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Minimal Covers of $Q^+(2n+1, q)$ by $(n-1)$-Dimensional Subspaces

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Abstract. A t-cover of a quadric $\mathcal{Q}$ is a set $\mathcal{C}$ of t-dimensional subspaces contained in $\mathcal{Q}$ such that every point of $\mathcal{Q}$ is contained in at least one element of $\mathcal{C}$.

We consider $(n-1)$-covers of the hyperbolic quadric $Q^+(2n+1, q)$. We show that such a cover must have at least $q^{n+1} + 2q + 1$ elements, give an example of this size for even q and describe what covers of this size should look like.

Keywords: covers, partial spreads, quadrics

1. Introduction

Let $\mathcal{Q}$ be a quadric. A *spread* in $\mathcal{Q}$ is a set $\mathcal{S}$ of generators of $\mathcal{Q}$ such that each point of $\mathcal{Q}$ is contained in exactly one element of $\mathcal{S}$.

If $\mathcal{Q} = Q^+(2n+1, q)$ is the hyperbolic quadric in PG$(2n+1, q)$, then it is known that $Q^+(4n+1, q)$ does not have a spread, while $Q^+(4n+3, q)$, q even, does have a spread. The existence of a spread in $Q^+(4n+3, q)$, q odd, is still open [8].

If no spreads exist, the natural question arises what are the sets of generators on $\mathcal{Q}$ being closest to a spread. This leads more generally to the following definitions:

Definition 1.1

(a) A *t-cover* of a quadric $\mathcal{Q}$ is a set $\mathcal{C}$ of t-dimensional subspaces contained in $\mathcal{Q}$ such that each point of $\mathcal{Q}$ is contained in at least one element of $\mathcal{C}$. If $t = 1$, we speak also of a *line cover*; if $t = 2$, we speak of a *plane cover*.

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(b) A *partial t-spread* is a set $\mathcal{S}$ of t-dimensional subspaces contained in $\mathcal{Q}$ such that each point of $\mathcal{Q}$ is contained in at most one element of $\mathcal{S}$.

In [4], the authors determined the 2-covers of the Klein quadric $Q^+(5, q)$ having minimum size. A lower bound for the size of a 1-cover of the Klein quadric was given, as well as examples reaching that bound. Similarly, large partial 1-spreads on the Klein quadric were constructed.

In this article, we continue the study started in [4] by studying the smallest $(n-1)$-covers of $Q^+(2n+1, q)$. We show that the smallest possible cardinality for a minimal $(n-1)$-cover of $Q^+(2n+1, q)$ is $q^{n+1} + 2q + 1$, and give examples of that size for q even.

We prove a theorem stating what an $(n-1)$-cover of $Q^+(2n+1, q)$ of that size $q^{n+1} + 2q + 1$ should *look like*. To achieve this, we need results on *minihypers in projective spaces* [6].

Definition 1.2 Let $\mathcal{F}$ be a set of points of PG (t, q) and let w be a mapping from $\mathcal{F}$ into $\mathbf{Z}^+$, where $t \geq 2$ and where $\mathbf{Z}^+$ denotes the set of all non-negative integers. Let $\mathcal{H}$ denote the set of all hyperplanes of PG (t, q).

If $\mathcal{F}$ and w satisfy the conditions

$$\sum_{P \in \mathcal{F}} w(P) = f \quad \text{and} \quad \min\left\{ \sum_{P \in \mathcal{F} \cap H} w(P) \middle\| H \in \mathcal{H} \right\} = m,$$

for given integers $f \geq 1$ and $m \geq 0$, then $(\mathcal{F}, w)$ is called an $\{f, m; t, q\}$-*minihyper*. In the special case $w(P) = 1$ for all $P \in \mathcal{F}$, we denote the minihyper simply by $\mathcal{F}$.

The article concludes with an upper bound for the size of a maximal partial $(n-1)$-spread of $Q^+(2n+1, q)$. For q even, partial $(n-1)$-spreads of $Q^+(2n+1, q)$ of size $q^{2n+1} + 1$ are constructed.

These results contribute to the study of blocking sets, spreads and covers in polar spaces, as discussed by Metsch [10]. For a table containing the known results on the existence and non-existence of spreads in polar spaces, we refer to [10, Table 2]. We would like to mention the following recent results on line covers of $H(3, q^2)$ and $Q(4, q)$.

