We now show that a 1-cochain c satisfying (1) can be used to construct a GQ $\mathcal{S}$ of order (q, q^2) such that $Q(4, q)$ is classically embedded in $\mathcal{S}$. In the following result we show that the geometry $\mathcal{T}^*$ and the GQ $\mathcal{S}$ containing $Q(4, q)$ as a classically embedded subquadrangle can be reconstructed from $\mathcal{T}$ and c satisfying (1). The result is a special case of [3, Theorem 3.2].

Lemma 3.2 *Let c be a* 1*-cochain of the simplicial complex* Γ_G *over* $\mathbb{Z}_2$ *such that*

$$\delta c(\mathcal{O}, \mathcal{O}', \mathcal{O}'') = 0 \Longleftrightarrow \mathcal{O}, \mathcal{O}', \mathcal{O}'' \text{ are collinear in } \mathcal{T},$$

and let $\mathcal{T}^*$ *be the algebraic* 2*-fold cover of* $\mathcal{T}$ *defined by c.*
Let $\mathcal{S}$ *be the following incidence structure.*

Points	(i)	*Points of* $Q(4, q)$.
	(ii)	*Points of* $\mathcal{T}^*$.
Lines	(a)	*Lines of* $Q(4, q)$.
	(b)	*The sets of points* $\ell \cup P$ *where* ℓ *is a line of* $\mathcal{T}^*$ *and* P *the base point of the subtended rosette covered by* ℓ.
Incidence	(i), (a)	*As in* $Q(4, q)$.
	(i), (b)	*A point* P *of type* (i) is incident with a line $\ell \cup Q$ of *type* (b) if and only if $P = Q$.
	(ii), (a)	*None.*
	(ii), (b)	*A point* P *of type* (ii) is incident with a line $\ell \cup Q$ of *type* (b) if and only if P is incident with ℓ in $\mathcal{T}^*$

Then $\mathcal{S}$ *is a GQ of order* (q, q^2) *with a classically embedded subquadrangle* $Q(4, q)$.

Proof: The proof that $\mathcal{S}$ is a GQ of order (q, q^2) is straightforward apart from showing that the third GQ axiom holds for a non-incident point-line pair $(P, \ell \cup Q)$ where P is of type (ii) and $\ell \cup Q$ is of type (b). We consider this case. Let $\mathcal{O}$ be the ovoid of $Q(4, q)$ corresponding to P and $R = \{\mathcal{O}_1, \ldots, \mathcal{O}_q\}$ the subtended rosette of $Q(4, q)$ corresponding to ℓ. Without loss of generality suppose that $P = (\mathcal{O}, 0)$. There are two possibilities for ℓ, either $\ell = \{(\mathcal{O}_1, 0), (\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2)), \ldots, (\mathcal{O}_q, c(\mathcal{O}_1, \mathcal{O}_q))\}$ or $\ell = \{(\mathcal{O}_1, 1), (\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2) + 1),$

$\ldots, (\mathcal{O}_q, c(\mathcal{O}_1, \mathcal{O}_q)+1)\}$. Suppose that $\mathcal{O} \in R$ and that without loss of generality $\mathcal{O} = \mathcal{O}_1$. Then since $(\mathcal{O}, 0)$ is not incident with ℓ we have that $\ell = \{(\mathcal{O}_1, 1), (\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2) + 1), \ldots, (\mathcal{O}_q, c(\mathcal{O}_1, \mathcal{O}_q) + 1)\}$ and $(\mathcal{O}, 0)$ is collinear with none of the points on ℓ. Thus Q is the unique point on $\ell \cup Q$ that is collinear with P. Now suppose that $\mathcal{O} \notin R$ and that without loss of generality $\ell = \{(\mathcal{O}_1, 0), (\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2)), \ldots, (\mathcal{O}_q, c(\mathcal{O}_q, \mathcal{O}_1))\}$. If $Q \in \mathcal{O}$ then $\mathcal{O}$ meets each of the $\mathcal{O}_i$ in $q + 1$ points and is contained in a unique subtended rosette with Q as the base point, which gives a unique line incident with P and a point of $\ell \cup Q$. If $Q \notin \mathcal{O}$, then there are two ovoids of R that meet $\mathcal{O}$ in precisely one point. Without loss of generality let these ovoids be $\mathcal{O}_1$ and $\mathcal{O}_2$. Now $(\mathcal{O}_1, 0)$ is collinear to $(\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2))$ (on ℓ) and $(\mathcal{O}_1, 1)$ is collinear to $(\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2) + 1)$, while $(\mathcal{O}, 0)$ is collinear to exactly one point of the form $(\mathcal{O}_1, -)$ and one of the form $(\mathcal{O}_2, -)$. So $(\mathcal{O}, 0)$ is collinear to exactly one point on $\ell \cup Q$ if and only if either $(\mathcal{O}, 0)$ is collinear to $(\mathcal{O}_1, 0)$ and $(\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2) + 1)$ or $(\mathcal{O}, 0)$ is collinear to $(\mathcal{O}_1, 1)$ and $(\mathcal{O}_2, c(\mathcal{O}_1, \mathcal{O}_2))$. This occurs if and only if $c(\mathcal{O}, \mathcal{O}_2) = c(\mathcal{O}, \mathcal{O}_1) + c(\mathcal{O}_1, \mathcal{O}_2) + 1$. That is, if and only if

$$\delta c(\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2) = c(\mathcal{O}, \mathcal{O}_1) + (c(\mathcal{O}, \mathcal{O}_1) + c(\mathcal{O}_1, \mathcal{O}_2) + 1) + c(\mathcal{O}_1, \mathcal{O}_2) = 1,$$

which c satisfies. Thus $\mathcal{S}$ is a GQ of order (q, q^2).

