Journal of Algebraic Combinatorics **15** (2002), 211–221

A Combinatorial Algorithm Related to the Geometry of the Moduli Space of Pointed Curves

G. BINI* gbini@math.lsa.umich.edu
Department of Mathematics, University of Michigan, 525 East University Ave., Ann Arbor, MI, 48109, USA

Received November 1, 2000; Revised September 13, 2001

Abstract. As pointed out in Arbarello and Cornalba (*J. Alg. Geom.* **5** (1996), 705–749), a theorem due to Di Francesco, Itzykson, and Zuber (see Di Francesco, Itzykson, and Zuber, *Commun. Math. Phys.* **151** (1993), 193–219) should yield new relations among cohomology classes of the moduli space of pointed curves. The coefficients appearing in these new relations can be determined by the algorithm we introduce in this paper.

Keywords: Schur Q-polynomials, projective representations, moduli space of curves

1. Introduction

Let $\mathcal{H}_N$ be the vector space of Hermitian matrices and denote by U_N the unitary group acting on $\mathcal{H}_N$ by conjugation. If $\mathbf{Z} = (z_{ab}) \in \mathcal{H}_N$, $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, i.e. $z_{ab} = x_{ab} + iy_{ab}$, we can regard $\mathcal{H}_N$ as a real euclidean space of dimension N^2 with coordinates x_{ab}, y_{ab} $(1 \leq a < b \leq N)$ and endowed with the inner product

$$(\mathbf{Z}_1, \mathbf{Z}_2) = Tr(\mathbf{Z}_1\mathbf{Z}_2), \quad \mathbf{Z}_i \in \mathcal{H}_N,$$

where $Tr(\mathbf{Z})$ denotes the trace of the matrix $\mathbf{Z}$.

Let Λ be a positive definite hermitian matrix. Then

$$d\mu_\Lambda := \frac{\exp(-(1/2)Tr(\Lambda\mathbf{X}^2))}{(2\pi)^{N^2/2}}d\mathbf{X}, \tag{1}$$

($d\mathbf{X}$ being the standard Lebesgue measure on $\mathcal{H}_N$) is a U_N-invariant measure. The techniques described in [2] provide a method of computing integrals such as

$$\langle F(\mathbf{X})\rangle := \frac{\int_{\mathcal{H}_N} F(\mathbf{X})\, d\mu_\Lambda}{\int_{\mathcal{H}_N} d\mu_\Lambda}, \tag{2}$$

where $F(\mathbf{X})$ is a U_N-equivariant function on $\mathcal{H}_N$. In fact, one can give an asymptotic expansion of (2) through a collection of graphs known as ‘ribbon graphs’. A theorem due to

*This paper is part of the author’s *Tesi di Perfezionamento* at Scuola Normale Superiore, Pisa, July 2000.

Di Francesco et al. [3] (hereafter referred to as the DFIZ Theorem) relates the asymptotic expansions of such integrals for different choices of $F(\mathbf{X})$.

More precisely, let

$$Z^{(N)}(\Lambda) := \frac{\int_{\mathcal{H}_N} \exp\left(\frac{\sqrt{-1}}{6} Tr(\mathbf{X}^3)\right) d\mu_\Lambda}{\int_{\mathcal{H}_N} d\mu_\Lambda}$$

be a function of the matrix Λ. As shown in [3], $Z^{(N)}(\Lambda)$ admits, as $N \to \infty$, an asymptotic expansion $Z(\theta_1, \dots, \theta_{2k+1}, \dots)$, where

$$\theta_{2k+1} = -\frac{2}{2k+1} Tr(\Lambda^{-2k-1})$$

are independent variables for large N. Similarly, for any non-constant polynomial

$$P(\mathbf{X}) = P(Tr(\mathbf{X}), \dots, Tr(\mathbf{X}^{2k+1}), \dots) \tag{3}$$

in the odd traces of $\mathbf{X}$, the integral

$$\langle\langle P(\mathbf{X}) \rangle\rangle = \frac{\int_{\mathcal{H}_N} P(\mathbf{X}) \exp((i/6) Tr(\mathbf{X}^3))\, d\mu_\Lambda}{\int_{\mathcal{H}_N} d\mu_\Lambda} \tag{4}$$

