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More on Geometries of the Fischer Group Fi_{22}

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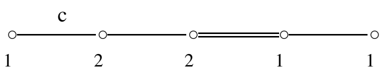
Abstract. We give a new, purely combinatorial characterization of geometries $\mathcal{E}$ with diagram

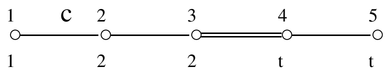
$c.F_4(1)$:


identifying each under some "natural" conditions—but not assuming any group action a priori—with one of the two geometries $\mathcal{E}(Fi_{22})$ and $\mathcal{E}(3 \cdot Fi_{22})$ related to the Fischer 3-transposition group Fi_{22} and its non-split central extension $3 \cdot Fi_{22}$, respectively. As a by-product we improve the known characterization of the c-extended dual polar spaces for Fi_{22} and $3 \cdot Fi_{22}$ and of the truncation of the c-extended 6-dimensional unitary polar space.

Keywords: Fischer group, diagram geometry, extended building

Introduction

In this article we carry on the classification project started in [5] of geometries $\mathcal{E}$ with diagram

$c.F_4(t)$:


where $t = 1, 2,$ or 4,

(types are indicated above the nodes). We do not assume that $\mathcal{E}$ is necessarily flag-transitive but instead that it satisfies the Interstate Property and the following two conditions.

(I) (a) Any two elements of type 1 in $\mathcal{E}$ are incident to at most one common element of type 2.

(b) Any three elements of type 1 in $\mathcal{E}$ are pairwise incident to common elements of type 2 if and only if all three of them are incident to a common element of type 5.

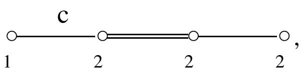
It was shown in [5] that for $t = 4$ there exists a unique such geometry, which is flag-transitive with automorphism group isomorphic to the Baby Monster sporadic simple group F_2. Here we deal with the case $t = 1$. There are two examples $\mathcal{E}(Fi_{22})$ and $\mathcal{E}(3 \cdot Fi_{22})$ of such

*The research was carried out while the second author was working at Imperial College under an EPSRC grant.

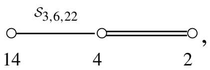
geometries admitting flag-transitive actions of the Fischer 3-transposition group Fi_{22} and its non-split central extension $3 \cdot Fi_{22}$, respectively. Both examples possess the property that their collinearity graph is locally isomorphic to the commuting graph of central involutions in the group $\Omega_8^+(2)$ and one of our main results is

Theorem 1 *Let $\mathcal{E}$ be a flag-transitive $c.F_4(1)$-geometry such that the collinearity graph of $\mathcal{E}$ is locally the commuting graph of central involutions in $\Omega_8^+(2)$. Then $\mathcal{E}$ is isomorphic either to $\mathcal{E}(Fi_{22})$ or to $\mathcal{E}(3 \cdot Fi_{22})$.*

In order to prove Theorem 1, we establish and study the relationships of $\mathcal{E}(Fi_{22})$ and $\mathcal{E}(3 \cdot Fi_{22})$ with other geometries of Fi_{22} and $3 \cdot Fi_{22}$. Precisely, we construct geometries $\mathcal{G}$ and $\mathcal{B}$ with diagrams



and



respectively, and try to characterize those instead of $\mathcal{E}$. In Theorem 2 (cf. Section 7) we prove that if $\mathcal{G}$ satisfies certain properties, which hold if it comes from a geometry like $\mathcal{E}$ satisfying (I), and if $\mathcal{B}$ satisfies the Intersection Property (IP) then $\mathcal{G}$ belongs to the group Fi_{22}. From this we derive in Theorem 3 (cf. again Section 7) a group-free version of Theorem 1.

We recall that (IP) is the following property (where X is a geometry containing a set of elements $P(X)$ called "points" and, for any object $y \in X$, $P(y)$ denotes the set of points incident to y).

(IP) For any $y, z \in X$, $P(y) \cap P(z) = P(u)$ for some $u \in X$ and $P(y) = P(z)$ if and only if $y = z$.

To understand the rest of the paper, it will be useful to have some knowledge about the different geometries for Fi_{22} and $3 \cdot Fi_{22}$ and about the relationships between them. This information will be provided in Sections 2 and 3. In Section 1 we review some general results on $c.F_4(1)$-geometries. The remaining Sections 4 to 7 contain, respectively, the construction from $\mathcal{E}$ to $\mathcal{G}$, the characterizations of $\mathcal{G}$ and $\mathcal{B}$, and the proofs of Theorems 1, 2, 3.

We emphasize again that all our proofs and constructions will be purely combinatorial and that we do not assume any group-action. However, for some lemmas we have much easier and shorter proofs in the case of flag-transitivity and for the interested reader we supply them in the appendix.

1. Some general results on $c.F_4(1)$-geometries

In this section, we review some general results on $c.F_4(1)$-geometries satisfying (I).

In what follows $\mathcal{E}$ denotes a $c.F_4(1)$-geometry satisfying (I) and $\mathcal{E}^i$ denotes the set of elements of type i in $\mathcal{E}$.

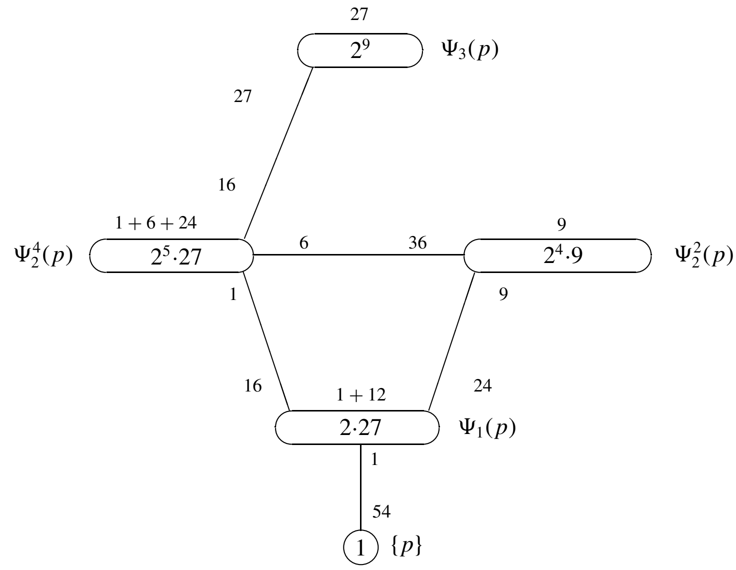
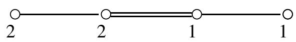


Figure 1. The suborbit diagram of Ψ related to $\Omega_8^+(2) : \Sigma_3$.

Let $\mathcal{F} = \mathcal{F}_4(1)$ be the building with diagram

$F_4(1)$: 

and flag-transitive automorphism group $F \cong \Omega_8^+(2) : \Sigma_3$. The elements from left to right on the diagram of $\mathcal{F}_4(1)$ will be called points, lines, planes, and symplecta, respectively. By [15, 10.14] $\mathcal{F}$ can be defined as follows.

Let $\mathcal{D} = \mathcal{D}_4(2)$ be the D_4-building with automorphism group $F^\infty \cong \Omega_8^+(2)$. Let the types of objects of $\mathcal{D}$ be labelled by the integers $1, 2, 3, 4$ where 2 corresponds to the central node in the Dynkin diagram. Then the points of $\mathcal{F}$ are the objects of type 2 in $\mathcal{D}$, the lines are the flags of type $\{1, 3, 4\}$ in $\mathcal{D}$, the planes are the flags of types $\{1, 3\}$, $\{1, 4\}$, and $\{3, 4\}$, and the symplecta are the objects of $\mathcal{D}$ whose type is unequal to 2. A point is incident to another element of $\mathcal{F}$ if their union is a flag in $\mathcal{D}$ and incidence between lines, planes, and symplecta is defined by inclusion.

Let Ψ be the collinearity graph of $\mathcal{F}_4(1)$ (i.e., the graph on the set of points in which two of them are adjacent if they are incident to a common line). Then the vertices of Ψ (the points of $\mathcal{F}_4(1)$) can be identified with the central involutions in F in such a way that two involutions p, q are adjacent if and only if $p \in O_2(C_F(q))$ (equivalently $q \in O_2(C_F(p))$). The suborbit diagram of Ψ with respect to the action of F is given in figure 1 (cf. [5, figure 2]).

If $q \in \Psi\backslash\{p\}$ then the order of the product pq is 2, 2, 4, and 3 for $q \in \Psi_1(p)$, $\Psi_2^2(p)$, $\Psi_2^4(p)$, and $\Psi_3(p)$, respectively. In particular, p commutes with $q \in \Psi\backslash\{p\}$ if and only if $q \in \Psi_1(p) \cup \Psi_2^2(p)$.

Let Δ denote the graph on the vertex set of Ψ in which two vertices p and q are adjacent if $q \in \Psi_1(p) \cup \Psi_2^2(p)$. In other terms, Δ is the commuting graph of the central involutions in F. The following result was established in [5, Lemmas 3.1 and 3.3]. (Recall that a graph X is said to be *locally* Y if for any vertex $x \in X$ the subgraph induced on the neighbourhood $X(x)$ of x in X is isomorphic to the graph Y.)

Lemma 1.1 *Let $\Gamma = \Gamma(\mathcal{E})$ be the graph on the set of elements of type* 1 *in $\mathcal{E}$ in which two of them are adjacent if they are incident to a common element of type* 2*. Then Γ is locally Δ and every graph which is locally Δ is $\Gamma(\mathcal{E})$ for some $c.F_4(1)$-geometry $\mathcal{E}$ satisfying (I).*

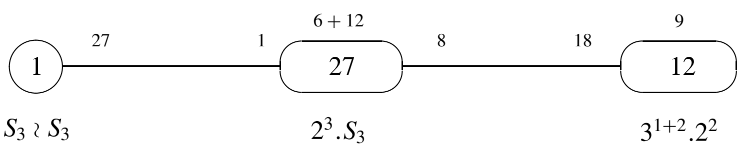
Thus studying the geometries $\mathcal{E}$ is equivalent to studying the graphs Γ which are locally Δ.

In what follows Γ stands for $\Gamma(\mathcal{E})$. The elements of type i in $\mathcal{E}$ can be identified with certain complete subgraphs in Γ on 1, 2, 4, 8, and 36 vertices for $i = 1, 2, 3, 4$, and 5, respectively, so that the incidence relation is via inclusion. If $x \in \Gamma$ then by (1.1) we can fix a bijection i_x from $\Gamma(x)$ onto the vertex set of Δ which induces an isomorphism from the subgraph of Γ induced on $\Gamma(x)$ onto Δ.

The graph Γ contains an important family of subgraphs which can be described as follows. Let $\tilde{\Xi}$ be the graph on the set of elements of type 2 in $\mathcal{E}$ (equivalently the graph on the set of edges of Γ) where two such elements are adjacent if they are incident to a common element of type 3 but not to a common element of type 1 (i.e., their (disjoint) union is a clique of size four in Γ). For $e \in \mathcal{E}^2$ let $\tilde{\Xi}^e$ be the connected component of $\tilde{\Xi}$ containing e and let Ξ^e be the subgraph in Γ induced on the set of vertices incident to those edges of Γ which are the vertices of $\tilde{\Xi}^e$. Then by Proposition 5.2 and Lemma 5.3 in [5] we have the following (where a graph X is called a 2-*clique extension* of a graph Y if there exists a mapping ψ from the vertex set of X onto the vertex set of Y such that $|\psi^{-1}(y)| = 2$ for every $y \in Y$ and two distinct vertices x_1, x_2 in X are adjacent if and only if their images $\psi(x_1)$ and $\psi(x_2)$ are either equal or adjacent in Y).

Lemma 1.2

(i) *$\tilde{\Xi}^e$ is the complement of the collinearity graph of the generalized quadrangle of order* (3, 3) *associated with the group $U_4(2).2 \cong Sp_4(3).2$ and has the suborbit diagram*



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(ii) *Ξ^e is the* 2*-clique extension of $\tilde{\Xi}^e$.*

The following lemma will be needed in Section 4.