Since the generalized quadrangle $H(3, q^2)$ arising from the non-singular Hermitian variety in PG $(3, q^2)$ is the dual of the quadrangle arising from the non-singular elliptic quadric in PG $(5, q)$, a cover of $H(3, q^2)$ is the dual of a blocking set in $Q^-(5, q)$, that is, a set of points of $Q^-(5, q)$ intersecting every line of $Q^-(5, q)$. In [9], Metsch proved that the smallest blocking sets of $Q^-(5, q)$ are equal to the set of points of $Q^-(5, q)$ in a tangent cone of $Q^-(5, q)$, different from the vertex of the tangent cone. Hence, dualizing this result, the smallest covers of $H(3, q^2)$ are equal to the set of lines of $H(3, q^2)$ intersecting a given line of $H(3, q^2)$ in exactly one point.

For covers of the parabolic quadric $Q(4, q)$ in PG $(4, q)$, Eisfeld et al. [5] proved that a cover $\mathcal{C}$ of $Q(4, q)$, q odd, contains more than $q^2 + 1 + (q-1)/3$ lines, and a cover of $Q(4, q)$, q even, $q \geq 32$, of cardinality $q^2 + 1 + r$, where $0 < r \leq \sqrt{q}$, always contains a spread of $Q(4, q)$.

The analogous question for covers of projective spaces has already been answered (cf. [1, 3]). The lower bound on the size of a cover of a given projective space was found by Beutelspacher [1]; the description of the covers of minimal size was given by Eisfeld [3].

Theorem 1.3 *Let $\mathcal{S}$ be a t-cover of $\mathcal{P} = \mathrm{PG}(d, q)$, where $d \geq t \geq 0$. Let $d = k(t+1) + r$ with $k, r \in \mathbb{N}_0$ and $r \leq t$.*
(a) $|\mathcal{S}| \geq q^{r+1} \cdot \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$.
(b) *If equality holds in (a), then there is a subspace U of dimension $t - r - 1$, such that every point of $\mathcal{P} \backslash U$ is contained in exactly one element of $\mathcal{S}$, whereas every point of U is contained in exactly $q^{r+1} + 1$ elements of $\mathcal{S}$.*

In this article, let $\theta_i = (q^{i+1} - 1)/(q - 1)$.

2. The lower bound

Let $\mathcal{Q} = Q^+(2n+1, q)$ be embedded in $\mathcal{P} = \mathrm{PG}(2n+1, q)$. If we have an $(n-1)$-cover $\mathcal{C}$ of $\mathcal{Q}$, we define the *excess* of a point $P \in \mathcal{Q}$ to be the number of elements of $\mathcal{C}$ through P minus one. The excess of a point of $\mathcal{P} \setminus \mathcal{Q}$ is defined as zero. Since $\mathcal{C}$ is a cover, all excesses are non-negative.

The *excess* of any point set of $\mathcal{P}$ is defined as the sum of the excesses of its points. A point with positive excess is called an *excess point*.

Theorem 2.1 *Let $\mathcal{C}$ be an $(n-1)$-cover of $\mathcal{Q} = Q^+(2n+1, q)$. Then $|\mathcal{C}| \geq q^{n+1} + 2q + 1$.*

Proof: Since $\mathcal{Q}$ has exactly $(q^{n+1} - 1)(q^n + 1)/(q - 1)$ points (see e.g. [7]), the $(n-1)$-cover $\mathcal{C}$ must have at least $\frac{(q^{n+1}-1)(q^n+1)}{q^n-1} = q^{n+1} + 2q - 1 + 2\frac{q-1}{q^n-1}$ elements.

Suppose that $\mathcal{C}$ has $q^{n+1} + 2q + \varepsilon$ elements. Then the total excess of $\mathcal{Q}$ is

$$(q^{n+1} + 2q + \varepsilon)\frac{q^n - 1}{q - 1} - \frac{(q^{n+1} - 1)(q^n + 1)}{q - 1} = (\varepsilon + 1)\frac{q^n - 1}{q - 1} - 2.$$

Consider a subspace U of $\mathcal{P}$ that has dimension $n+2$. The subspace U intersects each element of $\mathcal{C}$ in a non-empty subspace, that is, in 1 (mod q) points. Furthermore, it intersects $\mathcal{Q}$ in a quadric, and so in 1 (mod q) points. Hence the excess of U is $q^{n+1} + 2q + \varepsilon - 1 \equiv \varepsilon - 1 \pmod q$.

Suppose that $\varepsilon = 0$. Then the excess of any $(n+2)$-dimensional subspace is congruent to $q - 1 \pmod q$, hence it is at least $q - 1$. In particular, the set of excess points must intersect each $(n+2)$-dimensional subspace. By the Theorem of Bose and Burton [2], this means that there are at least $(q^n - 1)/(q - 1)$ excess points with equality if and only if the excess points are just the points of an $(n-1)$-dimensional subspace of $\mathcal{P}$. This contradicts the fact that the total excess is $\frac{q^n-1}{q-1} - 2$.