Clearly $\mathcal{S}$ contains $Q(4, q)$ as a subquadrangle, so it remains to show that $Q(4, q)$ is classically embedded in $\mathcal{S}$, that is, every subtended ovoid is an elliptic quadric.

Let P be a point of $\mathcal{S} \backslash Q(4, q)$, that is, a point of $\mathcal{T}^*$. The lines of $\mathcal{S}$ incident with P are the lines of the form $\ell \cup Q$ where $P \in \ell$ and Q is the base point of the rosette covered by ℓ. A line of this form meets $Q(4, q)$ at Q. So if $\mathcal{O}$ is the ovoid of $Q(4, q)$ subtended by the point P and $\mathcal{O}'$ is the elliptic quadric covered (as a point of $\mathcal{T}$) by P, then $\mathcal{O}$ is the set of base points of rosettes containing $\mathcal{O}'$. Hence $\mathcal{O} = \mathcal{O}'$ and $Q(4, q)$ is classically embedded in $\mathcal{S}'$. □

From Lemma 3.1 and Lemma 3.2 it follows that considering GQs of order (q, q^2) containing a classically embedded subquadrangle $Q(4, q)$ is equivalent to considering 1-cochains of Γ_G over $\mathbb{Z}_2$ that satisfy the condition (1). We now consider when two such 1-cochains give rise to isomorphic GQs.

Lemma 3.3 *Let c be a 1-cochain of Γ_G over $\mathbb{Z}_2$ satisfying* (1). *If c' is the 1-cochain $c + \delta b$ for some 1-coboundary δb, then c' also satisfies* (1). *Furthermore, the GQs $\mathcal{S}(c)$ and $\mathcal{S}(c')$, defined from c and c', respectively, as in Lemma* 3.1, *are isomorphic.*

Proof: Since $\delta c' = \delta(c + \delta b) = \delta c + \delta^2 b = \delta c$ (by Lemma 2.2) it follows that c' satisfies (1) if and only if c satisfies (1).

The isomorphism i from $\mathcal{S}(c)$ to $\mathcal{S}(c')$ acts on points of $\mathcal{S}(c)$ as follows. If P is a point of type (i) of $\mathcal{S}(c)$, that is $P \in Q(4, q)$, then $i : P \mapsto P$ (that is, i maps $P \in Q(4, q)$ as a point of $\mathcal{S}(c)$ to $P \in Q(4, q)$ as a point of $\mathcal{S}(c')$). If $(\mathcal{O}, \alpha)$ is a point of type (ii) of $\mathcal{S}(c)$, where $\mathcal{O}$ is an elliptic quadric ovoid of $Q(4, q)$ and $\alpha \in \mathbb{Z}_2$, then $i : (\mathcal{O}, \alpha) \mapsto (\mathcal{O}, \alpha + b(\mathcal{O}))$. □

The above result is a special case of [3, Theorem 3.3].

Lemma 3.3 means that if all 1-cochains c satisfy (1) are cohomologous, then all corresponding GQs $\mathcal{S}(c)$ are isomorphic (necessarily to $Q(5, q)$).

Now suppose that c and c' are two 1-cochains that satisfy (1). Both δc and $\delta c'$ are zero on any 2-simplex of Γ_G corresponding to a set of three collinear points of $\mathcal{T}$; and both are non-zero on any 2-simplex of Γ_G corresponding to a set of three pairwise collinear points of $\mathcal{T}$ not incident with a common line of $\mathcal{T}$. For the latter case we must have $\delta c = \delta c' = 1$, since both δc and $\delta c'$ map into $\mathbb{Z}_2$. Thus we have the following result.

Lemma 3.4 *If c and c' are 1-cochains of Γ_G over $\mathbb{Z}_2$ satisfying (1), then $\delta(c + c') = 0$.*

This leads to the following result.

Theorem 3.5 *If the group $H^1(\Gamma_G, \mathbb{Z}_2)$ is trivial, then any GQ $\mathcal{S}$ with subquadrangle $Q(4, q)$ such that $Q(4, q)$ is classically embedded in $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

Proof: Let c be the 1-cochain satisfying (1) constructed from $Q(5, q)$ as in Lemma 3.1 (so $\mathcal{S}(c)$ constructed as in Lemma 3.1 is isomorphic to $Q(5, q)$) and let c' be another 1-cochain satisfying (1). Now by Lemma 3.4 $\delta(c + c') = 0$ and by the hypothesis of the theorem $H^1(\Gamma_G, \mathbb{Z}_2)$ is trivial so $c + c'$ is a 1-coboundary, say δb. As $c' = c + \delta b$ it follows from Lemma 3.3 that $\mathcal{S}(c') \cong \mathcal{S}(c)$, and so $\mathcal{S}(c') \cong Q(5, q)$. Since every GQ of order (q, q^2) containing $Q(4, q)$ as a classically embedded subquadrangle may be represented as $\mathcal{S}(c)$ for some c satisfying (1) it follows that any such GQ is isomorphic to $Q(5, q)$. □

So we have now reformulated our characterisation problem to calculating the size of $H^1(\Gamma_G, \mathbb{Z}_2)$. By Theorem 2.4 this is equivalent to calculating the size of $H_1(\Gamma_G, \mathbb{Z}_2)$ which we do in the next section.