admits an asymptotic expansion, as $N \to \infty$, $\langle\langle P \rangle\rangle (\theta_1, \dots, \theta_{2k+1}, \dots)$. By the DFIZ Theorem this asymptotic expansion can be uniquely recovered from $Z(\theta_1, \dots, \theta_{2k+1}, \dots)$ by applying a differential operator R in the derivatives $\{\partial_{\theta_1}, \dots, \partial_{\theta_{2k+1}}, \dots\}$. In this paper we give a combinatorial algorithm to describe explicitly the terms appearing in the polynomial differential operator R. Although rather technical, this algorithm provides general formulae for the geometric interpretation of the DFIZ Theorem in the theory of the moduli space of stable pointed curves [1].

1.1. Geometric background

In this subsection we recall some basic facts on the geometry of the moduli space of stable pointed curves in order to clarify the geometric interpretation of the DFIZ Theorem.

For any pair of non-negative integers g, n, $n > 2 - 2g$, we denote by $\bar{\mathcal{M}}_{g,n}$ the moduli space of stable curves: its points are in one-to-one correspondence with isomorphism classes of n-pointed genus g stable curves $[C; p_1, \dots, p_n]$, i.e. curves with simple nodes and finitely many automorphisms. The complete description of the cohomology ring $H^*(\bar{\mathcal{M}}_{g,n}; \mathbb{Q})$ is still an open problem. For instance, the only classes known in even degrees are the so called 'tautological classes' (cf. [4]): among them there are the first Chern classes ψ_i, $1 \le i \le n$, of the vector bundles $\mathcal{L}_i$ whose fiber over $[C; p_1, \dots, p_n]$ is the cotangent space to C at the smooth point p_i. Alternative methods to construct new elements in $H^{2*}(\bar{\mathcal{M}}_{g,n}; \mathbb{Q})$ were proposed by M. Kontsevich in [6]; he also conjectured that these new elements might be expressed in terms of tautological classes. As suggested, and explained in details by Arbarello and Cornalba in [1], the DFIZ Theorem may be a basic tool for the proof of Kontsevich's conjecture. For this purpose one needs to work out algorithms for an accessible use of the DFIZ Theorem.

1.2. Notation and preliminaries

In this subsection we briefly recall some basic notation (cf. [7]) and some basic facts of algebraic combinatorics which will be used to state results in Section 2. For a non-negative integer n we denote by $\mathcal{P}(n)$ the set of partitions of n; when $n = 0$, the unique partition of length zero will be called the empty partition and denoted by $(\emptyset)$. A partition $\sigma = (\sigma_1, \ldots, \sigma_r, \ldots.)$ is called *strict* (*odd*) when all its parts are distinct (odd integers). In the sequel, we shall denote the sets of strict and odd partitions of weight n by $\mathcal{S}(n)$ and $\mathcal{O}(n)$, respectively. Finally, if we adopt the notation

$$\sigma = \left(1^{m_1} 2^{m_2} \ldots r^{m_r} \ldots\right),$$

then we set

$$z_\sigma = \prod_{i \geq 1} i^{m_i} m_i!$$

and

$$\varepsilon_\sigma = (-1)^{|\sigma| - l(\sigma)}.$$

In particular, notice that, for any odd partition σ,

$$|\sigma| \equiv l(\sigma) \bmod 2,$$

where $l(\sigma)$ is the length of σ.

The combinatorial algorithm in Section 2 will relate two families of symmetric functions, namely the power sums and the Schur Q-functions: for the purpose of what follows, we recall the definition of the latter ones. Let $X = \{x_1, \ldots, x_N\}$ be a set of N indeterminates and σ a partition in $\mathcal{S}(n)$ of length k, $k \leq N$.

Definition 1.1

$$Q_\sigma(X) := 2^k \sum_{\tau \in \mathfrak{S}_N / ((\mathfrak{S}_1)^k \times \mathfrak{S}_{N-k})} \tau\left(x_1^{\lambda_1} \ldots x_k^{\lambda_k}\right) \prod_{\substack{1 \leq i < j \leq N \\ i \leq k}} \frac{x_i + x_j}{x_i - x_j}. \tag{5}$$

Next, for any positive integer n, consider the two sets

$$\begin{aligned} \mathcal{B}_1 &:= \{Q_\sigma : \sigma \in \mathcal{S}(n)\}, \\ \mathcal{B}_2 &:= \{p_\pi : \pi \in \mathcal{O}(n)\}. \end{aligned} \tag{6}$$

The relation between $\mathcal{B}_1$ and $\mathcal{B}_2$ can be given in terms of projective representations of the symmetric group $\mathfrak{S}_n$.