Lemma 1.3 *Let $K := \tilde{\Xi}^e$ be a connected component of $\tilde{\Xi}$ and let $C \subseteq K$ be a clique of size* 8*. Then any vertex in $K \backslash C$ is incident to at least four vertices in C.*

Proof: By (1.2) we can identify K with the graph on the 40 points of the generalized $O_5(3)$-quadrangle $GQ(4, 3)$ in which two points are adjacent if they are not incident to a common line. So any line of $GQ(4, 3)$ can contain at most one point from C. Hence, as any point from $GQ(4, 3)$ is incident to exactly 4 lines, it can be collinear to at most 4 points from C which implies that it must be adjacent in K to at least $8 - 4 = 4$ vertices of C. □

The next result (Lemma 7.2 in [5]) describes possible intersections of the subgraphs Ξ^e.

Lemma 1.4 *Let* $e = \{x, y\}$, $f = \{x, z\}$ *be distinct elements of type* 2 *in* $\mathcal{E}$ *and set* $\Pi := \Xi^e \cap \Xi^f$. *Then the following assertions hold*:
(i) *if* $i_x(z) \in \Psi_1(i_x(y))$ *then* $|\Pi| = 16$ *and* $z \in \Xi^e$;
(ii) *if* $i_x(z) \in \Psi_2^2(i_x(y))$ *then* $|\Pi| = 10$ *and* $|\Gamma(z) \cap \Xi^e| = 20$;
(iii) *if* $i_x(z) \in \Psi_2^4(i_x(y))$ *then* $|\Pi| = 2$ *and* $|\Gamma(z) \cap \Xi^e| = 7$;
(iv) *if* $i_x(z) \in \Psi_3(i_x(y))$ *then* $|\Pi| = 1$ *and* $\Gamma(z) \cap \Gamma(x) \cap \Xi^e = \emptyset$.

Recall that a *μ-graph* in a graph X is a subgraph $X(x, y)$ induced on the set of common neighbours of two vertices x and y at distance 2 in X. We will say that a 2-path (x, z, y) in Γ is of D_6- or D_8-type if $i_z(y) \in \Psi_3(i_z(x))$ or $i_z(y) \in \Psi_2^4(i_z(x))$, respectively. In the next two lemmas we summarize the results on μ-graphs established in Section 6 in [5].

Lemma 1.5 *Let* (x, z, y) *be a* 2*-path of* D_8*-type in* Γ. *Then*
(i) *there is a unique edge* $e = \{x, v\}$ *incident to* x *such that* y *is contained in* Ξ^e;
(ii) $\Gamma(x, y) \cap \Xi^e$ *is a connected component of* $\Gamma(x, y)$ *of size* 36.

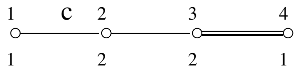
Set $p := i_x(v)$, $\Omega := i_x(\Gamma(x, y) \cap \Xi^e)$, *and let* I *be the stabilizer of* Ω *in* $F := Aut(\Delta)$. *Then*
(iii) I *stabilizes* p, *contains* $O_2(F_p)$, *and* $I \cong 2_+^{1+8}.3^{1+2}.2^2$;
(iv) I *has two orbits* (Ω *and its complement*) *with lengths* 36 *and* 18 *on* $\Psi_1(p)$, *two orbits on* $\Psi_2^4(p)$ *with lengths* 288 *and* 576, *and acts transitively on* $\Psi_2^2(p)$ *and on* $\Psi_3(p)$;
(v) *if* $r \in \Delta\backslash\Psi_3(p)$ *and* $r = i_x(u)$ *for some* $u \in \Gamma$ *then* r *is adjacent in* Δ *to a vertex from* Ω *and hence the distance from* y *to* u *in* Γ *is at most* 2;
(vi) *if* $w \in \Gamma(x, y)\backslash\Xi^e$ *then* (x, w, y) *is of* D_6*-type and* $i_x(w) \in \Psi_3(p)$.

Lemma 1.6 *Let* (x, z, y) *be a* 2*-path of* D_6*-type and let* Π *be the connected component containing* z *of the* μ*-graph* $\Gamma(x, y)$. *Then* Π *is the complete* 4*-partite graph* $K_{4\times 3}$ *on* 12 *vertices. The stabilizer of* $i_x(\Pi)$ *in* F *induces the full automorphism group of* Π *isomorphic to* $\Sigma_3 \wr \Sigma_4$.

1.1. The residue of an element of type 5

In this short subsection we state some facts about the residue of an element of type 5 in $\mathcal{E}$. These facts will be useful in Section 4. The reader can verify them by direct calculations.

If $x \in \mathcal{E}^5$ then $res_{\mathcal{E}}(x)$ is isomorphic to the geometry $\mathcal{H}$ with diagram

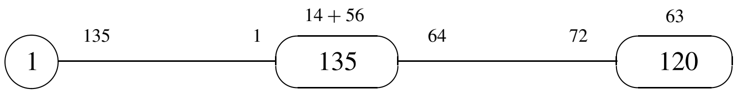


and flag-transitive automorphism group $H := Sp_6(2)$ (cf. [5]). There are several ways how to describe the geometry $\mathcal{H}$ and its relation to the symplectic polar space $\mathcal{P}(Sp_6(2))$ of H but the one which is most suitable for our purpose is probably the one in the context of affinization.

Let $O := O_8^+(2)$ and let V be an 8-dimensional $GF(2)$-vector space equipped with a non-degenerate quadratic form q of plus-type which is preserved by O. Let Ω be the graph on V with edges

$$E(\Omega) := \{\{u, v\} \mid u, v \in V, q(u + v) = 0\}.$$

Then the intersection array of Ω is



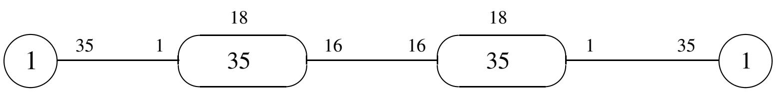
For $v \in V$ and $i \geq 0$, we denote as usual by $\Omega_i(v)$ the set of vertices at distance i from v in Ω. Set $\Omega_i := \Omega_i(0)$ (where 0 is the zero vector in V). Then Ω_1 consists of the isotropic and Ω_2 of the non-isotropic vectors in $V\backslash\{0\}$.

The automorphism group of Ω is a semidirect product $Aut(\Omega) = 2^8.O \cong 2^8.O_8^+(2)$ and $O = Aut(\Omega)_0$ is the stabilizer of 0 in $Aut(\Omega)$. If $v \in \Omega_2$ then

$$Aut(\Omega)_0 \cap Aut(\Omega)_v = O_v \cong Sp_6(2)$$

(cf. [1]). So we can consider H as the stabilizer in $Aut(\Omega)_0$ of a fixed vector $v \in \Omega_2$ and we will do this from now on.

For $i, j \in \{1, 2\}$, set $\Omega_{ij} := \Omega_i \cap \Omega_j(v)$. Then $|\Omega_{11}| = 72$ by the intersection array of Ω and from the properties of orthogonal and symplectic groups one can calculate that the intersection array of Ω_{11} is



In particular, there exists a natural pairing on Ω_{11} in 36 pairs. (These pairs are of the shape $\{u, u + v\}$, $u \in \Omega_{11}$, and the two vectors in a pair are at distance 3 in Ω_{11}.)

For $u \in \Omega_{11}$, let $\bar{u}$ be the pair containing u and let $\bar{\Omega}_{11}$ be the graph whose vertices and edges are the images under $\bar{\ }$ of the vertices and edges of Ω_{11}. Then $\bar{\Omega}_{11}$ is the complete graph on 36 vertices. We take the vertices and edges of $\bar{\Omega}_{11}$ as the objects of $\mathcal{H}$ of type 1 and 2, respectively. The elements of type 4 in $\mathcal{H}$ are all the 8-cliques in $\bar{\Omega}_{11}$ which are images of

8-cliques in Ω_{11} and the elements of type 3 are those 4-cliques which are contained in more than one 8-clique. The incidence relation on $\mathcal{H}$ is defined by inclusion. Then one can show that $\mathcal{H}$ has the desired diagram. Notice that the 8-cliques in Ω_{11} are the maximal cliques in Ω_{11} and that they correspond to maximal totally isotropic subspaces of V.

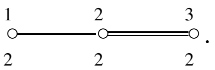
One can show that each edge and each 4- or 8-clique of $\bar{\Omega}_{11}$ which is an object of $\mathcal{H}$ corresponds to a totally isotropic 1-, 2-, or 3-dimensional subspace of V consisting only of vectors in $\{0\} \cup \Omega_{12}$. Furthermore, disjoint edges which are contained in a 4-clique correspond to the same vector in Ω_{12} and disjoint 4-cliques which are contained in a common 8-clique (where all are assumed to be objects of $\mathcal{H}$) determine the same 2-space.

For $i = 2, 3$, define a graph Γ_i on the set of objects of type i in $\mathcal{H}$ in which two such objects are adjacent if they are incident to a common element of type $i + 1$ but not to a common element of type 1. For $x \in \mathcal{H}^i$ denote by Γ_i^x the connected component of Γ containing x. Let $\mathcal{P}$ be the rank 3 geometry whose objects of types 1 and 2 are, respectively, the connected components of Γ_2 and Γ_3 and whose objects of type 3 are the objects of type 4 in $\mathcal{H}$. For $i \in \{2, 3\}$ and $x \in \mathcal{H}^i$, denote by Γ_i^x the connected component of Γ_i containing x. Define $y_j \in \mathcal{P}^j$, $j = 1, 2$, to be incident in $\mathcal{P}$ if there are $x_i \in \mathcal{H}^i$, $i = 2, 3$, such that $y_j = \Gamma_{j+1}^{x_{j+1}}$ and x_2, x_3 are incident in $\mathcal{H}$. Define y_j and $z \in \mathcal{P}^3$ to be incident if $y_j = \Gamma_{j+1}^x$ for some $x \in res_{\mathcal{H}}(z)$.

Then in view of the previous paragraph one can show

Lemma 1.7

(i) *$\mathcal{P} \cong \mathcal{P}(Sp_6(2))$ and $\mathcal{P}$ has the diagram*



(ii) *Each connected component of* Γ_2 *has* 10 *vertices and the connected components of* Γ_3 *are cliques of size* 3.

2. Geometries $\mathcal{E}(Fi_{22})$ and $\mathcal{E}(3 \cdot Fi_{22})$

In this section we describe two $c.F_4(1)$ geometries denoted $\mathcal{E}(Fi_{22})$ and $\mathcal{E}(3 \cdot Fi_{22})$ which satisfy (I). Existence of $\mathcal{E}(Fi_{22})$ was first noticed by D.V. Pasechnik. In [5] $\mathcal{E}(Fi_{22})$ is described in terms of the Baby Monster graph Θ and we start with a review of that description. By G we denote the sporadic simple group Fi_{22} and by $\hat{G} := G:2$ its extension by a nontrivial involutory outer automorphism.

Recall that the vertices of Θ are the $\{3, 4\}$-transpositions in the Baby Monster group F_2. Two vertices are adjacent if their product is a central involution in F_2. For $a \in \Theta$, let $F_2(a) = C_{F_2}(a) \cong 2.^2E_6(2).2$ be the stabilizer of a in F_2 and let $\Theta_i(a)$ be the set of vertices at distance i from a in Θ. Then the diameter of Θ is 3, $b \in \Theta \setminus \{a\}$ commutes with a if and only if $b \in \Theta_1(a) \cup \Theta_3(a)$, $F_2(a)$ acts transitively on $\Theta_1(a)$ and $\Theta_3(a)$ and has two orbits $\Theta_2^3(a)$ and $\Theta_2^4(a)$ on $\Theta_2(a)$ with stabilizers $Fi_{22}2$ and $2^{1+20}.U_4(3).2^2$, respectively (if $b \in \Theta_2^m(a)$ then the product ab is of order m). Let $b \in \Theta_2^3(a)$ and let Γ be the subgraph in Θ induced by $\Theta_3(a) \cap \Theta_3(b)$. Then Γ is a connected component of the subgraph in Θ induced on the set of vertices fixed by the order 3 element ab. Furthermore, Γ has 61 776 vertices, it is locally Δ—the commuting graph of central involutions in $\Omega_8^+(2):\Sigma_3$, and $F_2(a)\cap F_2(b)\cong Fi_{22}.2$ acts transitively on the vertex set of Γ with vertex stabilizer isomorphic to $\Omega_8^+(2):\Sigma_3\times 2$. By (1.1) this implies that $\Gamma=\Gamma(\mathcal{E})$ for a $c.F_4(1)$-geometry $\mathcal{E}=\mathcal{E}(Fi_{22})$ satisfying (I).