4. Calculating the homology

In this section we show that if $\mathcal{T}$ is the SPG of Metz, G the point graph of $\mathcal{T}$ and Γ_G the simplicial complex of G, then $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial. By Theorem 3.5 and Theorem 2.4 this characterises $Q(5, q)$ as a GQ of order (q, q^2) with a subquadrangle isomorphic to $Q(4, q)$ that is classically embedded.

By Theorem 2.8 we know that the group $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial if and only if each induced 1-cycle of Γ_G is a 1-boundary. We proceed to show that $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial by first showing that the problem may be simplified to showing that each induced 1-cycle consisting of four 1-simplexes is a 1-boundary and then use the graph G to show this is indeed the case.

From Section 2.2 we know that the 1-simplexes of an induced 1-cycle of Γ_G are the edges of an induced circuit of the graph G. For convenience we shall use both representations interchangeably and often abuse definitions by saying that two vertices of Γ_G are adjacent to mean they are the boundary of a 1-simplex.

Lemma 4.1 *Let σ be an induced 1-cycle of Γ_G consisting of at least four 1-simplexes. Then there exist induced 1-cycles $\sigma_1, \sigma_2, \ldots, \sigma_r$ such that each σ_i, $i = 1, \ldots, r$, consists of four 1-simplexes and σ is homologous to the sum of the σ_i.*

Proof: Let σ consist of n 1-simplexes; we proceed by induction on n. If $n = 4$, then the result is immediate. If $n \geq 5$, then let σ be the 1-cycle $(\mathcal{O}_1\mathcal{O}_2 \ldots \mathcal{O}_n)$, where we recall that each $\mathcal{O}_i$ is an elliptic quadric ovoid of $Q(4, q)$. Now suppose that $\mathcal{O}_1 \cap \mathcal{O}_2 = \{P\}$ and that $\mathcal{R}$ is the rosette, with base point P, containing both $\mathcal{O}_1$ and $\mathcal{O}_2$. By the proof of Theorem 2.5 of [3] or by elementary properties of the quadric $Q(4, q)$, if $\mathcal{O}_4$ does not contain P, then there are precisely two ovoids of $\mathcal{R}$ that intersect $\mathcal{O}_4$ in exactly one point. If $\mathcal{O}_4$ does contain P then $\mathcal{O}_4$ does not intersect any ovoid of $\mathcal{R}$ in exactly one point. Suppose first that P is not a point of $\mathcal{O}_4$. Since σ is an induced 1-cycle and we have assumed that $n \geq 5$, it follows that $\mathcal{O}_4$ is adjacent in G to neither $\mathcal{O}_1$ nor $\mathcal{O}_2$. Thus there exists an ovoid $\mathcal{O}$, $\mathcal{O} \neq \mathcal{O}_1, \mathcal{O}_2$, such that $\mathcal{O}$ is in $\mathcal{R}$ and $\mathcal{O}$ intersects $\mathcal{O}_4$ in exactly one point. Hence the induced 1-cycle σ may be expressed as the sum of the 1-cycles $(\mathcal{O}_1\mathcal{O}\mathcal{O}_4\mathcal{O}_5 \ldots \mathcal{O}_n)$, $(\mathcal{O}_1\mathcal{O}_2\mathcal{O})$ and $(\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O})$. If $\mathcal{O}_3$ is adjacent to $\mathcal{O}$, then $(\mathcal{O}_2\mathcal{O}_3\mathcal{O})$ and $(\mathcal{O}_4\mathcal{O}_3\mathcal{O})$ are triangles in G and so the 1-cycle $(\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O})$ is null homologous. If $\mathcal{O}_3$ is not adjacent to $\mathcal{O}$, then $(\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O})$ is an induced 1-cycle. Now the induced 1-cycle σ contains n 1-simplexes, and the 1-cycles $(\mathcal{O}_1\mathcal{O}\mathcal{O}_4\mathcal{O}_5 \ldots \mathcal{O}_n)$ and $(\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O})$ contain $n - 1$ and four 1-simplexes, respectively. Thus the induced 1-cycle σ is homologous to the sum of an elementary 1-cycle containing $n - 1$ 1-simplexes and a 1-cycle containing four 1-simplexes (both of which may or may not be induced). By Lemma 2.6 we can write both of these 1-cycles as the sum of induced 1-cycles, each consisting of fewer than n 1-simplexes. Consequently, σ may be written as the sum of induced 1-cycles, each of which contains fewer than n 1-simplexes.