Hence $\varepsilon \geq 1$, and the theorem is proved. □

Example 2.2 Suppose that q is even. Then there exists a spread $\mathcal{S}$ of the parabolic quadric $Q(2n+2, q)$ (see e.g. [10, Section 6]).

Let $\mathcal{Q} = Q^+(2n+1, q)$ be a hyperplane section of $Q(2n+2, q)$. Then a counting argument shows that $\mathcal{Q}$ contains exactly two elements of $\mathcal{S}$ and intersects the other elements of $\mathcal{S}$ in $(n-1)$-dimensional subspaces. These form a partial spread $\mathcal{C}_0$ of $Q^+(2n+1, q)$ covering all points except the points of two disjoint n-dimensional subspaces U_1, U_2. Let W_i be an $(n-2)$-dimensional subspace of U_i $(i = 1, 2)$. If we add to $\mathcal{C}_0$ the $q+1$ $(n-1)$-dimensional subspaces in U_i through W_i, we get an $(n-1)$-cover $\mathcal{C}$. The excess points of $\mathcal{C}$ are the points of W_1 and W_2, each having excess q. From this we see that $|\mathcal{C}| = q^{n+1}+2q+1$. In the following section we shall see that all covers of this size *look like* this example.

3. A characterization

In [3], the structure of excess points of minimum covers of projective spaces was determined (See Introduction). In this section, we do the same for minimum $(n-1)$-covers of $Q^+(2n+1, q)$, using similar methods.

From now on, let $\mathcal{Q} = Q^+(2n+1, q)$, $q \geq 3$, be embedded into $\mathcal{P} = \mathrm{PG}(2n+1, q)$, and let $\mathcal{C}$ be an $(n-1)$-cover of $\mathcal{Q}$ with $|\mathcal{C}| = q^{n+1} + 2q + 1$.

Lemma 3.1 *Let U, V be subspaces of $\mathcal{P}$ such that U is a hyperplane of V. Then there exist integers a, a', b, b' with* $\dim V - n - 2 \leq b \leq a \leq n-1$ *and* $a - 1 \leq a' \leq a$ *and* $b - 1 \leq b' \leq b$ *such that*

(a) *the excess of V is $q\frac{q^a-1}{q-1} + q\frac{q^b-1}{q-1}$.*

(b) *the excess of U is $q\frac{q^{a'}-1}{q-1} + q\frac{q^{b'}-1}{q-1}$.*

Proof: The proof is by backward induction on $\dim V$.
At first we consider the case $\dim V = 2n+1$, that is, $V = \mathcal{P}$. In this case, the excess of V is the total excess of $\mathcal{Q}$, that is, $2q\frac{q^{n-1}-1}{q-1}$, which yields (a).

Let U be a hyperplane of $\mathcal{P}$. Then U intersects $\mathcal{Q}$ either in a parabolic quadric with $(q^{2n}-1)/(q-1)$ points or in a cone over a hyperbolic quadric, containing $1 + q(q^n - 1)(q^{n-1}+1)/(q-1) = (q^{2n}-1)/(q-1) + q^n$ points. Hence U contains $(q^{2n}-1)/(q-1)$ (mod q^n) points of $\mathcal{Q}$. On the other hand, U contains $(q^{n-1}-1)/(q-1)$ (mod q^{n-1}) points of any element of $\mathcal{C}$. Hence the excess of a hyperplane U is congruent to

$$(q^{n+1}+2q+1)\frac{q^{n-1}-1}{q-1} - \frac{q^{2n}-1}{q-1} \equiv 2q\frac{q^{n-2}-1}{q-1} \pmod{q^{n-1}}.$$

As the excess must be a non-negative number being at most equal to the total excess, it must be either $2q\frac{q^{n-1}-1}{q-1}$ or $q\frac{q^{n-1}-1}{q-1} + q\frac{q^{n-2}-1}{q-1}$ or $2q\frac{q^{n-2}-1}{q-1}$, from which (b) follows.

Now let $\dim V < 2n+1$, and we assume that the assertion holds for bigger values of $\dim V$. In particular, the induction hypothesis yields (a). Furthermore, for any subspace $W \supseteq V$ with $\dim W = \dim V + 1$ we know from the induction hypothesis that the excess of W is $q\frac{q^{\tilde{a}}-1}{q-1} + q\frac{q^{\tilde{b}}-1}{q-1}$ with $a \leq \tilde{a} \leq a+1$ and $b \leq \tilde{b} \leq b+1$.