Alternatively, Γ can be defined as a graph on the set of $2D$-involutions in $\hat{G}\cong Fi_{22}.2$ in which two such involutions are adjacent if and only if they commute. We recall that the class $2D$ consists of outer involutions.

In [4] the subdegrees of $\hat{G}$ acting on Γ and the corresponding intersection numbers are calculated. Taking Γ as the connected component of the subgraph in the Baby Monster graph induced on the set of vertices fixed by an element of order 3 and in view of (1.5)(iv) one gets the suborbit diagram of Γ given in figure 2 (notice that $\hat{G}(x)$ induces the full automorphism group of the local graph $\Gamma(x)\cong\Delta$ with kernel of order 2).

The Schur multiplier of Fi_{22} is of order 6. Let $\tilde{G}\cong 3\cdot Fi_{22}.2$ be the non-split extension of $Fi_{22}.2$ by a normal subgroup of order 3. Let $\tilde{\Gamma}$ be the graph on the conjugacy class of involutions in $\tilde{G}$ which maps onto vertices of Γ under the natural homomorphism $\varphi:\tilde{G}\to\hat{G}$. Two involutions in $\tilde{\Gamma}$ again are adjacent if and only if they commute. As $C_{\tilde{G}}(O_3(\tilde{G}))=\tilde{G}'$ and the class $2D$ consists of involutions in $\hat{G}\backslash G$, the involutions in $\tilde{\Gamma}$ are not centralized by $O_3(\tilde{G})$. From this it is easy to check that φ induces a covering $\psi:\tilde{\Gamma}\to\Gamma$ such that each triangle from Γ lifts under ψ to a triangle of $\tilde{\Gamma}$. This means that $\tilde{\Gamma}$ is also locally Δ and by (1.1) $\tilde{\Gamma}=\Gamma(\mathcal{E}(3\cdot Fi_{22}))$ for a $c.F_4(1)$-geometry $\mathcal{E}(3\cdot Fi_{22})$ satisfying (I) and possessing a covering onto $\mathcal{E}(Fi_{22})$. Since one knows the orbits of $G(x,y)$ on $\Gamma(x)$ for x,y at distance two in $\tilde{\Gamma}$ from (1.5) and (1.6) one can deduce from the suborbit diagram of Γ that the suborbit diagram of $\tilde{\Gamma}$ with respect to the action of $\tilde{G}$ is the one given in figure 3.

Every flag-transitive automorphism group of $\mathcal{F}_4(1)$ contains $\Omega_8^+(2)$ and the latter acts primitively on the set of points in $\mathcal{F}_4(1)$. From this fact it is easy to deduce that (I)(a) holds for every flag-transitive $c.F_4(1)$-geometry. On the other hand, there exists a large class of

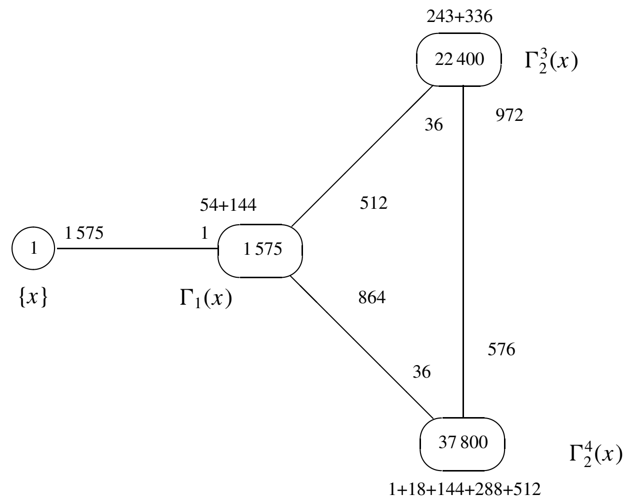


Figure 2. The suborbit diagram of Γrelated to Fi_{22}.

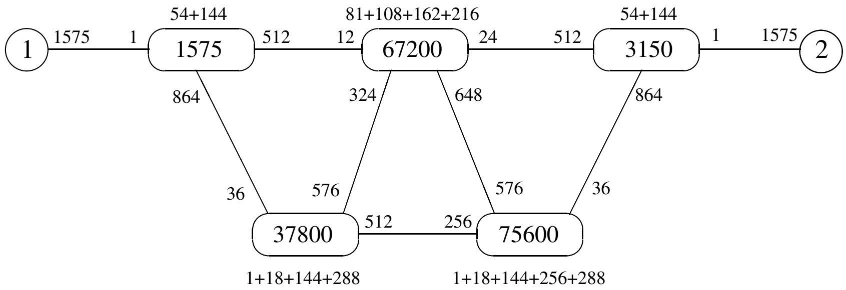


Figure 3. The suborbit diagram of $\tilde{\Gamma}$ related to $3 \cdot Fi_{22}.2$.

$c.F_4(1)$-geometries which do not satisfy (b). This class includes flag-transitive as well as not flag-transitive examples. The easiest such example can be constructed as follows.

Consider the action of $G = Fi_{22}$ on the set $\hat{\Gamma}$ of cosets of a subgroup $\Omega_8^+(2):3$ (which is an index 2 subgroup in the vertex stabilizer of the action of G on Γ). Then it is easy to check that the action has two symmetric subdegrees of length 1575. Moreover, if $\hat{\Gamma}^1$ and $\hat{\Gamma}^2$ are the corresponding orbital graphs then (up to renumbering) an edge of $\hat{\Gamma}^i$ is contained in 54 and 144 triangles for $i = 1$ and 2, respectively. Then $\hat{\Gamma}^1$ is the collinearity graph of a $c.F_4(1)$-geometry in which (b) fails.

Another class of examples comes from representations of $\mathcal{F}_4(1)$. A *separable* $\Omega_8^+(2)$-*admissible representation* of $\mathcal{F}_4(1)$ is a group R and an injective mapping φ from the point set of $\mathcal{F}_4(1)$ into the set of involutions of R such that $R = \langle Im\varphi \rangle$ and that $\varphi(p)\varphi(q)\varphi(r) = 1$ whenever $\{p, q, r\}$ is a line of $\mathcal{F}_4(1)$. The term $\Omega_8^+(2)$-*admissibility* means that the action of every $g \in \Omega_8^+(2)$ on the point set of $\mathcal{F}_4(1)$ can be extended to an automorphism of R.

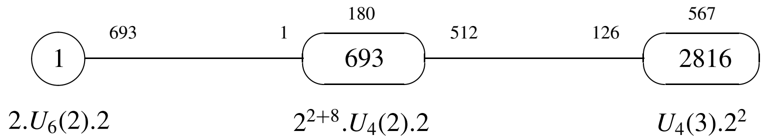
Let $\Gamma(R, \varphi)$ be the Cayley graph of R with respect to $Im\varphi$. Then $\Gamma(R, \varphi)$ is the collinearity graph of a $c.F_4(1)$-geometry for which (b) fails. It follows from the definition of $\mathcal{F}_4(1)$ that the group $\Omega_8^+(2)$ itself can be taken as R. In this case φ is the identity map. But we can also take the universal non-abelian representation which is non-trivial (since $\mathcal{F}_4(1)$ contains geometric hyperplanes) (cf. [13]) and contains, for instance, a 26-dimensional quotient isomorphic to the exterior square of the natural module of $\Omega_8^+(2)$. Certain quotients of the 26-dimensional module provide non-flag-transitive examples.

In general, if a representation is $\Omega_8^+(2)$-admissible then the corresponding $c.F_4(1)$-geometry is flag-transitive, if not we can obtain non-flag-transitive examples. By the way, we do not know what is the universal representation of $\mathcal{F}_4(1)$ and whether it is finite or infinite.

3. Some related geometries of Fi_{22}

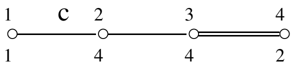
In this section we review some other geometries of $G \cong Fi_{22}$ and their relationships and characterizations (compare [7]). We start with the description of the 3-transposition graph of G.

The group Fi_{22} contains a conjugacy class ($2A$ in notation of [1]) of 3 510 involutions possessing the property that the order of the product of any two of them is 1, 2, or 3. The involutions in $2A$ are called 3-*transpositions* and the 3-transposition graph of G is defined as the graph A with vertices the set of $2A$-involutions in G and edges the set of all pairs of commuting involutions. The full automorphism group of A is $Aut(A) = \hat{G} \cong Fi_{22}.2$ and the suborbit diagram with respect to the action of $\hat{G}$ is the following.



The graph A is locally the collinearity graph of the polar space $\mathcal{P}(U_6(2))$ of $U_6(2)$. In particular, the maximal cliques in A are of size 22. If K is such a clique, then the involutions in K generate an elementary abelian subgroup Q of order 2^{10}. The normalizer of Q in $\hat{G}$ coincides with the setwise stabilizer $\hat{G}[K]$ of K and $\hat{G}[K] \cong 2^{10}.M_{22}.2$. The action of $\hat{G}[K]$ preserves on K a unique Steiner system $\mathcal{S} := \mathcal{S}(K)$ of type $\mathcal{S}(3, 6, 22)$. If K_1 and K_2 are distinct cliques with non-empty intersection then $K_1 \cap K_2$ is a vertex, an edge, or a 6-clique which is a block in $\mathcal{S}(K_1)$ and in $\mathcal{S}(K_2)$.

Let $\mathcal{F}$ be the geometry whose elements of type 1, 2, 3, and 4 are the vertices, the edges, the 6-cliques contained in more than one maximal clique, and the maximal cliques in A; the incidence is defined by inclusion. Then $\mathcal{F}$ belongs to the following diagram.

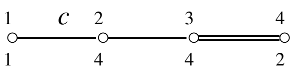


It is easy to see that every graph which is locally the collinearity graph of $\mathcal{P}(U_6(2))$ leads to a geometry with the above diagram. The following characterization was established in [10, 11] (earlier the result was proved in [9] under the assumption of flag-transitivity).

Lemma 3.1 *Up to isomorphism A is the only graph which is locally the collinearity graph of $\mathcal{P}(U_6(2))$.*

Let $\tilde{\mathcal{F}}$ be a geometry with the above diagram. Suppose that the residue of any element of type 1 in $\tilde{\mathcal{F}}$ is isomorphic to the polar space of $U_6(2)$ and that the collinearity graph $\tilde{A}$ of $\tilde{\mathcal{F}}$ is locally the collinearity graph of $\mathcal{P}(U_6(2))$. Then $\tilde{A} \cong A$ by (3.1). Since $\mathcal{F}$ can be uniquely reconstructed from A by taking the maximal cliques (which are of size 22), the 6-cliques which are contained in more than one maximal clique, the edges and the vertices of A as objects of $\mathcal{F}$ of types 4, 3, 2, 1, respectively, and defining incidence by inclusion we also get $\tilde{\mathcal{F}} \cong \mathcal{F}$. This gives the following

Corollary 3.2 *Up to isomorphism $\mathcal{F}$ is the only geometry with diagram*



in which the residue of any element of type 1 *is isomorphic to $\mathcal{P}(U_6(2))$ and whose collinearity graph is locally the collinearity graph of $\mathcal{P}(U_6(2))$.*

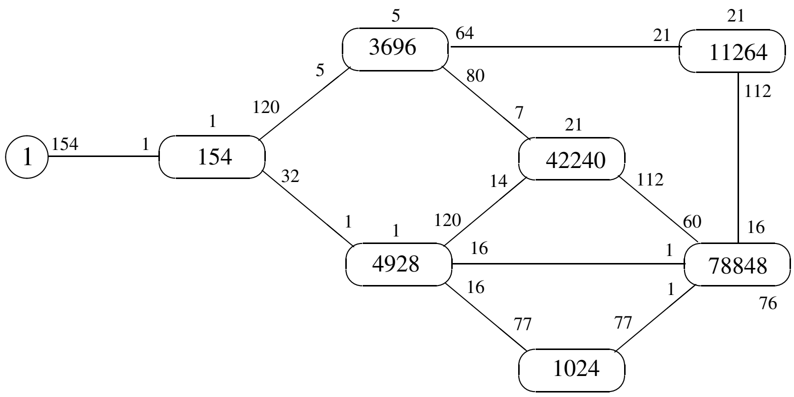
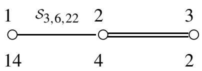
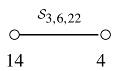


Figure 4. The suborbit diagram of Λ, the graph on elements of type 4 in $\mathcal{F}(Fi_{22})$.