Now suppose that P is a point of the ovoid $\mathcal{O}_4$. Let Q be a point, distinct from P, contained in the ovoid $\mathcal{O}_1$ but in neither $\mathcal{O}_3$ nor $\mathcal{O}_4$. From the geometry of $Q(4, q)$ we know that there is a (unique) subtended rosette $\mathcal{R}'$, containing $\mathcal{O}_1$ and having base point Q. Since $\mathcal{O}_3$ does not contain the base point of $\mathcal{R}'$ it follows that $\mathcal{O}_3$ is adjacent to two ovoids in $\mathcal{R}'$. If we let one such ovoid be $\mathcal{O}'$, then σ can be expressed as the sum of the 1-cycles $(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}')$ and $(\mathcal{O}_1\mathcal{O}'\mathcal{O}_3\mathcal{O}_4 \ldots \mathcal{O}_n)$. Now since $\mathcal{O}_4$ does not contain the base point of $\mathcal{R}'$ and $\mathcal{O}_4$ is not adjacent to $\mathcal{O}_1$, by above arguments $(\mathcal{O}_1\mathcal{O}'\mathcal{O}_3\mathcal{O}_4 \ldots \mathcal{O}_n)$ and consequently σ may be written as the sum of induced 1-cycles, each consisting of fewer than n 1-simplexes.

Finally, the result follows by induction. □

Given Lemma 4.1 we now need a method for showing that a given induced 1-cycle of Γ_G that contains four 1-simplexes is null homologous. The following lemma provides this method. In the work that follows the term *four-circuit* refers to a circuit of G consisting of four vertices and an *induced four-circuit* is an induced circuit of G with four vertices. Also, for A and B vertices of G we will use the notation $G_{\{A,B\}}$ to refer to the subgraph of G induced by the vertices adjacent to both A and B.

Lemma 4.2 *Let* $\sigma = (\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4)$ *be an induced* 1*-cycle of* Γ_G. *If* $G_{\{\mathcal{O}_1,\mathcal{O}_3\}}$ *is connected, then* σ *is null homologous.*

Proof: Suppose that $G_{\{\mathcal{O}_1,\mathcal{O}_3\}}$ is connected and let $\mathcal{O}_2\Omega_0\Omega_1 \ldots \Omega_n\mathcal{O}_4$ be a path connecting $\mathcal{O}_2$ and $\mathcal{O}_4$ in $G_{\{\mathcal{O}_1,\mathcal{O}_3\}}$. Then σ is equal to the sum of the 1-cycles $(\mathcal{O}_1\mathcal{O}_2\Omega_0)$, $(\mathcal{O}_3\mathcal{O}_2\Omega_0)$, $(\mathcal{O}_1\Omega_0\Omega_1)$, $(\mathcal{O}_3\Omega_0\Omega_1)$, $\ldots (\mathcal{O}_1\Omega_{n-1}\Omega_n)$, $(\mathcal{O}_3\Omega_{n-1}\Omega_n)$, $(\mathcal{O}_1\Omega_n\mathcal{O}_4)$ and $(\mathcal{O}_3\Omega_n\mathcal{O}_4)$. Since each of these is a triangle in G, the corresponding 1-cycles are all 1-boundaries and hence σ is null homologous. □

We now show that if $(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4)$ is an induced four-circuit, then the graph $G_{\{\mathcal{O}_1,\mathcal{O}_3\}}$ is connected and so the corresponding 1-cycle is null homologous. To do this we recall the Hirschfeld-Thas representation of $\mathcal{T}$, the SPG of Metz.

Let $\mathcal{Q} = Q^-(5, q)$ be the non-singular elliptic quadric of PG(5, q) and let the polarity of $\mathcal{Q}$ be represented by $\perp$. Let P be a point of PG(5, q) not on $\mathcal{Q}$ and Π a hyperplane of PG(5, q) not containing P. The geometry $\mathcal{T}$ may be represented in Π with point set the projection of $\mathcal{Q}\backslash P^{\perp}$ onto Π and line set the set of lines of Π containing q points of $\mathcal{T}$. Alternatively we may think of the point set of $\mathcal{T}$ as the set of intersections of secants to $\mathcal{Q}$ on P with Π and the line set as the set of intersections of planes on P meeting $\mathcal{Q}$ in two lines with Π. This representation will allow us to study the geometry of $\mathcal{T}$ in the quadric $\mathcal{Q}$. We first make the following observation.

Lemma 4.3 *The geometry of $\mathcal{Q}\backslash P^{\perp}$ is a 2-fold cover of $\mathcal{T}$.*

Proof: Either verify directly or note that applying the polarity $\perp$ of $\mathcal{Q}$ to the secants and double line planes of $\mathcal{Q}$ on P gives the Metz representation of $\mathcal{T}$ in $P^{\perp} \cap \mathcal{Q} \cong Q(4, q)$. □

Now let $(ABCD)$ be an induced four-circuit of G. Let $\langle A, P\rangle \cap \mathcal{Q} = \{A_1, A_2\}$, $\langle B, P\rangle \cap \mathcal{Q} = \{B_1, B_2\}$, $\langle C, P\rangle \cap \mathcal{Q} = \{C_1, C_2\}$ and $\langle D, P\rangle \cap \mathcal{Q} = \{D_1, D_2\}$; that is the fibres of A, B, C and D, respectively, with respect to the 2-fold cover. Let $\overline{G_{\{A,C\}}}$ be the subgraph of the point graph of $\mathcal{Q}\backslash P^{\perp}$ (which is the 2-fold cover of G) induced by the set $\{A_1, C_1\}^{\perp} \cup \{A_1, C_2\}^{\perp} \cup \{A_2, C_1\}^{\perp} \cup \{A_2, C_2\}^{\perp}$; that is $\overline{G_{\{A,C\}}}$ is the fibre of $G_{\{A,C\}}$.