Theorem 1.1 (*Schur*) *Let σ be a partition in $\mathcal{S}(n)$ and define*

$$c(\sigma, \pi) = 2^{\frac{l(\sigma)+l(\pi)+\epsilon(\sigma)}{2}}.$$

Then

$$Q_\sigma = \sum_{\pi \in \mathcal{O}(n)} \frac{c(\sigma, \pi)\langle\sigma\rangle(\pi)}{z_\pi} p_\pi, \tag{7}$$

where $\langle\sigma\rangle(\pi)$ denotes the value of the projective character of $\mathfrak{S}_n$ indexed by σ on the conjugacy class of type π.

If we fix the reverse lexicographic ordering on the set

$$\bigcup_n \mathcal{P}(n),$$

we can consider the column vectors $\mathbf{Q} = (Q_\sigma)$, for each $\sigma \in \mathcal{S}(n)$, $n \geq 1$, and $\mathbf{P} = (p_\pi)$, $\pi \in \mathcal{O}(n)$, $n \geq 1$. If $\mathbf{B} = (b_{\sigma\pi})$ is the diagonal block matrix with each block of size $|\mathcal{P}(n)|$, for $n \geq 1$, and the entries are given by

$$b_{\sigma\pi} = \frac{c(\sigma, \pi)\langle\sigma\rangle(\pi)}{z_\pi},$$

then Proposition 1.1 can be expressed in terms of linear algebra as

$$\mathbf{Q} = \mathbf{BP}. \tag{8}$$

Note that $\mathbf{B}$ is a non-singular matrix, since the entries in each block of size $|\mathcal{P}(n)|$ are multiples of the projective characters of $\mathfrak{S}_n$. In the sequel, we shall denote the entries of $\mathbf{B}^{-1}$ by $b^{\sigma\pi}$.

We finally recall that Schur Q-functions can be extended to any sequence $J = (j_1, \ldots, j_k)$ of integers. To this end, we introduce some additional notation. If, for some $r > 0$, the subsequence of those j_p for which $|j_p| = r$ does not have one of the forms $(r, -r, \ldots, -r, r)$ or $(-r, r, \ldots, -r, r)$, then

$$Q_J = 0.$$

Alternatively, there is a permutation w which rearranges J into a sequence of the form

$$L, (-r_1, r_1), \ldots, (-r_s, r_s), 0, \ldots,$$

where L is a strict partition and $r_p > 0$.

Definition 1.2

$$Q_J := sgn(w)(-1)^{r_1+\cdots+r_s} 2^s Q_L,$$

where Q_L is the symmetric function defined in (5).

2. The combinatorial algorithm and its geometric interpretation

In this section we first introduce a collection of graphs to describe general formulae related to the proof of the DFIZ Theorem. Secondly, following [1], we show how to interpret these results in the framework of moduli spaces of pointed curves.

Given a partition σ with distinct parts, let $k_j = |\sigma| - 3j$, $0 \leq j \leq t$, where $t = \lfloor |\sigma|/3 \rfloor$.

Definition 2.1 Let λ be a partition of weight k_j, $0 \leq j < t$. Each sequence $\underline{s}_j(\lambda) = (s_1, \ldots, s_{l(\lambda)})$ of non-negative integers which satisfies one of the equations

$$k_j - 3\big(s_1 + \cdots + s_{l(\lambda)}\big) = k_a, \quad j \leq a \leq t,$$

will be referred to as a λ-reductive sequence.