Let Λ be the graph on the set of elements of type 4 in $\mathcal{F}$ (the maximal cliques in A) in which two of them are adjacent if they are incident to a common element of type 3 (intersect in a 6-clique). Then Λ is of valency $154 = 2 \cdot 77$, every edge is in a unique triangle, and if $u \in \Lambda$ corresponds to a 22-clique K then the 77 triangles of Λ containing u are naturally indexed by the blocks of the Steiner system $\mathcal{S}(K)$. By [4, 2.17(iv)] the suborbit diagram of Λ is the one given in figure 4.

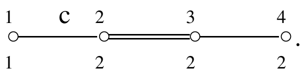
The truncation of $\mathcal{F}$ by the elements of type 1 is a geometry $\mathcal{F}^T$ with diagram



where


denotes the geometry on the 231 pairs and the 77 blocks of the Steiner system $\mathcal{S}(3, 6, 22)$ with incidence defined by inclusion. Obviously, Λ is equal to the graph on the set of elements of type 3 in $\mathcal{F}^T$ in which two such elements are adjacent if they are incident to a common element of type 2.

Finally, G acts flag-transitively on a geometry $\mathcal{G}$ which is a c-extension of the dual polar space of the symplectic group $Sp_6(2)$, i.e., which has the diagram

$(c.C_3^*)$:


The residue of an element of type 4 in $\mathcal{G}$ is the $c.C_2$-geometry of $U_4(2)$. In the flag-transitive case we have the following characterization of $\mathcal{G}$ ([7, (5.3)]).

Lemma 3.3 *Let $\mathcal{H}$ be any flag-transitive $c.C_3^*$-geometry with $c.C_2$-residues belonging to $U_4(2)$ and let $H \leq Aut(\mathcal{H})$ be flag-transitive. Then either $\mathcal{H} \cong \mathcal{G}$ and $H \cong Fi_{22}$ or $Fi_{22}.2$ or $\mathcal{H} \cong 3.\mathcal{G}$ and $H \cong 3 \cdot Fi_{22}$ or $3 \cdot Fi_{22}.2$ (non-split extensions).*

Let Φ be the collinearity graph of $\mathcal{G}$, i.e., the graph with vertices $V(\Phi) = \mathcal{G}^1$ and edges $E(\Phi) = \mathcal{G}^2$. By [4, 2.17(v)] the suborbit diagram of Φ is as in figure 5.

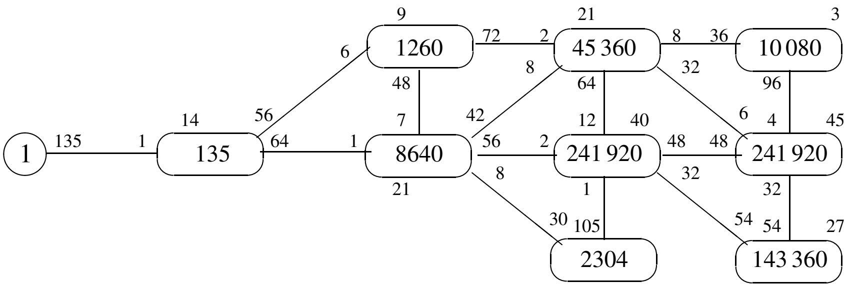
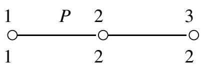


Figure 5. The suborbit diagram of Φ, the collinearity graph of $\mathcal{G}(Fi_{22})$.

The characterization of $\mathcal{G}$ in [7] was achieved by recovering the geometry $\mathcal{F}^T$ and the graph Λ from $\mathcal{G}$ and we will partly follow those lines here. On a first step, we will construct a $c.C_3^*$-geometry $\mathcal{G}$ with $c.C_2$-residues belonging to $U_4(2)$ from the geometry $\mathcal{E}$ we actually want to determine. Then we will show that we can recover a geometry with the same diagram as $\mathcal{F}^T$ from $\mathcal{G}$. However, since in our case neither this geometry nor $\mathcal{G}$ must necessarily be flag-transitive this will not suffice to determine $\mathcal{G}$ (or $\mathcal{E}$). We will even have to reconstruct the geometry $\mathcal{F}$ resp. the graph A, so that finally we can appeal to (3.1) resp. (3.2).

Most of our constructions will require far more subtle arguments than the flag-transitive case. A crucial role in the determination of some of the graphs and geometries constructed in the sequel will be played by the following characterization of not-necessarily flag-transitive rank 3 P-geometries by Hall and Shpectorov [3].

Lemma 3.4 *Suppose $\mathcal{P}$ is a P-geometry with diagram*



such that

(1) *any two different elements of type* 2 *are incident to at most one common element of type* 3;

(2) *any three elements of type* 3 *which are pairwise incident to a common element of type* 2 *are all incident to a common element of type* 1.

Then $\mathcal{P}$ is either the 2*-local geometry of the group M_{22} or the geometry of the group* $3{\cdot}M_{22}$.

Throughout the rest of the paper we denote the Petersen geometries for M_{22} and $3{\cdot}M_{22}$ by $\mathcal{P}_{22}$ and $3\mathcal{P}_{22}$, respectively.

For any P-geometry $\mathcal{P}$, the *derived graph* of $\mathcal{P}$ is defined as the graph on the set of elements of type 1 in $\mathcal{P}$ in which two elements are adjacent if they are incident to a common element of type 2 (see e.g. [8, p. 27, 308]). We denote the derived graphs of $\mathcal{P}_{22}$ and $3\mathcal{P}_{22}$ by Π_{22} and $3\Pi_{22}$, respectively. The intersection arrays of Π_{22} and $3\Pi_{22}$ are presented in [8, p. 27].

4. The reduction from $\mathcal{E}$ to $\mathcal{G}$

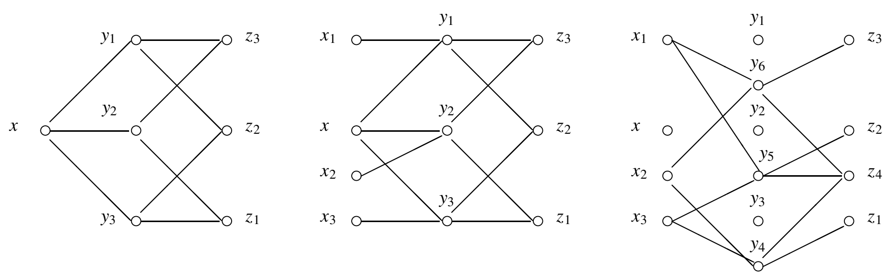
In this section we achieve the first step in our characterization of $c.F_4(1)$-geometries $\mathcal{E}$, i.e., we show how to construct a $c.C_3^*$-geometry $\mathcal{G}$ with $c.C_2$-residues belonging to $U_4(2)$ from $\mathcal{E}$. For this purpose it will be useful to define three graphs on the sets of objects of $\mathcal{E}$ of types 2, 3, and 5, respectively.

The first graph is the analogue of the graph $\tilde{\Xi}$ described in the introduction and it will be denoted by the same letter. So $\tilde{\Xi}$ is the graph with vertices $V(\tilde{\Xi}) := \mathcal{E}^2$ and edges all the pairs $\{x, y\} \subseteq \mathcal{E}^2$ such that x and y are incident to a common element of type 3 but not to a common element of type 1. Similary, $\tilde{\Delta}$ is the graph with vertices $V(\tilde{\Delta}) := \mathcal{E}^3$ and edges all the pairs $\{x, y\} \subseteq \mathcal{E}^3$ with x, y incident to a common element of type 4 but not to a common element of type 1. (Equivalently, in terms of the collinearity graph Γ of $\mathcal{E}$, x, y are two disjoint 4-cliques which come from elements of type 3 and whose union is an 8-clique coming from an element of type 4 in the residue of both of them.) Finally, Φ is the graph with vertices $V(\Phi) := \mathcal{E}^5$ and edges all the pairs $\{x, y\} \subseteq \mathcal{E}^5$ such that x and y are incident to a common element in $\mathcal{E}^4$.

For $x \in \mathcal{E}^2$ (resp. $\mathcal{E}^3$) let $\tilde{\Xi}^x$ (resp. $\tilde{\Delta}^x$) denote the connected component of $\tilde{\Xi}$ (resp. $\tilde{\Delta}$) containing x. It will be convenient later to introduce also the subgraphs Ξ^x and Δ^x of Γ induced on the sets of vertices of Γ which are incident in $\mathcal{E}$ to vertices of $\tilde{\Xi}^x$ resp. $\tilde{\Delta}^x$.

Lemma 4.1 *Let $\tilde{\Delta}^x, x \in \mathcal{E}^3$, be a connected component of $\tilde{\Delta}$. Then $|V(\tilde{\Delta}^x)| = 4$. For $i = 4, 5$, set $\tilde{\Delta}^{x,i} := \{y \in \mathcal{E}^i \mid y \in res(z)$ for some $z \in V(\tilde{\Delta}^x)\}$. Then $|\tilde{\Delta}^{x,4}| = 6$, $|\tilde{\Delta}^{x,5}| = 4$, and the subgraph of Φ with vertices $\tilde{\Delta}^{x,5}$ and edges $\tilde{\Delta}^{x,4}$ is the complete graph on 4 vertices.*

Proof: By the diagram of $\mathcal{E}$ we can write $res(x) \cap \mathcal{E}^4 = \{y_1, y_2, y_3\}$ and $res(x) \cap \mathcal{E}^5 = \{z_1, z_2, z_3\}$ where the numeration is chosen in such a way that $y_i, y_j \in res(z_k)$ for all triples $\{i, j, k\} = \{1, 2, 3\}$. Let x_i, $i = 1, 2, 3$, be the unique neighbour of x in $\tilde{\Delta}$ determined by y_i. Then we see in $res(z_k)$ that $y_{3+k} := x_i \cup x_j$ is an element in $\mathcal{E}^4$. Let $z_4 \in \mathcal{E}^5 \cap res(y_4) \cap res(y_5)$ (z_4 exists by the structure of $res(x_3)$). Then $x_1, x_2 \in res(z_4)$, too, and there must be an element in $y \in \mathcal{E}^4$ incident to x_1, x_2, and z_4. But x_1, x_2 are both already incident to three elements in $\{y_1, \ldots, y_6\}$. So $y = y_6$ and this easily implies the assertion (cf. the picture below).



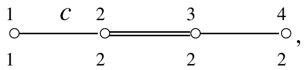
∎

Now we can define the geometry $\mathcal{G}$. The sets of objects of $\mathcal{G}$ are the four sets

$$\mathcal{G}^1 := \mathcal{E}^5, \quad \mathcal{G}^2 := \mathcal{E}^4, \quad \mathcal{G}^3 := \{\tilde{\Delta}^x \mid x \in \mathcal{E}^3\}, \quad \text{and} \quad \mathcal{G}^4 := \{\tilde{\Xi}^x \mid x \in \mathcal{E}^2\}.$$

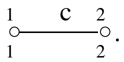
The incidence relation on $\mathcal{G}$ is defined as follows: Between elements of $\mathcal{G}^1, \mathcal{G}^2$ we take the incidence relation induced from $\mathcal{E}$. An element $x \in \mathcal{G}^1 \cup \mathcal{G}^2$ is incident to an element $y \in \mathcal{G}^3 \cup \mathcal{G}^4$ if $y = \tilde{\Delta}^z$ (resp. $\tilde{\Xi}^z$) for some $z \in res_{\mathcal{E}}(x)$. Finally, $x \in \mathcal{G}^3$, $y \in \mathcal{G}^4$ are incident if there are $x_1 \in \mathcal{E}^3$, $x_2 \in \mathcal{E}^2 \cup res_{\mathcal{E}}(x_1)$ such that $x = \tilde{\Delta}^{x_1}$, $y = \tilde{\Xi}^{x_2}$. Notice that Φ is just the collinearity graph of $\mathcal{G}$ (where elements of type 1 are considered as points and elements of type 2 as lines).