Consider the $q+1$ subspaces V_i with $U \leq V_i \leq W$ and $\dim V_i = \dim V$ (one of them being V). By the induction hypothesis, each V_i has an excess $q\frac{q^{a_i}-1}{q-1} + q\frac{q^{b_i}-1}{q-1}$ with $\tilde{a} - 1 \leq a_i \leq \tilde{a}$ and $\tilde{b} - 1 \leq b_i \leq \tilde{b}$. Let α be the number of a_i equal to $\tilde{a}$, and let β be the number of b_i equal to $\tilde{b}$. Let x be the excess of U. Counting the sum of the excesses in two ways, we get

$$(q+1)\left(q\frac{q^{\tilde a-1}-1}{q-1}+q\frac{q^{\tilde b-1}-1}{q-1}\right)+\alpha q\frac{q^{\tilde a}-q^{\tilde a-1}}{q-1}+\beta q\frac{q^{\tilde b}-q^{\tilde b-1}}{q-1}$$
$$=q\frac{q^{\tilde a}-1}{q-1}+q\frac{q^{\tilde b}-1}{q-1}+qx.$$

Hence

$$x=q\frac{q^{\tilde a-2}-1}{q-1}+q\frac{q^{\tilde b-2}-1}{q-1}+\alpha q^{\tilde a-1}+\beta q^{\tilde b-1}. \qquad (*)$$

Clearly, x cannot be bigger than the minimum of the excesses of the V_i. We discuss the possible values of α and β.

- $\alpha=\beta=0$. Then all a_i, including a, are equal to $\tilde a-1$. That is, $a=\tilde a-1$ and similarly $b=\tilde b-1$. By $(*)$, (b) is fulfilled with $a'=\tilde a-2$ and $b'=\tilde b-2$.
- $\alpha=1,\beta=0$. Then $a\in\{\tilde a-1,\tilde a\}$ and $b=\tilde b-1$. By $(*)$, $a'=\tilde a-1$ and $b'=\tilde b-2$, and (b) holds.
- $\alpha=0,\beta=1$. This case works as the previous case.
- $\alpha=\beta=1$. Then $a\in\{\tilde a-1,\tilde a\}$ and $b\in\{\tilde b-1,\tilde b\}$. By $(*)$, (b) holds with $a'=\tilde a-1$ and $b'=\tilde b-1$.
- $\alpha=2,\beta=0,\tilde a=\tilde b$. This case is identical with the previous case.
- $\tilde a=\tilde b,\alpha+\beta>2$. By $(*)$, the excess x of U is bigger than $2q\frac{q^{\tilde a-1}-1}{q-1}$. Hence also the excesses of the V_i are bigger than this value. Hence the excess of V_i is $q\frac{q^{\tilde a}-1}{q-1}+q\frac{q^{b_i}-1}{q-1}$, that is, we can assume that $\alpha=q+1$. This yields

  $$x=q\frac{q^{\tilde a}-1}{q-1}+q\frac{q^{\tilde b-2}-1}{q-1}+\beta q^{\tilde b-1}.$$

  – If $\beta=0$, then $b=\tilde b-1$, and (b) holds with $b'=\tilde b-2$.
  – If $\beta=1$, then $b\in\{\tilde b-1,\tilde b\}$, and (b) holds with $b'=\tilde b-1$.
  – If $\beta\geq 2$, then the excess of U is bigger than $q\frac{q^{\tilde a}-1}{q-1}+q\frac{q^{\tilde a-1}-1}{q-1}$, which means that also the excesses of the V_i must be bigger than this value. Consequently, $b_i=\tilde b,\beta=q+1$, and (b) holds with $b'=b=\tilde b$.

- $\tilde a>\tilde b,\alpha\geq 2$. Then the excess of each V_i must be at least $x>q\frac{q^{\tilde a-1}-1}{q-1}+q\frac{q^{\tilde b-1}-1}{q-1}$, whence $\alpha+\beta\geq q+1$.

  – $\alpha\geq 3$. Then $\alpha q^{\tilde a-1}>q^{\tilde a-1}+(q+1)q^{\tilde b-1}$, that is, the excess of U (and hence of V_i) is bigger than $q\frac{q^{\tilde a-1}-1}{q-1}+q\frac{q^{\tilde b}-1}{q-1}$. Hence $\alpha=q+1$, and so $x=q\frac{q^{\tilde a}-1}{q-1}+q\frac{q^{\tilde b-2}-1}{q-1}+\beta q^{\tilde b-1}$. Now (b) follows as in the previous case, distinguishing between the cases $\beta=0$, $\beta=1$ and $\beta\geq 2\Rightarrow\beta=q+1$.
  – $\alpha=2,\tilde a>\tilde b+1$. Then again $\alpha q^{\tilde a-1}>q^{\tilde a-1}+(q+1)q^{\tilde b-1}$, from which we get $\alpha=q+1$ as above, which is a contradiction.