If we now consider the generalized Schur Q-functions

$$\varphi_{(\lambda_1 - 3s_1, \ldots, \lambda_{l(\lambda)} - 3s_{l(\lambda)})},$$

by Definition 1.2 the non-zero φ's stem from those sequences of integers

$$\big(\lambda_1 - 3s_1, \ldots, \lambda_{l(\lambda)} - 3s_{l(\lambda)}\big)$$

whose coordinates can be permuted by a bijection w into the form

$$\lambda_1, (-r_1, r_1), \ldots (-r_s, r_s), 0, 0, 0,$$

with λ_1 strict partition and s possibly zero. We say that λ_1 is a λ-reduced partition or a λ-reduction.

Moreover, we give the following

Definition 2.2 Let λ be a partition of weight k_j, $1 \leq j < t$, and $\underline{s}_j(\lambda)$ a λ-reductive sequence yielding a λ-reduction λ_1 of weight k_a. Then we set

$$L[\underline{s}_j(\lambda)] = -2^s \mathrm{sgn}(w)^{r_1+\cdots+r_s} \prod_{m=1}^{l(\lambda)} (-1)^{s_m} c_{s_m,\lambda_m},$$

where

$$c_{s_m,\lambda_m} = \sum_{l=0}^{2s_m} \frac{1}{2^l} \binom{\lambda_m - 3s_m + l - 1}{l} \frac{(6s_m - 2l - 1)!!}{(2s_m - l)! 6^{2s_m - l}}.$$

For any integer j in the interval $[0, t)$, we now associate with σ a labelled graph $T_j(\sigma)$. For this purpose we introduce the following collection of graphs.

Definition 2.3 Let λ be a partition of weight k_j, $0 \leq j < t$. A λ-graph G_λ is a labelled graph with vertices in bijection with the set of λ-reduced partitions of weight k_a, $j \leq a \leq t$. Each vertex is joined only to the vertex corresponding to (λ) (the root of G_λ), possibly with more than one edge. Moreover, every edge joining vertices corresponding to λ-reductions of weight k_t with the root are labelled with $L[\underline{s}_j(\lambda)]$, where $\underline{s}_j(\lambda)$ is the corresponding λ-reductive sequence.

In the sequel, we shall denote by $V(j, \sigma)$ the set of vertices of $T_j(\sigma)$. The graph $T_j(\sigma)$ is constructed in several steps.

STEP 1 First draw G_σ as in Definition 2.3: see figure 1 for an example.

STEP 2 For each π-reduction γ of weight k_a, $1 \leq a < t$, draw G_γ. Keeping in mind that γ is a π-reduction corresponding to a π-reductive sequence $\underline{s}_a(\pi)$, we replace the labels of G_γ by $L[\underline{s}_a(\pi)]L[\underline{s}_t(\gamma)]$, where $\underline{s}_t(\gamma)$ is a γ-reductive sequence giving rise to a partition of weight k_t: see figure 2.

STEP 3 Do the same as in *STEP 2* for each γ-reduction ν of weight k_m, $n + 1 \geq m \geq t$. In this case, the labels of the edges for the graphs we obtain are given by

$$L[\underline{s}_a(\pi)]L[\underline{s}_m(\gamma)]L[\underline{s}_t(\nu)],$$

where ν is a γ-reduction of weight k_t.

STEP h At this step, the edges of each graph will be labelled with the products of h factors for each subsequent reduction: see figure 3.

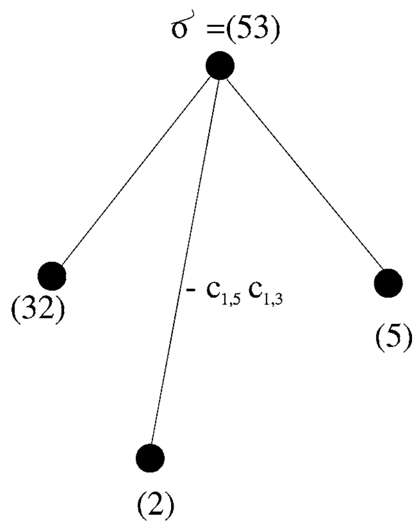

Figure 1. The graph G_σ with $k_j = 8$.