Lemma 4.4 *If $\overline{G_{\{A,C\}}}$ is connected, then $G_{\{A,C\}}$ is connected.*

Proof: Let X and Y be any two vertices of $G_{\{A,C\}}$. Let X_1 and Y_1 be in the fibres of X and Y, respectively. Under the covering map a path from X_1 to Y_1 is mapped to a path from X to Y. Hence $G_{\{A,C\}}$ is connected. □

We now work towards proving the following proposition.

Proposition 4.5 *The graph $\overline{G_{\{A,C\}}}$ is connected.*

Let π be the plane $\langle P, A, C\rangle$ and let H be the subgraph of $\overline{G_{\{A,C\}}}$ induced by the vertex set

$$((\langle A_1, C_1\rangle^{\perp} \cap \mathcal{Q}) \cup (\langle A_2, C_2\rangle^{\perp} \cap \mathcal{Q}))\backslash \mathcal{C},$$

where $\mathcal{C} = \pi^{\perp} \cap \mathcal{Q}$. For convenience we let $\mathcal{E}_1 = \langle A_1, C_1\rangle^{\perp} \cap \mathcal{Q}$ and $\mathcal{E}_2 = \langle A_2, C_2\rangle^{\perp} \cap \mathcal{Q}$. Note that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-singular elliptic quadrics and that since $\langle A_1, C_1\rangle$ and $\langle A_2, C_2\rangle$ are in π it follows that $\mathcal{C} = \mathcal{E}_1 \cap \mathcal{E}_2$.

Lemma 4.6 *If H is connected, then $\overline{G_{\{A,C\}}}$ is connected.*

Proof: Since both $\mathcal{E}_1\backslash\mathcal{C}$ and $(\langle A_1, C_2\rangle^\perp \cap \mathcal{Q})\backslash\mathcal{C}$ are contained in $A_1^\perp$ it follows that each point of $\mathcal{E}_1\backslash\mathcal{C}$ is adjacent to a unique point of $(\langle A_1, C_2\rangle^\perp \cap \mathcal{Q})\backslash\mathcal{C}$. Similarly each point of $\mathcal{E}_1\backslash\mathcal{C}$ is adjacent to a unique point of $(\langle A_2, C_1\rangle^\perp \cap \mathcal{Q})\backslash\mathcal{C}$, and each point of $\mathcal{E}_2\backslash\mathcal{C}$ is adjacent to a unique point of $(\langle A_2, C_1\rangle^\perp \cap \mathcal{Q})\backslash\mathcal{C}$ and a unique point of $(\langle A_1, C_2\rangle^\perp \cap \mathcal{Q})\backslash\mathcal{C}$. From this we see that H being connected forces $\overline{G_{\{A,C\}}}$ to be connected. □

Let μ_P be the involutory collineation of PG$(5, q)$ that fixes $\mathcal{Q}$ and acts on points of $\mathcal{Q}$ by $\mu_P(Q) = Q$ if $Q \in P^\perp$ and $\mu_P(Q) = R$ if $\langle Q, P\rangle \cap \mathcal{Q} = \{Q, R\}$, $R \neq Q$ (see [9, Section 22.6]). The map μ_P interchanges A_1 and A_2, interchanges C_1 and C_2, interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$ and fixes $\mathcal{C}$ pointwise. So "geometrical" statements about $\mathcal{E}_1$ are also true for $\mathcal{E}_2$. Given a statement involving $\mathcal{E}_1$ we will often assume the equivalent statement for $\mathcal{E}_2$ without explicitly mentioning it.

We now consider the graph H and the corresponding geometry on the quadric $\mathcal{Q}$. The geometry of H separates into two classes depending on whether $\langle\mathcal{E}_1, A_2, C_2\rangle$ and $\langle\mathcal{E}_2, A_1, C_1\rangle$ are both four dimensional subspaces of PG$(5, q)$ or both PG$(5, q)$ itself.

Lemma 4.7 *For $R \in \mathcal{E}_1\backslash\mathcal{C}$, $\mathcal{C}_R = R^\perp \cap (\mathcal{E}_2\backslash\mathcal{C})$ is a conic and one of the following two cases holds*:

(i) $\text{Dim}(\langle\mathcal{E}_1, A_2, C_2\rangle) = \text{Dim}(\langle\mathcal{E}_2, A_1, C_1\rangle) = 4$, *and so* $\langle\mathcal{E}_1\rangle \cap \langle A_2, C_2\rangle = W_1$ *and* $\langle\mathcal{E}_2\rangle \cap \langle A_1, C_1\rangle = W_2$, *for points* $W_1, W_2 \in$ PG(5, q)$\backslash\mathcal{Q}$. *For* $R, S \in \mathcal{E}_1\backslash\mathcal{C}$, $R \neq S$, $\mathcal{C}_R = \mathcal{C}_S$ *if and only if* $W_1 \in \langle R, S\rangle$. *In this case* $W_2 \in \langle\mathcal{C}_R\rangle$.
Furthermore, if q is odd, then $W_1 \neq W_2$, W_1, W_2 *are external points of the conic* $\pi \cap \mathcal{Q}$ *and* $W_1, W_2 \notin \pi^\perp$. *If q is even, then* $W_1 = W_2 = N$, *the nucleus of the conic* $\pi \cap \mathcal{Q}$, *and* $N \in \pi^\perp$.