– $\alpha = 2, \tilde{a} = \tilde{b} + 1$. Because of $\beta \geq q - 1$ and $q \geq 3$, we have $\alpha q^{\tilde{a}-1} + \beta q^{\tilde{b}-1} > q^{\tilde{a}-1} + (q+1)q^{\tilde{b}-1}$, which again gives $\alpha = q + 1$, being a contradiction.

- $\tilde{a} > \tilde{b}, \alpha = 0, \beta \geq 2$. We show that this case can be avoided choosing W in an intelligent way.

  Count the incidences (s, W^*), where s is an excess point outside of V and $W^* = \langle s, V \rangle$. Starting from s, we see that there are $q\frac{q^{n-1}-q^a}{q-1} + q\frac{q^{n-1}-q^b}{q-1}$ such incidences. Let $d = \dim V$. The number of $(d+1)$-dimensional subspaces W^* containing V is $\frac{q^{2n+1-d}-1}{q-1}$. Each of these W^* contributes $0, q \cdot q^a, q \cdot q^b$ or $q \cdot (q^a + q^b)$ incidences. The average contribution of a W^* is $q^{a+1}\frac{q^{n-1-a}-1}{q^{2n+1-d}-1} + q^{b+1}\frac{q^{n-1-b}-1}{q^{2n+1-d}-1}$.

  – Suppose that $a > b$. Because of $b \geq d - n - 2$, we have $n - 1 - a < 2n + 1 - d$. Hence the average contribution of W^* is smaller than $q^a + q^{b+1}$. This means that there must exist a W^* contributing either 0 or $q \cdot q^b$, which means that $\tilde{a} = a$. This avoids the current case.
  – If $a = b$, then there exists a choice of W^* with $\tilde{a} = \tilde{b}$, avoiding the current case. For otherwise all W^* would contribute $q \cdot q^a$ to the number of incidences (s, W^*), whence $2q^{a+1}\frac{q^{n-1-a}-1}{q^{2n+1-d}-1} = q^{a+1}$, which gives a contradiction.

- $\tilde{a} > \tilde{b}, \alpha = 1, \beta \geq 2$. As in the case $\alpha \geq 3$, we see that $\alpha + \beta \geq q + 1$, that is, $\beta \in \{q, q+1\}$.

  – $\beta = q + 1$. Then (b) holds with $a' = \tilde{a} - 1$ and $b' = b = \tilde{b}$.
  – $\beta = q$. Then $x = q\frac{q^{\tilde{a}-1}-1}{q-1} + q\frac{q^{\tilde{b}}-1}{q-1} - q^{\tilde{b}-1}$. This is a value that cannot be written in the form $q\frac{q^{a'}-1}{q-1} + q\frac{q^{b'}-1}{q-1}$.

    Let V' be one of the V_i with an excess of $q\frac{q^{\tilde{a}-1}-1}{q-1} + q\frac{q^{\tilde{b}}-1}{q-1}$. Doing the same argument with V' in place of V, we must get the same (exceptional) value of x, that is, we must fall again into the case $\tilde{a} > \tilde{b}, \alpha = 1$ with the same parameters. However, as in the case $\alpha = 0$, we see that it is possible to choose W^* such that $\tilde{a}^* = \tilde{b}^*$ (leading immediately to a different case) or $\tilde{a}^* = a = \tilde{a} - 1$ (leading possibly to the same case, but with a different parameter $\tilde{a}$). This yields a contradiction.

This discussion concludes the proof. □

In the case $\dim V = 0$, Lemma 3.1 shows that the excess points of the cover have excess congruent to 0 (mod q). If we now divide the excess of every excess point by q, then we remain with a set of $2(q^{n-1} - 1)/(q - 1)$ points intersecting every hyperplane in at least $2(q^{n-2} - 1)/(q - 1)$ points. Hence, the excess points form a weighted $\{2(q^{n-1} - 1)/(q - 1), 2(q^{n-2} - 1)/(q - 1); 2n + 1, q\}$-minihyper $\mathcal{F}$.

For $n = 2$, this means that $\mathcal{F}$ is either a point with multiplicity two or two points with multiplicity one (see also [4, Theorem 3.1]). Assume $n \geq 3$.