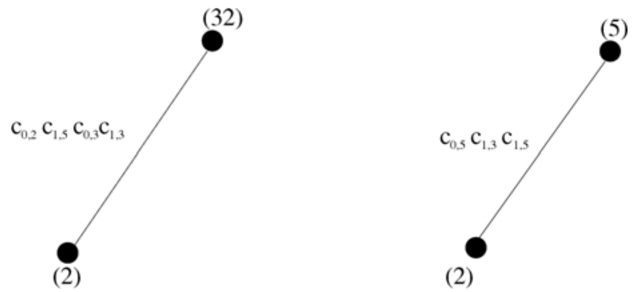

Figure 2. The graphs $G_{(32)}$ and $G_{(5)}$.

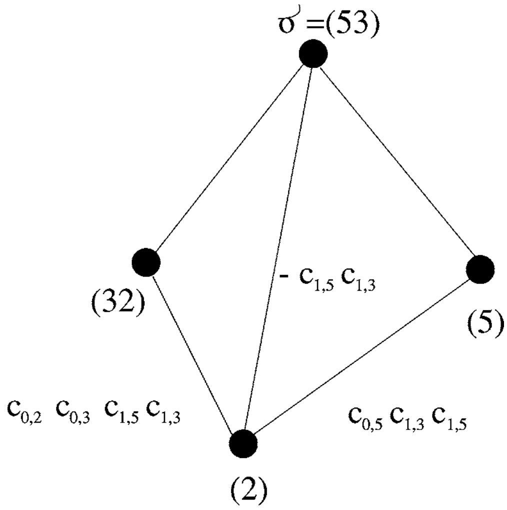

Figure 3. The graph $T_0(\sigma)$ with $\sigma = (53)$.

Since at each step we have partitions with decreasing weight bounded by k_t, the process described above leads to the construction of the graph $T_j(\sigma)$: we paste together all the graphs obtained at each step by superimposing those vertices corresponding to common partitions.

Let v be a vertex of $T_j(\sigma)$ which corresponds to a partition ρ_v: we denote by $C_{\rho_v}(\sigma)$ the product of all the labels assigned to each edge starting at v.

Then we have

Theorem 2.1 *Let* $\nu = (1^{\nu_0} \dots (2k+1)^{\nu_k} \dots)$ *be an odd partition. The asymptotic expansion, as* $N \to \infty$, *of the integral*

$$\frac{\int_{\mathcal{H}_N} (Tr(\mathbf{X})^{\nu_0} \cdot \dots \cdot Tr(\mathbf{X}^{2k+1})^{\nu_k} \cdot \dots) \exp((\sqrt{-1}/6) Tr(\mathbf{X}^3))\, d\mu_\Lambda}{\int_{\mathcal{H}_N} d\mu_\Lambda}$$

is equal to

$$
\begin{aligned}
&(2\sqrt{-1})^{|\nu|} \sum_{\sigma\in\mathcal{S}(n)} \sum_{j=0}^{t} \sum_{v\in V(j,\sigma)} \sum_{\pi\in\mathcal{D}(|\rho_v|)} (-1)^{|\sigma|+|\rho_v|} b^{\nu\sigma} C_{\rho_v}(\sigma)\, b_{\rho_v\pi} \cdot \\
&\quad\cdot \left(\prod_{r=0} \frac{-2}{\pi_r!(2r+1)} \frac{\partial}{\partial\theta_{2r+1}}\right)^{\pi_{2r+1}} Z(\theta_1, \ldots, \theta_{2r+1}, \ldots),
\end{aligned}
\tag{9}
$$

where t is the integral part of $\frac{|\nu|}{3}$ and $b^{\nu\sigma}$, $b_{\rho_v\pi}$ are the entries of the matrices $\mathbf{B}^{-1}$ and $\mathbf{B}$ in (8).

Proof: By the arguments used in [3] to prove 'Proposition (K)', the polynomial $P_\nu(\mathbf{X})$ is replaced by a linear combination of symmetric polynomials $f_\sigma(\mathbf{X})$, where σ is a strict partition of weight $|\nu|$. As observed by Jósefiak in [5], the functions $f_\sigma(\mathbf{X})$ are multiples of the Schur generalized functions $Q_\sigma(\mathbf{X})$ as defined in (5), i.e.