(ii) $\text{Dim}(\langle\mathcal{E}_1, A_2, C_2\rangle) = \text{Dim}(\langle\mathcal{E}_2, A_1, C_1\rangle) = 5$. *For* $R, S \in \mathcal{E}_1\backslash\mathcal{C}$ *and* $R \neq S$ *we have that* $\mathcal{C}_R \neq \mathcal{C}_S$.

Proof: First let $W = \langle A_1, C_1\rangle \cap \langle A_2, C_2\rangle$, then $W^\perp = \langle\langle A_1, C_1\rangle^\perp, \langle A_2, C_2\rangle^\perp\rangle = \langle\mathcal{E}_1, \mathcal{E}_2\rangle$. In $W^\perp \cap \mathcal{Q} \cong Q(4, q)$ the tangent space at $R \in \mathcal{E}_1\backslash\mathcal{C}$ is a quadratic cone meeting $\mathcal{E}_2$ in a conic $\mathcal{C}_R$ which contains no points of $\mathcal{C}$ since R and $\mathcal{C}$ are contained in the nonsingular elliptic quadric $\mathcal{E}_1$.

The collineation μ_P fixes $\mathcal{Q}$ and maps $\langle\mathcal{E}_1, A_2, C_2\rangle$ onto $\langle\mathcal{E}_2, A_1, C_1\rangle$ and so it follows that they have the same dimension, either four or five. If $\text{Dim}(\langle\mathcal{E}_1, A_2, C_2\rangle) = \text{Dim}(\langle\mathcal{E}_2, A_1, C_1\rangle) = 4$, then it follows that $\langle\mathcal{E}_1\rangle \cap \langle A_2, C_2\rangle$ and $\langle\mathcal{E}_2\rangle \cap \langle A_1, C_1\rangle$ are points W_1 and W_2, respectively. If $\text{Dim}(\langle\mathcal{E}_1, A_2, C_2\rangle) = \text{Dim}(\langle\mathcal{E}_2, A_1, C_1\rangle) = 5$, then $\langle\mathcal{E}_1\rangle$ and $\langle A_2, C_2\rangle$ are skew as are $\langle\mathcal{E}_2\rangle$ and $\langle A_1, C_1\rangle$.

Now if $R \in \mathcal{E}_1\backslash\mathcal{C}$ and $\mathcal{C}_R = R^\perp \cap \mathcal{E}_2$, then $\langle\mathcal{C}_R\rangle = R^\perp \cap \langle A_2, C_2\rangle^\perp = \langle R, A_2, C_2\rangle^\perp$ and so for $S \in \mathcal{E}_1\backslash\mathcal{C}$ with $R \neq S$ it follows that $\mathcal{C}_R = \mathcal{C}_S$ if and only if $\langle R, A_2, C_2\rangle = \langle S, A_2, C_2\rangle$. This is the case if and only if $\langle A_2, C_2\rangle$ meets $\langle\mathcal{E}_1\rangle$ in a point W_1 and $W_1 \in \langle R, S\rangle$. In this case we know from the above that $\langle\mathcal{E}_2\rangle$ and $\langle A_1, C_1\rangle$ intersect in a point W_2. Since $\langle\mathcal{C}_R\rangle = R^\perp \cap \langle A_2, C_2\rangle^\perp$ and $\langle A_1, C_1\rangle \subset R^\perp$, it follows that $W_2 \in \langle\mathcal{C}_R\rangle$. Also in this case since $W_1 = \langle A_1, C_1\rangle^\perp \cap \langle A_2, C_2\rangle$ it follows that W_1 is on π and is also on the tangents to $\pi \cap \mathcal{Q}$ at A_1 and C_1. Similarly, W_2 is on π and on tangents to $\pi \cap \mathcal{Q}$ at A_2 and C_2. If q is odd, then W_1 and W_2 are external to the conic $\pi \cap \mathcal{Q}$, $W_1 \neq W_2$ and since $\pi \not\subset W_1^\perp, W_2^\perp$ it follows that $W_1, W_2 \notin \pi^\perp$. If q is even, then since W_1 and W_2 are each on two tangents to a conic, they coincide in the nucleus N of the conic. Since $\pi \subset N^{\perp}$ it also follows that $N \in \pi^{\perp}$. □

At this point we make the comment that for $T \in \mathcal{E}_2 \backslash \mathcal{C}$ we will use $\mathcal{C}_T$ to represent the conic $T^{\perp} \cap \mathcal{E}_1$.

Given Lemma 4.7 we are now equipped to show that H is connected.

Lemma 4.8 *The graph H is connected.*

Proof: If every two distinct points of $R, S \in \mathcal{E}_1 \backslash \mathcal{C}$, $R \neq S$, are connected, then by applying the automorphism of H induced by the collineation μ_P it follows that any two distinct points of $\mathcal{E}_2 \backslash \mathcal{C}$ are connected. As we have already seen, for $R \in \mathcal{E}_1 \backslash \mathcal{C}$ we have that $R^{\perp} \cap \mathcal{E}_2 = R^{\perp} \cap (\mathcal{E}_2 \backslash \mathcal{C})$ and so there is an edge between $\mathcal{E}_1 \backslash \mathcal{C}$ and $\mathcal{E}_2 \backslash \mathcal{C}$. It follows that H is connected.