If all the points in this minihyper have weight one, then Hamada has proved that this set is the union of two disjoint subspaces PG$(n-2, q)$ [6, Theorem 4.1]. It is however possible that some of the points have weight bigger than one. We will now show that, in general, this set is the union of two subspaces Π_1 and Π_2 of dimension $n - 2$, where the points of $\Pi_1 \cap \Pi_2$ have weight two and where the remaining points of $\Pi_1 \cup \Pi_2$ have weight one.

Lemma 3.2 *Let $\mathcal{F}$ be the $\{2(q^{n-1}-1)/(q-1), 2(q^{n-2}-1)/(q-1); 2n+1, q\}$-minihyper of excess points of $\mathcal{C}$. Then the points of $\mathcal{F}$ have weight one or two.*

Proof: Consider a subspace Π of dimension $n+2$ skew to $\mathcal{F}$. There are θ_{n-2} spaces Ω of dimension $n+3$ passing through Π. By Lemma 3.1, each one of them must have at least two points in common with $\mathcal{F}$; so must have exactly two points in common with $\mathcal{F}$. This shows that no points of $\mathcal{F}$ have excess bigger than two. □

Lemma 3.3 *Let $\mathcal{F}$ be the $\{2(q^{n-1}-1)/(q-1), 2(q^{n-2}-1)/(q-1); 2n+1, q\}$-minihyper of excess points of $\mathcal{C}$. If $\mathcal{F}$ contains a point P with weight two, then $\mathcal{F}$ consists of a union of lines through P.*

Proof: Suppose that a line l through P contains $x \geq 3$ points of $\mathcal{F}$. Then there are $2\theta_{n-2} - x$ points left. Suppose that there is a point $R \in l \setminus \mathcal{F}$. Then R lies in q^{2n} hyperplanes not containing l.

A point $S \in (\mathcal{F} \setminus l)$ lies in q^{2n-1} hyperplanes through RS not containing l. This shows that the average number of points of $\mathcal{F}$ in these hyperplanes is $(2\theta_{n-2} - x)q^{2n-1}/q^{2n} = 2\theta_{n-3} + (2-x)/q < 2\theta_{n-3}$. This means that there is a hyperplane through R containing less than $2\theta_{n-3}$ points of $\mathcal{F}$.

This is false; so $l \subset \mathcal{F}$. □

The following lemma follows from Lemma 3.1 if we now use the known fact that every point of $\mathcal{F}$ has an excess which is a multiple of q.

Lemma 3.4 *Let $\mathcal{F}$ be the $\{2(q^{n-1}-1)/(q-1), 2(q^{n-2}-1)/(q-1); 2n+1, q\}$-minihyper of excess points of $\mathcal{C}$. Then a t-dimensional subspace, $n+4 \leq t \leq 2n$, intersects $\mathcal{F}$ in a $\{(q^a-1)/(q-1)+(q^b-1)/(q-1), (q^{a-1}-1)/(q-1)+(q^{b-1}-1)/(q-1); t, q\}$-minihyper, with $t-n-2 \leq b \leq a \leq n-1$.*

Lemma 3.5 *Let $\mathcal{F}$ be the $\{2(q^{n-1}-1)/(q-1), 2(q^{n-2}-1)/(q-1); 2n+1, q\}$-minihyper of excess points of $\mathcal{C}$. Then the set of points of $\mathcal{F}$ with weight two is a subspace of* PG $(2n+1, q)$.

Proof: Let l be a line containing two points P_1 and P_2 of $\mathcal{F}$ having weight two. Let P_3 be a point of $\mathcal{F}$ on l with weight one. By induction, we will find a subspace Π_{n+4} of dimension $n+4$ through l intersecting $\mathcal{F}$ in a $\{2(q+1), 2; n+4, q\}$-minihyper.

Let x be the sum of the weights of the points of $l \cap \mathcal{F}$. Then counting the incidences of the points of $\mathcal{F} \setminus l$ with the hyperplanes through l, we get as sum of these incidences, the number

$$(2\theta_{n-2} - x)\,\theta_{2n-2}.$$

This implies that the average of the incidences over all hyperplanes through l is equal to $x + (2\theta_{n-2} - x)\,\theta_{2n-2}/\theta_{2n-1}$, which is equal to

$$x + \frac{2}{q-1}\left(q^{n-2} - \frac{q^{2n-1}+q^{n-1}-1-q^{n-2}}{q^{2n}-1}\right) - x\left(\frac{q^{2n-1}-1}{q^{2n}-1}\right).$$

Since $x \leq 2q + 1$, there is a hyperplane through l having at most

$$\frac{2}{q-1}\left(q^{n-2} - \frac{q^{2n-1} + q^{n-1} - 1 - q^{n-2}}{q^{2n} - 1}\right) - 2 + 2q + 1$$

points of $\mathcal{F}$. Since each hyperplane must have at least $2\theta_{n-3}$ points of $\mathcal{F}$, by using Lemma 3.4, there must be a hyperplane through l intersecting $\mathcal{F}$ in a $\{2\theta_{n-3}, 2\theta_{n-4}; 2n, q\}$-minihyper.