$$f_\sigma(\mathbf{X}) = (-1)^{|\sigma|} Q_\sigma(\mathbf{X}).$$

On the other hand, the polynomial $P_\nu(\mathbf{X})$ is the Newton symmetric function p_ν in the eigenvalues of $\mathbf{X}$. Therefore, Theorem 1.1 allows us to get a systematic way to pass from $p_\nu(\mathbf{X})$ to $f_\sigma(\mathbf{X})$ by means of projective characters of the symmetric group of degree $|\nu|$, namely

$$P_\nu(\mathbf{X}) = \sum_{\sigma\in\mathcal{S}(|\nu|)} (-1)^{|\sigma|} b^{\nu\sigma} f_\sigma(\mathbf{X}).$$

The properties of the graphs introduced in Definition 2.3 allow us to deduce Formula (9) in order to solve the recursive relations given in 'Proposition (W′)' in [3]. $\square$

Remark 2.1 Notice that the coefficient $(2\sqrt{-1})^{|\nu|}$ depends on 'Proposition (W′)' in [3]. Since $\mathcal{D}(\rho_v) = \emptyset$ for $\rho_v = (\emptyset)$, the differential operator

$$\left(\prod_{r=0} \frac{\partial}{\partial\theta_{2r+1}}\right)^{\pi_{2r+1}}$$

is set equal to 1.

Let us now translate the statement of Theorem 2.1 in geometric terms. Fix an integer n and a sequence $m_* = (0, m_1, \ldots)$ of non-negative integers almost all of which are zero. Following [6, 9], for non-negative integers $d_1, \ldots, d_n$ such that

$$\sum_{i=1}^{n} d_i = 3g - 3 + n - \sum_{i\geq 1}(i-1)m_i, \tag{10}$$

consider the integral

$$\langle\tau\rangle_{m_*} = \int_{W_{m_*,n}} \prod_{j=1}^{n} \psi_i^{d_j}, \tag{11}$$

where $W_{m_*,n}$ is the homology cycle of the moduli space of curves introduced by Arbarello and Cornalba in [1]. From a geometric point of view, the integral in (11) provides information about the intersection of the cohomology classes ψ_j along the cycle $W_{m_*,n}$.

Next, consider the formal power series

$$F(t_*, s_*) = \sum_{n_*,m_*} \langle\tau\rangle_{\underline{d},m_*} \frac{t_*^{n_*}}{n_*!} s_*^{m_*},$$

$$Z(t_*, s_*) = \exp(F(t_*, s_*)),$$

where

$$t_* = (t_0, t_1, t_2, \ldots), \qquad s_* = (s_0, s_1, s_2, \ldots),$$

are infinite sequences of indeterminates, and

$$m_* = (m_0, m_1, m_2, \ldots), \qquad n_* = (n_0, n_1, n_2, \ldots)$$

are infinite sequences of non-negative integers, almost all of which are zero. We have also set

$$n_*! = \prod_{j=0}^{\infty} n_j!, \qquad t_*^{n_*} = \prod_{j=0}^{\infty} t_j^{n_j},$$

and similarly for $s_*^{m_*}$. Notice that when $s_* = \hat{s}_* = (0, 1, 0, 0, \ldots)$ the terms which do not vanish in the generating function $F(t_*, s_*)_{s_*=\hat{s}_*}$ correspond to sequences of type $m_* = (0, m_1, 0, 0, \ldots)$. Thus the non-vanishing coefficients in this generating function arise from those integrals for which

$$\sum_{j=1}^{n} d_j = 3g - 3 + n.$$

The evaluation of $F(t_*, s_*)$ at $s_* = \hat{s}_* = (0, 1, 0, 0, \ldots)$ is denoted by

$$F(t_*) := F(t_*, s_*)_{s_*=\hat{s}_*} = \sum_{n_*} \left\langle \tau_0^{n_0} \tau_1^{n_1} \ldots \right\rangle \frac{t^{n_*}}{n_*!}. \tag{12}$$

Kontsevich proves in [6] that $Z(t_*, s_*)$ is an asymptotic expansion, as $N \to \infty$, of the Gaussian integral

$$\int_{\mathcal{H}_N} \exp\left(-i \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} s_j \frac{Tr(\mathbf{X}^{2j+1})}{2j+1}\right) d\mu_\Lambda. \tag{13}$$