Lemma 4.2 *Let $\mathcal{G}$ be a geometry constructed as above from a c.$F_4(1)$-geometry $\mathcal{E}$ satisfying (I). Then $\mathcal{G}$ has the diagram*



and if $x \in \mathcal{G}^4$, then $res_{\mathcal{G}}(x)$ is isomorphic to the c.C_2-geometry related to $U_4(2)$.

Proof: For $x \in \mathcal{G}^1 = \mathcal{E}^5$ we get from (1.7)(i) that $res_{\mathcal{G}}(x)$ is the symplectic (dual) polar space related to the group $Sp_6(2)$ and for $x \in \mathcal{G}^3$ we get from (4.1) that $res^-_{\mathcal{G}}(x)$ is the geometry of vertices and edges of the complete graph on 4 vertices, i.e., $res^-_{\mathcal{G}}(x)$ has the diagram



Let $y \in res^-_{\mathcal{G}}(x)$ and $z \in res_{\mathcal{G}}(x) \cap \mathcal{G}^4$ (where still $x \in \mathcal{G}^3$). Then by the definition of the incidence relation in $\mathcal{G}$, there is some $y_1 \in res_{\mathcal{E}}(y)$ with $x = \tilde{\Delta}^{y_1}$; on the other hand, there are also $y_i \in \mathcal{E}^i$, $i = 2, 3$, such that $x = \tilde{\Delta}^{y_3}$, $z = \tilde{\Xi}^{y_2}$, and y_2 and y_3 are incident in $\mathcal{E}$. It follows from the definition of $\tilde{\Delta}$ and $\tilde{\Xi}$ that we may assume $y_3 = y_1$ in which case $y_2 \in res_{\mathcal{E}}(y_1)$. Now the string diagram of $\mathcal{E}$ implies that $y_2 \in res_{\mathcal{E}}(y)$ and so $z = \tilde{\Xi}^{y_2} \in res_{\mathcal{G}}(y)$. Together with the above this shows that the diagram of $\mathcal{G}$ is as stated.

The only thing that remains to be shown is that the c.C_2-geometry in $res_{\mathcal{G}}(y)$, $y \in \mathcal{G}^4$, is the one related to $U_4(2)$. Let $y = \tilde{\Xi}^x$ for some $x \in \mathcal{E}^2$. From (1.2)(i) we know that $\tilde{\Xi}^x$ has 40 vertices and that it is the graph related to $U_4(2)$ with suborbit diagram as in that lemma. Let $z \in res_{\mathcal{G}}(y) \cap \mathcal{G}^1$. Then $|res_{\mathcal{E}}(z) \cap V(\tilde{\Xi}^x)| = 10$ by (1.7)(ii) and counting the number of pairs (z, u), $z \in res_{\mathcal{G}}(y) \cap \mathcal{G}^1$, $u \in V(\tilde{\Xi}^x) \cap res_{\mathcal{E}}(z)$ in two ways we calculate that

$$|res_{\mathcal{G}}(y) \cap \mathcal{G}^1| = \frac{|V(\tilde{\Xi}^x)| \cdot |res_{\mathcal{E}}(x) \cap \mathcal{E}^5|}{|V(\tilde{\Xi}^x) \cap res_{\mathcal{E}}(z)|} = \frac{40 \cdot 9}{10} = 36.$$

Now [2, (1.3)] yields the assertion. □

Before now turning to the determination of $\mathcal{G}$ in the general, i.e., not necessarily flag-transitive case we prove some properties of $\mathcal{G}$ which will turn out to be very useful.

Lemma 4.3 *Let $\mathcal{G}$ be as in* (4.2). *Then $\mathcal{G}$ satisfies the following properties.*

(i) *Any two different elements of $\mathcal{G}^3$ are incident to at most one common element in $\mathcal{G}^4$.*

(ii) *Any three elements of $\mathcal{G}^4$ which are pairwise incident to a common element in $\mathcal{G}^3$ are also incident to a common element in $\mathcal{G}^1$ and to a common element in $\mathcal{G}^2$.*

(iii) *If $y \in \mathcal{G}^1$ and $z \in \mathcal{G}^3$ are incident to two common elements in $\mathcal{G}^4$ then y and z are incident to each other.*

Proof: (i) Let $\tilde{\Delta}_1, \tilde{\Delta}_2 \in \mathcal{G}^3$, $\tilde{\Delta}_1 \neq \tilde{\Delta}_2$, $\tilde{\Xi}_1, \tilde{\Xi}_2 \in \mathcal{G}^4$ with $\tilde{\Delta}_i \in res_{\mathcal{G}}(\tilde{\Xi}_j)$, $1 \leq i, j \leq 2$. Since $\tilde{\Delta}_i$ consists of 4 pairwise disjoint 4-cliques of Γ we have $|\Delta_i| = 16$ and as $\tilde{\Delta}_i \in res_{\mathcal{G}}(\tilde{\Xi}_j)$ we have $\Delta_i \subseteq \Xi_j$. Further as $\tilde{\Delta}_1 \neq \tilde{\Delta}_2$ also $\Delta_1 \neq \Delta_2$ and hence $|\Xi_1 \cap \Xi_2| \geq |\Delta_1 \cup \Delta_2| > 16$. Now (1.4) yields $\Xi_1 = \Xi_2$ and $\tilde{\Xi}_1 = \tilde{\Xi}_2$.

(ii) Let $\tilde{\Xi}_1, \tilde{\Xi}_2, \tilde{\Xi}_3 \in \mathcal{G}^4$, $\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3 \in \mathcal{G}^3$ with $\tilde{\Xi}_i, \tilde{\Xi}_j \in res_{\mathcal{G}}(\tilde{\Delta}_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $\Delta_k = \Xi_i \cap \Xi_j$. Suppose first there exists some $x \in \Delta_1 \cap \Delta_2 \cap \Delta_3$. Then by [5, (7.12)] there exist y_i with $\tilde{\Xi}_i = \tilde{\Xi}^{e_i}$ for $e_i := \{x, y_i\}$ and such that there are $d_i \in \mathcal{E}^3$ (i.e., 4-cliques in Γ) with $e_i, e_j \subseteq d_k$, $\{i, j, k\} = \{1, 2, 3\}$. In terms of $res_{\mathcal{E}}(x) \cong \mathcal{F}_4(1)$ this means that e_1, e_2, e_3 correspond to three pairwise collinear points. Hence there exists $z \in \mathcal{E}^5 = \mathcal{G}^1$ which is incident to all of them and also to d_1, d_2, d_3. This implies $\tilde{\Xi}_i, \tilde{\Delta}_i \in res_{\mathcal{G}}(z)$ for all i. Now $res_{\mathcal{G}}(z)$ is isomorphic to the dual of the symplectic polar space $\mathcal{P}(Sp_6(2))$. So in $res_{\mathcal{G}}(z)$ we can identify $\tilde{\Xi}_1, \tilde{\Xi}_2, \tilde{\Xi}_3$ with three pairwise collinear points which implies that they are incident to a common element in $\mathcal{G}^2$.

So all we have to do is to find some $x \in \Delta_1 \cap \Delta_2 \cap \Delta_3$.

Suppose $\Delta_1 \cap \Delta_2 = \emptyset$; let $\tilde{\tilde{\Delta}}_k := \tilde{\Xi}_i \cap \tilde{\Xi}_j$. Then $\tilde{\tilde{\Delta}}_1, \tilde{\tilde{\Delta}}_2$ correspond to two disjoint 8-cliques in the graph $\tilde{\Xi}_3$. It is easy to see from the distribution diagram of $\tilde{\Xi}_3$ that there must be $d_i \in \tilde{\tilde{\Xi}}_i$, $i = 1, 2$ such that $(d_1, d_2) \in E(\tilde{\Xi}_3)$. By (1.3) any vertex outside an 8-clique is adjacent to some vertices in the 8-clique. So there is $d_3 \in \tilde{\tilde{\Delta}}_3$ such that $(d_2, d_3) \in E(\tilde{\Xi}_1)$. Let $u \in d_3$, $v \in d_2$, $x \in d_1$.

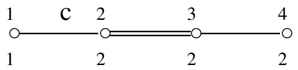
If $u \in \Gamma(x)$ then $\{u, v, x\}$ is a 3-clique in Γ, hence must be incident to an element $z \in \mathcal{E}^5$ which implies $\tilde{\Xi}_i \in res_{\mathcal{G}}(z)$ for all i and we are done. If $u \notin \Gamma(x)$ then $d_{\Gamma}(u, x) = 2$ and $v \in \Gamma(u, x)$. If $v \notin \Xi_1 \cap \Xi_2$, then by [5] the path u, v, x must be of D_6-type, i.e., $i_v(u) \in \Psi_3(i_v(x))$. But by the above diagram there is $d_4 \neq d_1$, $d_4 \in \tilde{\tilde{\Delta}}_1$ which is adjacent to d_3 and d_2. Let $w \in d_4$. Then $i_v(w) \in \Psi_1(i_v(x))$ and $w \in \Gamma(u)$ which contradicts $i_v(u) \in \Psi_3(i_v(x))$.

(iii) Let $x_1, x_2 \in \mathcal{G}^4$, $y \in \mathcal{G}^3$, and $z \in \mathcal{G}^1$ such that $x_1, x_2 \in res_{\mathcal{G}}(y) \cap res_{\mathcal{G}}(z)$. Then in $res(z)$, x_1, x_2 correspond to two points in the symplectic polar space for $Sp_6(2)$. So there exists $x_3 \in \mathcal{G}^4 \cap res(z)$ such that $x_i, x_3 \in res(y_i)$ for some $y_i \in \mathcal{G}^3$, $i = 1, 2$. Now by (ii) there is $w \in \mathcal{G}^2$ such that w is incident to all of $y, y_1, y_2, x_1, x_2, x_3$. Then $z, w \in res(y_1) \cap res(y_2)$ and it follows from the definition of the graph $\tilde{\Delta}$ and the geometry $\mathcal{G}$ that either $z \in res(w)$ or $y_1 = y_2$. Since in the latter case also $y = y_1 = y_2$ and because the diagram of $\mathcal{G}$ is a string, both possibilities yield the assertion. $\square$

As we will show at the beginning of Section 5 the condition (i) follows from (iii) and the diagram of $\mathcal{G}$.

5. The characterization of $\mathcal{G}$

In this section we characterize the geometry $\mathcal{G}$. Our characterization is more or less independent from the question whether $\mathcal{G}$ is obtained from a geometry like $\mathcal{E}$ or not. We just require a few properties of $\mathcal{G}$ which hold in our case by the results of the previous section (Lemmas 4.2 and 4.3). Precisely, we consider any geometry $\mathcal{G}$ with diagram



and which satisfies the following assumptions.

(II) (a) If $x \in \mathcal{G}^1$ then $res(x)$ is the (dual) polar space related to the symplectic group $Sp_6(2)$.

(b) If $x \in \mathcal{G}^4$ then $res(x)$ is the $c.C_2$-geometry (on 36 points) related to the group $U_4(2) \cong \Omega_6^-(2)$.

(c) Any two different elements of $\mathcal{G}^3$ are incident to at most one common element of $\mathcal{G}^4$.

(d) Any three different elements of $\mathcal{G}^4$ which are pairwise incident to a common element of $\mathcal{G}^3$ are all incident to a common element of $\mathcal{G}^2$ (equivalently of $\mathcal{G}^1$).

(e) If two different elements $x_1, x_2 \in \mathcal{G}^4$ are incident to common elements $y \in \mathcal{G}^1$ and $z \in \mathcal{G}^3$ then y and z are incident.

Condition (c) is not really needed; it follows easily from (e) and the diagram: Let $a, b \in \mathcal{G}^3$ be incident to $x_1, x_2 \in \mathcal{G}^4$, $x_1 \neq x_2$. Choose any $z \in res(a) \cap \mathcal{G}^1$. Then by (e) also $z \in res(b)$ and the structure of $res(z)$ implies that $a = b$. Nevertheless we state (c) seperately because it will be needed later.