By induction, there is a subspace PG $(n+4, q)$ through l intersecting $\mathcal{F}$ in a $\{2(q+1), 2; n+4, q\}$-minihyper. For, suppose there is a $(2n+1-i)$-dimensional subspace Π_{2n+1-i} through l, $n-i-2 \geq 2$, intersecting $\mathcal{F}$ into a $\{2(q^{n-1-i}-1)/(q-1), 2(q^{n-i-2}-1)/(q-1); 2n+1-i, q\}$-minihyper. Now the average number of points of $\mathcal{F}$ in a hyperplane of Π_{2n+1-i} through l is equal to $x + (2(q^{n-1-i}-1)/(q-1) - x)(q^{2n-i-1}-1)/(q^{2n-i}-1)$. According to Lemma 3.4, any hyperplane not intersecting $\mathcal{F}$ in a minihyper with the desired parameters must intersect $\mathcal{F}$ in at least $(q^{n-i-1}-1)/(q-1)+1 \geq 2(q^{n-i-2}-1)/(q-1)+2q$ points. The average number given is smaller than this number, hence there must be a hyperplane in Π_{2n+1-i} intersecting $\mathcal{F}$ as desired.

By induction, this implies that l lies in an $(n+4)$-dimensional subspace H sharing a $\{2q+2, 2; n+4, q\}$-minihyper with $\mathcal{F}$.

Now, by assumption, l contains at most $2q+1$ points of $\mathcal{F}$, so there is a point R of $\mathcal{F}$ lying in this subspace, but not lying on l. Then the three lines P_1P_2, RP_1, RP_2 are all contained in $\mathcal{F}$; but then $\mathcal{F}$ shares more than $2q+2$ elements with H.

So, all points of P_1P_2 have weight two.

This argument now implies that the points of $\mathcal{F}$ of weight two form a subspace. □

Theorem 3.6 *Let $\mathcal{F}$ be the $\{2(q^{n-1}-1)/(q-1), 2(q^{n-2}-1)/(q-1); 2n+1, q\}$-minihyper of excess points of $\mathcal{C}$, where $\mathcal{F}$ has a u-dimensional subspace Π of points having weight two. Then $\mathcal{F}$ consists of two $(n-2)$-dimensional subspaces intersecting in this subspace Π.*

Proof: If $u = -1$, then the theorem follows from [6, Theorem 4.1]. So, assume $u \geq 0$. Consider the quotient geometry of Π represented by an $(2n-u)$-dimensional space Π' skew to Π.

The minihyper $\mathcal{F}$ consists of subspaces of dimension $u+1$ passing through Π, so in Π', $\mathcal{F}$ defines a set $\mathcal{F}'$ of size $2\theta_{n-u-3}$.

Consider a hyperplane Π'' of Π' and suppose it shares x points with $\mathcal{F}'$. Then $\langle \Pi, \Pi'' \rangle$ shares $xq^{u+1} + 2\theta_u$ points with $\mathcal{F}$. Since every hyperplane shares at least $2\theta_{n-3}$ points with $\mathcal{F}$, necessarily $x \geq 2\theta_{n-u-4}$.

So, $\mathcal{F}'$ is a $\{2\theta_{n-u-3}, 2\theta_{n-u-4}; 2n-u, q\}$-minihyper only having points of weight one. So, by [6, Theorem 4.1], $\mathcal{F}'$ is the union of two disjoint subspaces of dimension $n-u-3$. This proves the theorem. □

The preceding results now imply the following description of the set of excess points of an $(n-1)$-cover, of size $q^{n+1}+2q+1$, of $\mathcal{Q} = Q^+(2n+1, q)$, $q \geq 3$. This corollary is also valid for $n = 2$ [4].