So let R and S be two distinct points of $\mathcal{E}_1 \backslash C$. We first show that R is connected to more than half the vertices of $\mathcal{E}_1 \backslash \mathcal{C}$. We case split according to Lemma 4.7.

Case (i) Let $R' \in \mathcal{E}_1 \backslash \mathcal{C}$, $R' \neq R$ be the point such that $\mathcal{C}_R = \mathcal{C}_{R'}$. Now for $T, T' \in \mathcal{C}_R$ either $\mathcal{C}_T = \mathcal{C}_{T'}$ or $\mathcal{C}_T \cap \mathcal{C}_{T'} = \{R, R'\}$. Hence the number of points of $\mathcal{E}_1 \backslash \mathcal{C}$ connected to a point of $\mathcal{C}_R$ is at least

$$\frac{(q+1)}{2}(q-1) + 2 > \frac{q^2 - q}{2} = \frac{|\mathcal{E}_1 \backslash \mathcal{C}|}{2}.$$

Case (ii) If $U \in \mathcal{E}_1 \backslash \mathcal{C}$, $U \neq R$, then it is connected to the points $\mathcal{C}_R \cap \mathcal{C}_U$ of $\mathcal{C}_R$, that is, at most two points. Hence the number of points of $\mathcal{E}_1 \backslash \mathcal{C}$ connected to a point of $\mathcal{C}_R$ is at least

$$\frac{q(q+1)}{2} + 1 > \frac{q^2 - q}{2} = \frac{|\mathcal{E}_1 \backslash \mathcal{C}|}{2}.$$

So in either of the two cases R is connected to more than half the points of $\mathcal{E}_1 \backslash \mathcal{C}$ and the same argument applies to S. It follows that if S is any other point of $\mathcal{E}_1 \backslash \mathcal{C}$ that R and S must be connected to a common point of $\mathcal{E}_1 \backslash \mathcal{C}$ and hence are connected. □

As we have shown that H is connected by Lemma 4.6 it follows that $\overline{G_{\{A,C\}}}$ is connected and Proposition 4.5 is proved. Finally, we have our main theorem.

Theorem 1.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a generalized quadrangle of order (q, q^2) having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, q)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ consisting of all of the points collinear with a given point X of $\mathcal{S} \backslash \mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

Proof: As usual let $\mathcal{T}$ be the SPG of Metz, G the point graph of $\mathcal{T}$ and Γ_G the associated simplicial complex. By Proposition 4.5 and Lemma 4.4 if $(ABCD)$ is any four-circuit of G, then B and D are connected in the subgraph of G induced by the common neighbours of A and C. Hence $(ABCD)$ may be decomposed into triangles and the corresponding

1-chain of Γ_G is a 1-boundary. By Lemma 4.1 this implies that each induced 1-cycle of Γ_G is a 1-boundary. By applying Theorem 2.8 we see that $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial, which by Theorem 2.4 means that $H^1(\Gamma_G, \mathbb{Z}_2)$ is trivial. Finally applying Theorem 3.5 the theorem is proved. □

5. Generalisation of techniques

The focus of this paper has been the proof of Theorem 1.2; that is to show that if a GQ $\mathcal{S}$ of order (q, q^2) contains a subquadrangle isomorphic to $Q(4, q)$ such that all of the subtended ovoids are elliptic quadrics, then $\mathcal{S} \cong Q(5, q)$. However, the techniques employed to prove the result may be extended as follows. Suppose that $\mathcal{S}$ is a GQ of order (s, s^2) containing a subquadrangle $\mathcal{S}'$ of order s such that each subtended ovoid of $\mathcal{S}'$ is subtended by precisely two points of $\mathcal{S}\backslash\mathcal{S}'$. Then by [3, Theorem 2.5] the incidence structure of subtended ovoids and subtended rosettes of $\mathcal{S}'$ is an SPG $\mathcal{T}$ with parameters $s = q - 1,\ t = q^2,\ \alpha = 2,\ \mu = 2q(q - 1)$. The subquadrangle $\mathcal{S}'$ is said to be *doubly subtended* in $\mathcal{S}$. In a manner directly analogous to that in Section 3 the geometry of $\mathcal{S}\backslash\mathcal{S}'$ defines a 2-fold cover of the SPG $\mathcal{T}$ and in fact an algebraic 2-fold cover. The condition (1) also determines precisely those 1-cochains that define a GQ via a construction analogous to that given in Lemma 3.2. Thus the number of embeddings of $\mathcal{S}'$ in a GQ of order (s, s^2) with the same subtended ovoid/rosette structure is determined by the group $H_1(\Gamma_G, \mathbb{Z}_2)$ where Γ_G is the simplicial complex defined by the point graph G of $\mathcal{T}$. (Two such embeddings of $\mathcal{S}'$ in GQs $\mathcal{S}_1$ and $\mathcal{S}_2$ are "distinct" if there is no isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ that maps $\mathcal{S}'$ onto $\mathcal{S}'$ and $\mathcal{T}$ onto $\mathcal{T}$.) If the group $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial, then the embedding of $\mathcal{S}'$ is unique. We will omit the precise connection between $|H_1(\Gamma_G, \mathbb{Z}_2)|$ and the number of embeddings of $\mathcal{S}'$ with subtended ovoid/rosette structure $\mathcal{T}$ (this is explained in detail in [2]) and just make the comment that if the group $H_1(\Gamma_G, \mathbb{Z}_2)$ is non-trivial, then it does not necessarily follow that there are distinct embeddings of $\mathcal{S}'$.