As already mentioned, flag-transitive $c.C_3^*$-geometries with $c.C_2$-residues belonging to $U_4(2)$ have been determined in [7] (see (3.3)). Here we do not assume the existence of any group of automorphisms acting on $\mathcal{G}$.

As in the previous section, by Φ we denote the collinearity graph of $\mathcal{G}$, i.e., the graph with $V(\Phi) := \mathcal{G}^1$ and $E(\Phi) := \mathcal{G}^2$, and we will often identify the objects of $\mathcal{G}$ with the corresponding vertices, edges (or 2-cliques), 4-cliques, and certain 36-vertex subgraphs of Φ.

The determination of $\mathcal{G}$ will be achieved in a series of steps which we present in several subsections. In the first subsection we define a graph Π and we show that each of its connected components is isomorphic to one of the derived graphs Π_{22} and $3\Pi_{22}$ of the P-geometries $\mathcal{P}_{22}$, $3\mathcal{P}_{22}$. In the second subsection we define another graph B (the analogue of the graph Λ defined in Section 3) which will help us to show that any two connected components of Π are isomorphic. Furthermore, we will use B to define a geometry $\mathcal{B}$. In Section 6 we will show that at least if we impose a more or less natural condition on B resp. $\mathcal{B}$ then $\mathcal{B}$ is isomorphic to the truncation $\mathcal{F}^T$ of the geometry $\mathcal{F}$ for Fi_{22}.

5.1. The graph Π and the geometry $\mathcal{P}$

The graph Π is constructed from $\mathcal{G}$ in the same way as the graph $\tilde{\Xi}$ was constructed from $\mathcal{E}$, i.e., Π is the graph with vertices $V(\Pi) := \mathcal{G}^2$ and edges

$$E(\Pi) := \{\{x, y\} \mid x, y \in \mathcal{G}^2, x \cup y \in \mathcal{G}^3 \text{ (and so } x \cap y = \emptyset)\}$$

(where again we identify the objects of $\mathcal{G}^2$ and $\mathcal{G}^3$ with the corresponding edges and 4-cliques in Φ). If $\{x, y\} \in E(\Pi)$ then by definition $x \cup y$ is a uniquely determined element in $\mathcal{G}^3$. So we can define a map

$$\begin{aligned} \alpha : E(\Pi) &\to \mathcal{G}^3 \\ \{x, y\} &\to x \cup y. \end{aligned}$$

The fibers of α are of size 3. We will call two edges of Π *α-equivalent* if they are in the same fiber of α.

For $x \in \mathcal{G}^2$, similarly as before, we denote by Π^x the connected component of Π containing x, and for $q \in \mathcal{G}^4$ we denote by Π_q the subgraph of Π with vertices $V(\Pi_q) := V(\Pi) \cap res(q)$ and edges $\{x, y\}$ such that x, y, and $x \cup y$ are contained in $res_{\mathcal{G}}(q)$. The following lemma shows that Π_q is actually the subgraph of Π induced on the set $V(\Pi_q)$.

Lemma 5.1 *If $q \in \mathcal{G}^4$ and $x, y \in V(\Pi_q)$ with $\{x, y\} \in E(\Pi)$ then $x \cup y \in res(q)$.*

Proof: Let $z := x \cup y \in \mathcal{G}^3$. Then $z, q \in res(x) \cap res(y)$ and so $r \in res(x) \cap res(y)$ for any $r \in res(z) \cap \mathcal{G}^4$. Choose some $r \neq q$. Since $res^+(x)$ and $res^+(y)$ are projective planes there exist $z_1, z_2 \in \mathcal{G}^3$ which are incident to x, q, r resp. y, q, r. Assumption (II)(c) yields that $z_1 = z_2$. So $z, z_1 = z_2$ are incident to x, y and r and we see in $res(r)$ that $z_1 = z$ which implies the assertion. □

Now part (i) of the next lemma follows from the properties of the $c.C_2$-geometry for $U_4(2)$ (see [7, (6.2)]), (ii) is just a consequence of (i) for Π.

Lemma 5.2
(i) *If $q \in \mathcal{G}^4$ then Π_q is a disjoint union of 27 Petersen graphs. If Π_0 is a connected component of Π_q and $x, y \in V(\Pi_0)$, $x \neq y$, then $x \cap y = \emptyset$.*
(ii) *If Π^1 is a connected component of Π and $q \in \mathcal{G}^4$ then $\Pi^1 \cap \Pi_q = \emptyset$ if $res(q) \cap V(\Pi^1) = \emptyset$ and $\Pi^1 \cap \Pi_q$ is a disjoint union of Petersen subgraphs of Π^1 if $res(q) \cap V(\Pi^1) \neq \emptyset$ where $\Pi^1 = \Pi^x$ for some $x \in res(q)$.*

We note that in the known examples $\Pi^1 \cap \Pi_q$ is a single connected component and we will prove in (5.6) that under our assumptions this is also true at least if $\Pi^1 \cong \Pi_{22}$.

Lemma 5.3

(i) *If* $x_1, x_2 \in V(\Pi)$, $x_1 \neq x_2$, *and* $d_\Pi(x_1, x_2) \leq 2$ *then* $x_1 \cap x_2 = \emptyset$.

(ii) *If* $x_1, x_2 \in V(\Pi)$, $x_1 \neq x_2$, $d_\Pi(x_1, x_2) \leq 4$ *and there is some* $q \in \mathcal{G}^4$ *incident to both of* x_1 *and* x_2 *then* $x_1 \cap x_2 = \emptyset$.

Proof: (i) If $d_\Pi(x_1, x_2) = 1$ the assertion is trivial since then $x_1 \cup x_2 \in \mathcal{G}^3$. So assume $d_\Pi(x_1, x_2) = 2$ and let $z \in \Pi(x_1) \cap \Pi(x_2)$. Let $e := \{x_1, z\}$, $f := \{z, x_2\} \in E(\Pi)$ be the corresponding edges. Then by definition $\alpha(e), \alpha(f) \in res(z) \cap \mathcal{G}^3$. Since $res^+(z)$ is the projective plane over $GF(2)$ there exists $q \in \mathcal{G}^4$ with $q \in res(z) \cap res(\alpha(e)) \cap res(\alpha(f))$. By the diagram of $\mathcal{G}$ then also $x_1, x_2 \in res(q)$ and (x_1, z, x_2) is a path in (a connected component of) the subgraph Π_q. Now the assertion follows from Lemma 5.2.

(ii) Let x_i, q be as assumed and suppose there exists $a \in x_1 \cap x_2$. Since $d_\Pi(x_1, x_2) \leq 4$ there is $z \in V(\Pi)$ with $d_\Pi(z, x_i) \leq 2$ for $i = 1, 2$. As in (i) we see that there are $q_i \in \mathcal{G}^4 \cap res(z) \cap res(x_i)$, $i = 1, 2$, such that there is a path from x_i to z in a connected component Π_i of Π_{q_i}. By (5.2) we must have $q_1 \neq q_2$. On the other hand, q and q_i are incident to x_i and we see in $res^+(x_i)$ that there is an element $l_i \in \mathcal{G}^3$ which is incident to q, q_i and x_i. Since $a \in x_i$, by the diagram of $\mathcal{G}$ then a must be incident to l_1, l_2, q_1, q_2, q. Further, in $res(z)$ we see that there is $l \in \mathcal{G}^3 \cap res(q_i)$, $i = 1, 2$, and by assumption (II)(e) a must also be incident to l. Now l and l_i determine two edges $\{z, z_i\}$ and $\{x_i, x_i'\}$ of the Petersen graph Π_i and $a \in l \cap x_i$. Lemma 5.2 shows that this is only possible if $z_i = x_i$ (since $a \notin z$). In particular, we must have $d_\Pi(z, x_i) = 1$ for $i = 1, 2$ and so $d_\Pi(x_1, x_2) \leq 2$ in contradiction to (i). $\square$

Let us now fix some $x \in \mathcal{G}^2$ and let us consider the connected component Π^x of Π which contains x. In order to identify the isomorphism type of the graph Π^x we define a rank 3 geometry $\mathcal{P} := \mathcal{P}^x$ whose objects of type 1, 2, 3 are respectively the vertices and the edges of Π^x and the Petersen subgraphs induced by elements of $\mathcal{G}^4$ and contained in Π^x; more formally,

$$\begin{aligned}
\mathcal{P}^1 &:= V(\Pi^x),\\
\mathcal{P}^2 &:= E(\Pi^x),\\
\mathcal{P}^3 &:= \{(q, \Pi_0) \mid q \in \mathcal{G}^4 \text{ such that } \Pi^x \cap \Pi_q \neq \emptyset\\
&\qquad \text{and } \Pi_0 \text{ is a connected component of } \Pi^x \cap \Pi_q\}.
\end{aligned}$$

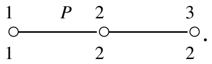
The incidence relation on $\mathcal{P}$ is defined in the obvious way, i.e., Petersen subgraphs of Π^x are incident to its edges and vertices, and the edges of Π^x are incident to its vertices.

The following map $\beta : \mathcal{P} \to \mathcal{G}$ (which maps $\mathcal{P}^i$ into $\mathcal{G}^{i+1}$) will be useful.

$$\begin{aligned}
\beta(y) &:= y, && \text{for } y \in \mathcal{P}^1,\\
\beta(y) &:= \alpha(y), && \text{for } y \in \mathcal{P}^2,\\
\beta(q, \Pi_0) &:= q, && \text{for } (q, \Pi_0) \in \mathcal{P}^3.
\end{aligned}$$

Obviously, the map β is incidence preserving and hence the image $\bar{\mathcal{P}} := \beta(\mathcal{P})$ induces a connected subgeometry of $\mathcal{G}$ (consisting of objects of $\mathcal{G}$ of types 2, 3, and 4).

Lemma 5.4 *$\mathcal{P}$ has the diagram*



Proof: By definition of the incidence relation the diagram of $\mathcal{P}$ is a string and by (5.2) $res_{\mathcal{P}}(y)$ is the rank-2 geometry of vertices and edges of the Petersen graph if $y \in \mathcal{P}^3$. Finally, if $y \in \mathcal{P}^1$ then the restriction $\beta_y := \beta|_{res_{\mathcal{P}}(y)}$ induces an isomorphism

$$\beta_y : res_{\mathcal{P}}(y) \to res^+_{\mathcal{G}}(y).$$

This yields the assertion. □

Lemma 5.5 *$\mathcal{P} \cong \mathcal{P}_{22}$ or $3\mathcal{P}_{22}$ and $\Pi^x \cong \Pi_{22}$ or $3\Pi_{22}$, respectively.*

Proof: We want to apply (3.4). So we have to show that conditions (1) and (2) from (3.4) hold in $\mathcal{P}$.

Let first $x_1, x_2 \in \mathcal{P}^2$, $x_1 \neq x_2$, be incident to $y_1, y_2 \in \mathcal{P}^3$. Then $\beta(x_i) \in res_{\mathcal{G}}(\beta(y_j))$ for $\{i, j\} = \{1, 2\}$ and as α-equivalent edges of Π are never contained in the same Petersen subgraph we have $\beta(x_1) \neq \beta(x_2)$. Now (II)(c) implies $\beta(y_1) = \beta(y_2) =: q$ for some $q \in \mathcal{G}^4$. So $y_1 \neq y_2$ would imply that y_1 and y_2 are two different Petersen subgraphs in Π_q. But then y_1 and y_2 would be disjoint subgraphs of Π_q which in terms of $\mathcal{P}$ means that they cannot be incident to any common element in $\mathcal{P}^2$. Hence $y_1 = y_2$ and (1) holds.