Corollary 3.7 *Let $\mathcal{C}$ be an $(n-1)$-cover of $\mathcal{Q} = Q^+(2n+1, q)$, $q \geq 3$, with $|\mathcal{C}| = q^{n+1}+2q+1$. Then there are two $(n-2)$-dimensional subspaces U_1, U_2 (possibly* *coinciding) on $\mathcal{Q}$ such that all points of $U_1 \cap U_2$ have excess $2q$, all points of $(U_1 \cup U_2)\backslash(U_1 \cap U_2)$ have excess q, and all points of $\mathcal{Q}\backslash(U_1 \cup U_2)$ have excess 0.*

Remark 3.8 Theorem 3.6 is also valid for arbitrary $\{2(q^{n-1}-1)/(q-1), 2(q^{n-2}-1)/(q-1); 2n+1, q\}$-minihypers of PG$(2n+1, q)$, $q \geq 3$. The proof that the points of such a minihyper have weight one or two follows from using [6, Theorem 2.5] in combination with the proof of Lemma 3.2.

4. Partial $(n-1)$-Spreads of $Q^+(2n+1, q)$

The construction made in Example 2.2 also shows that the hyperbolic quadric $Q^+(2n+1, q)$, q even, has partial $(n-1)$-spreads of size $q^{n+1}+1$. The question is whether larger partial $(n-1)$-spreads are possible. This question is studied in the following theorem. In this theorem, a *hole* of $\mathcal{S}$ is a point of $Q^+(2n+1, q)$ not lying on an element of $\mathcal{S}$.

Theorem 4.1 *Let $\mathcal{S}$ be a partial $(n-1)$-spread of $Q^+(2n+1, q)$. Then $|\mathcal{S}| \leq q^3+q$ for $n=2$ and $|\mathcal{S}| \leq q^{n+1}+q-1$ for $n > 2$.*

Proof: Let $\mathcal{S}$ be a partial $(n-1)$-spread of size $q^{n+1}+q$ of $\mathcal{Q} = Q^+(2n+1, q)$. For $n=2$, [4, Theorem 3.6] shows that this is the maximal possible cardinality of a line spread of $Q^+(5, q)$. Assume now that $n > 2$. A partial $(n-1)$-spread of $Q^+(2n+1, q)$ of size $q^{n+1}+q$ has q^n+1 holes.

Let U be a hyperplane of $\mathcal{P}$. Then U intersects $\mathcal{Q}$ either in a parabolic quadric with $(q^{2n}-1)/(q-1)$ points or in a cone over a hyperbolic quadric, containing $1+q(q^n-1)(q^{n-1}+1)/(q-1) = (q^{2n}-1)/(q-1)+q^n$ points. Suppose every element of $\mathcal{S}$ intersects U in an $(n-2)$-dimensional subspace. Since $\mathcal{S}$ has size $q^{n+1}+q$, then U would have q^n+1 holes in the first case and $2q^n+1$ holes in the second case. If however an element of $\mathcal{S}$ is completely contained in U, the number of holes reduces by q^{n-1}. So, the number of holes in a hyperplane is always 1 (mod q^{n-1}).

Suppose that x_i is the number of holes in the hyperplane π_i, $i = 1, \ldots, \theta_{2n+1}$, of PG$(2n+1, q)$. Then

$$\sum_{i=1}^{\theta_{2n+1}} 1 = \frac{q^{2n+2}-1}{q-1},$$

$$\sum_{i=1}^{\theta_{2n+1}} x_i = (q^n+1)\frac{q^{2n+1}-1}{q-1},$$

$$\sum_{i=1}^{\theta_{2n+1}} x_i(x_i-1) = (q^n+1)q^n\frac{q^{2n}-1}{q-1},$$

$$\sum_{i=1}^{\theta_{2n+1}} x_i(x_i-1)(x_i-2) \geq (q^n+1)q^n(q^n-1)\frac{q^{2n-1}-1}{q-1}.$$

Now

$$
\begin{aligned}
0 &\geq \sum_{i=1}^{\theta_{2n+1}} (x_i - 1)(x_i - q^{n-1} - 1)(x_i - q^n - 1) \\
&= \sum_{i=1}^{\theta_{2n+1}} x_i(x_i - 1)(x_i - 2) - (q^{n-1} + q^n) \sum_{i=1}^{\theta_{2n+1}} x_i(x_i - 1) \\
&\quad + (q^{n-1} + 1)(q^n + 1) \sum_{i=1}^{\theta_{2n+1}} (x_i - 1).
\end{aligned}
$$

Now replacing $\sum_{i=1}^{\theta_{2n+1}} x_i(x_i - 1)(x_i - 2)$ by the lower bound stated above gives the inequality $0 > q^{2n+1}(q^n + 1)(q^{n-2} - 1)$. This is false. □

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