In Section 4 it was shown that if $\mathcal{S}' = Q(4, q)$ and $\mathcal{T}$ is the SPG with points elliptic quadrics of $Q(4, q)$ and lines the rosettes of elliptic quadric ovoids, then every induced 1-cycle of Γ_G is homologous to the sum of induced 1-cycles each of which contains four 1-simplexes. The proof of this result was entirely combinatorial and extends to the general case of a GQ $\mathcal{S}'$ of order s doubly subtended in a GQ $\mathcal{S}$ of order (s, s^2) as does the interpretation of this result in the graph G.

The GQ of Kantor [11] contains a subquadrangle isomorphic to $Q(4, q)$ (q odd) and it was pointed out in [3] that this subquadrangle is doubly subtended with the subtended ovoids being ovoids constructed by Kantor in [10]. Thus, by the above, calculating the group $H_1(\Gamma_G, \mathbb{Z}_2)$ for this case will yield either a characterisation of the GQ of Kantor or a new GQ of order (q, q^2) that contains $Q(4, q)$ and subtends the Kantor ovoids of $Q(4, q)$. This is a possible avenue for future work, although the lack of a sufficiently "nice" representation of the graph G may make it more difficult than the calculation presented in this paper. In the case when s is odd it has been shown by Thas ([18, Theorem 7.1]) that if $\mathcal{S}$ is a GQ of order (s, s^2) and $\mathcal{S}'$ is a doubly subtended subquadrangle of $\mathcal{S}$, then $\mathcal{S}'$ must be a GQ with all points antiregular.

When s is even Thas [18, Theorem 7.1] also proved that $\mathcal{S}'$ must be isomorphic to $Q(4, s)$. For s even $Q(5, s)$ is the only known GQ of order (s, s^2) containing $Q(4, s)$ as a subquadrangle. (By [1] the existence of another such GQ of order (s, s^2), s even, implies the existence of a new ovoid of PG$(3, q)$.)

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References

1. M.R. Brown, "Generalized quadrangles of order (q, q^2), q even, containing $W(q)$ as a subquadrangle," *Geom. Dedicata* **56**(3) (1995), 299–306.
2. M.R. Brown, "Generalized Quadrangles and Associated Structures," Ph.D. Thesis, University of Adelaide, 1997.
3. M.R. Brown, "Semipartial geometries and generalized quadrangles of order (r, r^2)," *Bull. Belg. Math. Soc. Simon Stevin* **5**(2/3) (1998), 187–205.
4. P.J. Cameron, "Covers of graphs and EGQs," *Discrete Math.* **97**(1–3) (1991), 83–92.
5. F. De Clerck and H. Van Maldeghem, "Some classes of rank 2 geometries," in *The Handbook of Incidence Geometry*, Elsevier Science Publishers, B.V., Amsterdam, The Netherlands, 1995, pp. 433–475.
6. M.J. Greenberg and J.R. Harper, *Algebraic Topology, A First Course*, The Benjamin/Cummings Publishing, Reading, MA, 1981.
7. J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, 2nd ed., Clarendon Press, Oxford, 1998.
8. J.W.P. Hirschfeld and J.A. Thas, "Sets of type $(1, n, q+1)$ in PG(d, q)," *Proc. London Math. Soc*. **41**(3) (1980), 254–278.
9. J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Clarendon Press, Oxford, 1991.
10. W.M. Kantor, "Ovoids and translation planes," *Canad. J. Math.* **34** (1980), 1195–1203.
11. W.M. Kantor, "Some generalized quadrangles with parameters q^2, q," *Math. Z.* **192** (1986), 45–50.
12. S.E. Payne, "A restriction on the parameters of a subquadrangle," *Bull. Amer. Math. Soc.* **70** (1973), 747–748.
13. S.E. Payne and J.A. Thas, *Finite Generalized Quadrangles*, Pitman, London, 1984.
14. J.J. Seidel, "A survey of two-graphs," in *Accad. Naz. Lincei*. Roma, 1976, pp. 481–511.
15. H. Seifert and W. Threlfall, *A Textbook of Topology*, Academic Press, New York, 1980.
16. E.H. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1966.
17. J.A. Thas, "A remark concerning the restriction on the parameters of a 4-gonal configuration," *Simon Stevin* **48** (1974/75), 65–68.
18. J.A. Thas, "3-regularity in generalized quadrangles: A survey, recent results and the solution of a longstanding conjecture," *Rend. Circ. Mat. Palermo* 2 (Suppl. 53) (1998), 199–218. Combinatorics '98 (Mondello).
19. J.A. Thas and S.E. Payne, "Spreads and ovoids in finite generalized quadrangles," *Geom. Dedicata* **52**(3) (1994), 227–253.