Similarly, if $y_i \in \mathcal{P}^3$, $i = 1, 2, 3$, are pairwise incident to a common element in $\mathcal{P}^2$ then their images $\beta(y_i) \in \mathcal{G}^4$, $i = 1, 2, 3$, are pairwise incident to a common element of $\mathcal{G}^3$ and (II)(d) shows that they are incident to a common element $z \in \mathcal{G}^2$. If $\Pi^x = \Pi^z$ then $z \in \mathcal{P}^1 \cap res(y_i)$, $i = 1, 2, 3$, and we are done. So suppose $z \notin \Pi^x$. Let $x_i = \{e_i, f_i\} \in \mathcal{P}^2$ such that $x_i \in res(y_i) \cap res(y_3)$, $i = 1, 2$. Then $z \in res_{\mathcal{G}}(\alpha(x_i))$, $i = 1, 2$, which since $z \notin \Pi^x$ implies that $e_1 \cap e_2 \neq \emptyset$ (as subgraphs of Φ). As x_1, x_2 are edges of the Petersengraph y_3 we must have $x_1 = x_2$. Hence y_1, y_2 and y_3 are all incident to x_1 and by the diagram of $\mathcal{P}$ then also to e_1, e_2, i.e., (2) holds in this case as well. □

Notice that from (5.5) we just know that any connected component of Π is isomorphic either to Π_{22} or to $3\Pi_{22}$. But, in principle, (5.5) allows the possibility that one connected component of Π is isomorphic to Π_{22} and another one to $3\Pi_{22}$. Again, such a degeneracy does not occur in the known examples: all connected components of Π are isomorphic to Π_{22} if Π is constructed from $\mathcal{G}(Fi_{22})$ and all connected components are isomorphic to $3\Pi_{22}$ if Π is constructed from $\mathcal{G}(3 \cdot Fi_{22})$. In the next subsection, we will see that if Π is constructed from a geometry $\mathcal{G}$ which satisfies (II) then also all connected components of Π are isomorphic. But first, we prove one more lemma which will be quite important.

Lemma 5.6 *If $\Pi^x \cong \Pi_{22}$ then the map $\beta : \mathcal{P}^x \to \mathcal{G}$ is injective and hence the image $\beta(\mathcal{P}^x)$ is a subgeometry of $\mathcal{G}$ isomorphic to $\mathcal{P}^x \cong \mathcal{P}_{22}$.*

Proof: Let $\mathcal{P} := \mathcal{P}^x$. By definition $\beta|_{\mathcal{P}^1}$ is injective. Suppose there are $e_i \in \mathcal{P}^2$ with $\beta(e_1) = \beta(e_2) =: e \in \mathcal{G}^3$. Let $e_i = \{x_i, y_i\}$, $i = 1, 2$. Then $x_1 \cap x_2 \neq \emptyset$. On the other hand, $x_1, x_2 \in res_{\mathcal{G}}(e)$ and so $x_1, x_2 \in res_{\mathcal{G}}(q)$ for any $q \in \mathcal{G}^4 \cap res_{\mathcal{G}}(e)$. Since the diameter of Π_{22} is equal to 4 by [8, p. 27] this contradicts (5.3). So $\beta|_{\mathcal{P}^2}$ is injective, too.

Finally suppose there exists $q \in \mathcal{G}^4$ such that $\Pi^x \cap \Pi_q$ consists of at least two connected components Π_1, Π_2. Then it follows from (5.2) and the fact that $|res_{\mathcal{G}}(q) \cap \mathcal{G}^1| = 36$ that there are $x_i \in V(\Pi_i)$, $i = 1, 2$, with $x_1 \cap x_2 \neq \emptyset$. Since $x_1, x_2 \in res_{\mathcal{G}}(q)$ we get again a contradiction to (5.3).

So injectivity of β is proved and, since the map β is incidence preserving, $\beta(\mathcal{P})$ induces an isomorphism of geometries. $\square$

5.2. *The graph B and isomorphism of the* Π^x

The main goal of this subsection is to show that all connected components of Π are isomorphic. The idea to prove this is to define a graph B on the set of connected components of Π, to show that any two adjacent vertices of B are isomorphic, and finally, that B is connected. As already mentioned at the beginning of this section the graph B will also play an important role in Section 6.

We first recall a well-known property of the P-geometries of M_{22} and $3{\cdot}M_{22}$. Let $\mathcal{P}$ be a P-geometry isomorphic to $\mathcal{P}_{22}$ or $3\mathcal{P}_{22}$. Then $\mathcal{P}$ contains a set $\mathcal{H}(\mathcal{P})$ of 77 subgeometries consisting of elements of type 2 and 3 in $\mathcal{P}$. Each $H \in \mathcal{H}(\mathcal{P})$ is isomorphic to the generalized $Sp_4(2)$-quadrangle $GQ(2, 2)$ if $\mathcal{P} \cong \mathcal{P}_{22}$ and to its unique flag-transitive 3-fold cover $3GQ(2, 2)$ if $\mathcal{P} \cong 3\mathcal{P}_{22}$. Each element of type 2 in $\mathcal{P}$ is contained in a unique subgeometry $H \in \mathcal{H}(\mathcal{P})$ and all the three elements of type 3 incident to it in $\mathcal{P}$ then also belong to H (but any element of type 3 belongs to 5 subgeometries). There exists a bijection between the set of subgeometries $\mathcal{H}(\mathcal{P}_{22})$ and the set of blocks of the Steiner system $\mathcal{S}(3, 6, 22)$ with the property that if $H \in \mathcal{H}(\mathcal{P}_{22})$ is the subgeometry corresponding to the block h then the elements of type 3 of $\mathcal{P}_{22}$ which belong to H are the pairs contained in h and the elements of type 2 of $\mathcal{P}_{22}$ which belong to H correspond to the partitions of h into three disjoint pairs.

We define B as the graph with vertex set $V(B) := \{\Pi^x \mid x \in \mathcal{G}^2\}$ and edge set

$$\begin{aligned} E(B) := \{\{\Pi^1, \Pi^2\} \mid \Pi^i \in V(B),\ &\text{there are } e_i \in E(\Pi^i), i = 1, 2, \\ &\text{with } \alpha(e_1) = \alpha(e_2)\}. \end{aligned}$$

In other words, the vertices of B are the connected components of Π and two such connected components are adjacent in B if they arise from α-equivalent edges.

Notice that there might exist edges $\{\Pi^1, \Pi^2\} \in E(B)$ with $\Pi^1 = \Pi^2$ but that (5.6) immediately implies

Lemma 5.7 *If* $\{\Pi^1, \Pi^2\} \in E(B)$ *and* $\Pi^1 = \Pi^2$ *then* $\Pi^1 \cong \Pi^2 \cong 3\Pi_{22}$.

Let us fix an edge $\{\Pi^1, \Pi^2\} \in E(B)$ and, for $i = 1, 2$, let $\mathcal{P}_i$ be the P-geometry related to Π^i. For a subgeometry $H \in \mathcal{H}(\mathcal{P}_i)$ and for $j = 2, 3$, let $H^j := H \cap \mathcal{P}_i^j$ be the set of objects of $\mathcal{P}_i$ of type j which are contained in H. Further we set

$$E_1 := \{e \in E(\Pi^1) \mid \text{ there is } f \in E(\Pi^2) \text{ with } \alpha(e) = \alpha(f)\}.$$

Lemma 5.8 *Let $e \in E_1$ and let $H_e \in \mathcal{H}(\mathcal{P}_1)$ be the subgeometry of $\mathcal{P}_1$ containing e. Then $H_e^2 \subseteq E_1$. Moreover, there is a subgeometry $H \in \mathcal{H}(\mathcal{P}_2)$ such that*

$$H^2 = \left\{f \in E(\Pi^2) \mid \alpha(f) = \alpha(e_1)\, \textit{for some}\; e_1 \in H_e^2\right\}.$$

Proof: Let $(y, \Pi_0) \in H_e^3 \cap res_{\mathcal{P}_1}(e)$ and let $f \in E(\Pi^2)$ with $\alpha(e) = \alpha(f) =: x$. Then $x \in res_{\mathcal{G}}(y)$, Π_0 is a connected component of $\Pi^1 \cap \Pi_y$, and $e \in E(\Pi_0)$. Let Π_0' be the connected component of $\Pi^2 \cap \Pi_y$ with $f \in E(\Pi_0')$ and let $\{e, e_1, e_2\} = res_{H_e}(y, \Pi_0)$. Then by definition of $(3)GQ(2,2)$-subgeometries in P-geometries, e, e_1, e_2 are three pairwise opposite edges of the Petersen subgraph Π_0. On the other hand, it follows from $res_{\mathcal{G}}(y)$ that if $f_1, f_2 \in E(\Pi_0')$ are the edges opposite to f then (up to numeration) $\alpha(e_i) = \alpha(f_i)$ holds for $i = 1, 2$ (cf. [7]). Hence $e_1, e_2 \in E_1$ and $H_e^2 \subseteq E_1$ follows by connectivity of H_e. The last assertion is clear from this and the proof of the lemma. □

Applying (5.7) and (5.8) to the case $\Pi^1 = \Pi^2 = \Pi^x$ we get

Corollary 5.9 *Let Π^x be a connected component of Π and suppose there are $e, f \in E(\Pi^x)$, $e \neq f$, with $\alpha(e) = \alpha(f)$. Then $\Pi^x \cong 3\Pi_{22}$ and if $H_e, H_f \in \mathcal{H}(\mathcal{P}^x)$ are the subgeometries containing e and f, respectively, then for any $e_1 \in H_e^2$ there is $f_1 \in H_f^2$ with $\alpha(e_1) = \alpha(f_1)$.*

Lemma 5.10 *If $\{\Pi^1, \Pi^2\} \in E(B)$ then $\Pi^1 \cong \Pi^2$.*

Proof: For $i = 1, 2$, we define maps $\beta_i : \mathcal{P}_i \to \mathcal{G}$ similarly as the map β was defined in the previous subsection. Let e, H_e be as in (5.8).

If $\Pi^1 \cong \Pi_{22}$ then by (5.6) $\beta_1(H_e) \cong H_e$ induces a $GQ(2,2)$ subgeometry in $\mathcal{G}$. Again as in (5.8), let $f \in E(\Pi^2)$ with $\alpha(e) = \alpha(f) =: x$ and let $H_f \in \mathcal{H}(\mathcal{P}_2)$ with $f \in H_f^2$. Then $\beta_2(H_f) = \beta_1(H_e)$ by (5.8), connectivity of H_e, H_f, and definition of β. If $\Pi^2 \cong 3\Pi_{22}$ we would have $H_f \cong 3GQ(2,2)$ which would imply that $|\beta_2^{-1}(x)| = 3$. But $\beta_2^{-1}(x) = \alpha^{-1}(x) \cap E(\Pi^2)$, the fibers of α are of size 3, and we also have $\alpha(e) = x$. So this it not possible and the assertion follows. □

Lemma 5.10 implies that we also have a set $\mathcal{H}(\mathcal{G})$ of pairwise isomorphic subgeometries of $\mathcal{G}$ consisting of elements from $\mathcal{G}^3$ and $\mathcal{G}^4$. If all connected components of Π are type Π_{22} then each $H \in \mathcal{H}(\mathcal{G})$ is isomorphic to $GQ(2,2)$. But so far we have not determined the isomorphism type for the case $3\Pi_{22}$.

For $q \in \mathcal{G}^4$, by B_q we denote the subgraph of B with vertices

$$V(B_q) := \{\Pi^y \mid \Pi^y \in V(B), \Pi^y \cap \Pi_q \neq \emptyset\}$$

and edges

$$E(B_q) := \{\{\Pi^1, \Pi^2\} \mid \Pi^i \in V(B_q), \text{ there are } e_i \in E(\Pi^i), i = 1, 2, \text{ such that } \alpha(e_1) = \alpha(e_2) \in res_{\mathcal{G}}(q)\}.$$

Notice that B_q is not necessarily the induced subgraph of B on $V(B_q)$.

Lemma 5.11 *The graph B is connected; in particular, $\Pi^1 \cong \Pi^2$ for any two connected components Π^1, Π^2 of Π.*

Proof: By (5.10) it suffices to show that B is connected. So let Π^1, Π^2 be two arbitrary vertices of B. Let $e_i \in E(\Pi^i)$ and set $x_i := \alpha(e_i)$, $i = 1, 2$. If $x_1 = x_2$ then $\{\Pi^1, \Pi^2\} \in E(B)$ and we are done. Otherwise, by connectedness of $\mathcal{G}$ there exists a path $y_1 := x_1, q_1, y_2, q_2, \ldots, y_k, q_k, y_{k+1} := x_2$ with $y_i \in \mathcal{G}^3$, $q_i \in \mathcal{G}^4$, and $y_i, y_{i+1} \in res_{\mathcal{G}}(q_i)$ for all i. Since the graph on Petersen subgraphs in Π_{q_1} in which two such subgraphs are adjacent if they are constructed from equivalent edges is connected there exists a path in B_{q_1} (and hence also in B) from Π^1 to any $\Pi^z \in V(B)$ with $\alpha(z) = y_2$. Now the assertion follows by induction on k. $\square$

The graph on Petersen subgraphs in Π_{q_1} mentioned in the above proof is isomorphic to the dual of the collinearity graph of the generalized $U_4(2)$-quadrangle. The latter is commonly known as the *Schläfli graph* and we will also adopt this name in the following.