Here $d\mu_\Lambda$ is the measure defined in (1) and the positive definite diagonal $N \times N$ matrix Λ is linked to the t variables by the substitution

$$t_k = -(2k-1)!!Tr(\Lambda^{-2k-1}).$$

For the sake of simplicity, in the sequel we set

$$\langle f \rangle_\Lambda = \int_{\mathcal{H}_N} f d\mu_\Lambda,$$
$$\langle\langle f \rangle\rangle_\Lambda = \int_{\mathcal{H}_N} f \exp\left(i \frac{Tr\mathbf{X}^3}{6}\right) d\mu_\Lambda.$$

If we apply the differential operator

$$\prod_{j \geq 2} \frac{1}{m_j!} \left(\frac{\partial}{\partial s_j}\right)^{m_j}$$

to the integral in (13) and evaluate it at $s_* = \hat{s}_*$, we have

$$\left(\prod_j \left(\frac{\partial}{\partial s_j}\right)^{m_j} Z(t_*, s_*)\right)\Bigg|_{s_*=\hat{s}_*} = \left\langle\!\!\left\langle \left(\prod_j \left(-i\left(-\frac{1}{2}\right)^j \frac{Tr\mathbf{X}^{2j+1}}{2j+1}\right)^{m_j}\right)\right\rangle\!\!\right\rangle. \tag{14}$$

Let us denote by μ the partition $(3^{m_1} \ldots (2k+1)^{m_k} \ldots)$. Then, the statement of Theorem 2.1 can be adapted to the generating function $Z(t_*, s_*)$.

Thus, we can deduce information on the intersection theory of the moduli space $\bar{\mathcal{M}}_{g,n}$. More precisely, the following holds.

Corollary 2.2

$$\left(\prod_{j \geq 2} \frac{1}{m_j!} \left(\frac{\partial}{\partial s_j}\right)^{m_j} Z(t_*, s_*)\right)\Bigg|_{s_*=\hat{s}_*} = \left[\prod_{j \geq 2} (-2^j)^{m_j}\right].$$
$$\cdot \sum_{\sigma \in \mathcal{S}(|\mu|)} \sum_{h=0}^{t} \sum_{v \in V(h,\sigma)} \sum_{\pi \in \mathcal{D}(|\rho_v|)} b^{v\sigma} C_{\rho_v}(\sigma) b_{\rho_v \pi}.$$
$$\cdot \left(\prod_{r=0} \left(-\frac{(2r-1)!!}{\pi_r!}\right)^{\pi_r} \frac{\partial}{\partial t_r}\right)^{\pi_r} Z(t_*),$$

Proof: It follows directly from Theorem 2.1 after setting

$$t_k = \frac{(2k-1)!!}{2}\theta_{2k+1}. \qquad \square$$

Acknowledgments

I would like to thank Enrico Arbarello for suggestive and stimulating conversations. My acknowledgments also go to Marialuisa J. de Resmini for her remarks and careful reading of the manuscript.

References

1. E. Arbarello and M. Cornalba, "Combinatorial and algebro geometric cohomology classes on the moduli space of curves," *J. Alg. Geom.* **5** (1996), 705–749.
2. D. Bessis, C. Itzykson, and J.B. Zuber, "Quantum field theory techniques in graphical enumeration," *Adv. Appl. Math.* **1** (1980), 109–157.
3. P. Di Francesco, C. Itzykson, and J.-B. Zuber, "Polynomial averages in the Kontsevich model," *Commun. Math. Phys.* **151** (1993), 193–219.
4. C. Faber, "A conjectural description of the tautological ring of the moduli space of curves," in *Moduli of Curves and Abelian Varieties*, *The Dutch Intercity Seminar on Moduli*, C. Faber and E. Looijenga (Eds.), Aspects of Maths. E 33, Vieweg, 1999.
5. T. Jósefiak, "Symmetric functions in the Kontsevich-Witten intersection theory of the moduli space of curves," *Letters in Mathematical Physics* **33** (1995), 347–351.
6. M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function," *Commun. Math. Phys.* **147** (1992), 1–23.
7. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.
8. I. Schur, "Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionem," *J. Reine Angew. Math* **139** (1911), 155–250.
9. E. Witten, "Two dimensional gravity and intersection theory on moduli space," *Survey in Diff. Geom.* **1** (1991), 243–310.