To get a better understanding of the graph B it might be useful to consider also the "coloured" graph $\tilde{B}$ which we define as the graph with the same set of vertices as B and whose edges are indexed by the set $\mathcal{H}(\mathcal{G})$, i.e.,

$$E(\tilde{B}) := \left\{\{\Pi^1, \Pi^2\}_H \mid \{\Pi^1, \Pi^2\} \in E(B),\ H \in \mathcal{H}(\mathcal{G}), \beta_i^{-1}(H) \in \mathcal{H}(\mathcal{P}_i) \text{ for } i = 1, 2\right\}.$$

For $q \in \mathcal{G}^4$, $\tilde{B}_q$ then denotes the coloured graph with vertices $V(B_q)$ and edges

$$E(\tilde{B}_q) := \left\{\{\Pi^1, \Pi^2\}_H \mid \beta_i^{-1}(H) \cap res_{\mathcal{G}}(q) \neq \emptyset, i = 1, 2\right\}.$$

We will call a triangle $\{\Pi^1, \Pi^2, \Pi^3\} \subseteq B$ *short* if the three edges of the corresponding triangle in $\tilde{B}$ are all of the same colour, i.e., are indexed by the same subgeometry $H \in \mathcal{H}(\mathcal{G})$. Other triangles (if they exist) are called *long*. Observe that in the known (flag-transitive) examples all triangles are short.

The next little corollary is just the reformulation of (5.7) and (5.9) in terms of the graphs B and $\tilde{B}$.

Corollary 5.12 *If there exists a vertex $\Pi^1 \in B$ (resp. $\tilde{B}$) such that $\{\Pi^1, \Pi^1\} \in E(B)$ (resp. $\{\Pi^1, \Pi^1\}_H \in E(\tilde{B})$ for some $H \in \mathcal{H}(\mathcal{G})$) then $\Pi^1 \cong 3\Pi_{22}$.*

However, even in the case of Π_{22}, (5.7) and (5.9) do not exclude the possibility that the uncoloured graph underlying $\tilde{B}$ contains multiple edges, i.e., that there are $\Pi^1, \Pi^2 \in B$ and subgeometries $H_1, H_2 \in \mathcal{H}(\mathcal{G})$ such that $H_1 \neq H_2$ and $\{\Pi^1, \Pi^2\}_{H_i} \in E(\tilde{B})$ for both i.

For some of the results of subsections 5.1 and 5.2 we can get stronger versions or shorter proofs under the presence of a flag-transitive action on $\mathcal{G}$ (even without knowing the precise structure of the flag-transitive group acting). For the interested reader we include them in the Appendix A. But they are not needed in the rest of the paper.

5.3. Definition of the geometry $\mathcal{B}$

In this subsection we use the graph B to define a geometry $\mathcal{B}$. For the determination of $\mathcal{B}$ in Section 6 the following assumption will be useful.

(III) The intersection of any two different Schläfli subgraphs of B is either empty, or a vertex, or a triangle.

Let us first assume that all connected components of Π are isomorphic to the derived graph Π_{22} of the P-geometry $\mathcal{P}_{22}$ belonging to the Mathieu group M_{22}. By (5.6), for any $x \in \mathcal{G}^2$ and $\mathcal{P} := \mathcal{P}^x$ the corresponding P-geometry, we can identify the set $\mathcal{P}^3$ of objects of type 3 in $\mathcal{P}$ with the set $\{\Pi^x \cap \Pi_q \mid q \in \mathcal{G}^4, \Pi^x \cap \Pi_q \neq \emptyset\}$ or just with the set

$$\{q \in \mathcal{G}^4 \mid \Pi^x \cap \Pi_q \neq \emptyset\} = \{q \in \mathcal{G}^4 \mid res_{\mathcal{G}}(q) \cap V(\Pi^x) \neq \emptyset\}.$$

Furthermore, we have

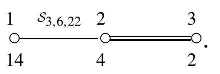
Lemma 5.13 *If* $\{\Pi^1, \Pi^2\}_H, \{\Pi^1, \Pi^{2'}\}_{H'} \in E(\tilde{B}_q)$ *and* $H \neq H'$ *then* $\Pi^2 \neq \Pi^{2'}$; *in particular*, $B_q \cong \tilde{B}_q$ *and both graphs are isomorphic to the Schläfli graph. In particular*, *all triangles in* B_q *are short.*

Proof: Let $\beta_1 : \mathcal{P}^1 \to \mathcal{G}$ be the map β defined for $\mathcal{P} = \mathcal{P}^1$. Then, in the considered situation, $\beta_1^{-1}(H^2) \cap E(\Pi^1 \cap \Pi_q)$ and $\beta_1^{-1}(H'^2) \cap E(\Pi^1 \cap \Pi_q)$ are different triples of pairwise opposite edges of $\Pi^1 \cap \Pi_q$. In $res_{\mathcal{G}}(q)$ we see that therefore $\Pi^2 \cap \Pi_q \neq \Pi^{2'} \cap \Pi_q$. So $\Pi^2 \neq \Pi^{2'}$.

Since B_q and $\tilde{B}_q$ are both isomorphic to the graph on the set of connected components of Π_q in which two such components are adjacent if they are constructed from α-equivalent edges contained in $res_{\mathcal{G}}(q)$, as already remarked in the proof of (5.11), we have the Schläfli graph. The last statement is just a property of that graph. $\square$

Let $\mathcal{B}$ be the rank 3 geometry whose objects of types 1, 2, 3 are respectively the Schläfli subgraphs, the short triangles, and the vertices of B with incidence defined by inclusion.

Lemma 5.14
(i) *$\mathcal{B}$ has the diagram*



(ii) *If B satisfies* (III) *then $\mathcal{B}$ satisfies hypothesis* (IV) *from Section* 6.

Proof: By definition, the diagram of $\mathcal{B}$ is a string and, for $B_q \in \mathcal{B}^1$, the diagram of $res_{\mathcal{B}}(B_q)$ follows from the fact that B_q is the Schläfli graph. For $\Pi^1 \in \mathcal{B}^3$, the elements of $res_{\mathcal{B}}(\Pi^1)$ are the Schläfli subgraphs and the short triangles through Π^1 in B. In the P-geometry $\mathcal{P}_1$ corresponding to Π^1 these can be identified with the set $\mathcal{P}_1^3$ of objects of type 3 of $\mathcal{P}_1$ and the set $\mathcal{H}(\mathcal{P}_1)$ of $GQ(2, 2)$-subgeometries of $\mathcal{P}_1$, i.e., with the pairs and the hexads of the Steiner system $\mathcal{S}(3, 6, 22)$, in such a way that a subgeometry $h \in \mathcal{H}(\mathcal{P}_1)$ contains an element $q \in \mathcal{P}_1^3$ (i.e., a hexad contains a pair) if and only if the triangle indexed by $\beta_1(h)$ in $\tilde{B}$ is contained in the Schläfli graph $\tilde{B}_q$. This gives $res_{\mathcal{B}}(\Pi^1)$ and (i) is shown.

For (ii) notice that (IV)(a) is just a reformulation of (III) in terms of $\mathcal{B}$ and that (IV)(b) follows from the construction of $\mathcal{B}$ and the results of the present section, especially (5.8). □

Now we assume that $\Pi^x \cong 3\Pi_{22}$ for all $x \in \mathcal{G}^2$. For a fixed connected component Π^x and $\mathcal{P}_x$ the corresponding P-geometry, let us call two elements $\Pi^x \cap \Pi_p, \Pi^x \cap \Pi_q \in \mathcal{P}_x^3$ *x-equivalent* and write $\Pi^x \cap \Pi_p \sim_x \Pi^x \cap \Pi_q$ if their images under the associated covering $\Pi^x \to \Pi_{22}$ are equal.

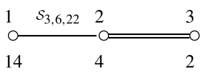
Lemma 5.15 *If* $\Pi^x \cap \Pi_p \sim_x \Pi^x \cap \Pi_q$ *then* $B_p = B_q$.

Proof: If $\Pi^x \cap \Pi_p \sim_x \Pi^x \cap \Pi_q$ then there exist exactly five subgeometries $H_x \in \mathcal{H}(\mathcal{P}_x)$ such that $\Pi^x \cap \Pi_p, \Pi^x \cap \Pi_q \in H_x^3$. Each of these subgeometries determines a triangle through Π^x in both graphs B_p, B_q. Hence the sets of neighbours of Π^x in B_p and B_q are the same. Furthermore, if Π^y is such a neighbour and $H_y \in \mathcal{H}(\mathcal{P}_y)$ is the subgeometry with $\beta_y(H_y) = \beta_x(H_x)$ for one of those H_x then $\Pi^y \cap \Pi_p$ and $\Pi^y \cap \Pi_q$ must be at maximal distance in the collinearity graph of H_y since the same holds for $\Pi^x \cap \Pi_p, \Pi^x \cap \Pi_q$, and H_x. This means that the images of $\Pi^y \cap \Pi_p$ and $\Pi^y \cap \Pi_q$ under the natural covering from $\Pi^y \to \Pi_{22}$ are also equal, i.e., $\Pi^y \cap \Pi_p \sim_y \Pi^y \cap \Pi_q$. So we can replace x by y and the assertion follows by the first part of the proof and connectivity of B_p and B_q. □

Notice that in spite of the fact that now $\Pi^x \cong 3\Pi_{22}$ for all x the short triangles through a given vertex Π^x of B can still be identified with the blocks of the Steiner system $\mathcal{S}(3, 6, 22)$ because now the corresponding subgeometries of $\mathcal{P}_x$ are isomorphic to $3GQ(2, 2)$. Furthermore, (5.15) implies that a similar statement also holds for the Schläfli subgraphs of B through Π^x, i.e., they can be identified with the pairs of $\mathcal{S}(3, 6, 22)$. So we can define the geometry $\mathcal{B}$ exactly as in the case of Π_{22} and the assertions of (5.14) will hold again.

6. Identification of $\mathcal{B}$ with $\mathcal{F}^T$

In this section we consider geometries $\mathcal{B}$ with diagram



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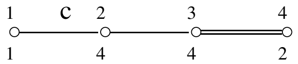
Again we do not assume the existence of any automorphisms of $\mathcal{B}$. The diagram of $\mathcal{B}$ is the same as that of the truncation $\mathcal{F}^T$ of the geometry $\mathcal{F} = \mathcal{F}(Fi_{22})$ with diagram



described in Section 3 and our goal is to identify $\mathcal{B}$ with $\mathcal{F}^T$. In contrast to the flag-transitive case, we do not know any characterization of geometries like $\mathcal{B}$ and, therefore, we will reconstruct the geometry $\mathcal{F}$ and then apply (3.1) resp. (3.2) to identify $\mathcal{B}$. Since the elements of type 1 in $\mathcal{F}$ are the 3-transpositions of the group Fi_{22} this means that we have to reconstruct the 3-transpositions of Fi_{22} just using information from the geometry $\mathcal{B}$.

By definition of $\mathcal{B}$, with any $x \in \mathcal{B}^3$ there is associated a Steiner system $\mathcal{S}_x \cong S(3, 6, 22)$ in such a way that the elements of $res(x) \cap \mathcal{B}^1$ can be identified with the pairs of different points of $\mathcal{S}_x$, the elements of $res(x) \cap \mathcal{B}^2$ can be identified with the blocks of $\mathcal{S}_x$, and $a \in res(x) \cap \mathcal{B}^1$ and $t \in res(x) \cap \mathcal{B}^2$ are incident in $\mathcal{B}$ if and only if the pair of $\mathcal{S}_x$ corresponding to a is contained in the block corresponding to t. In order to simplify notation we will sometimes identify elements of $\mathcal{B}^1 \cap res(x)$ and $\mathcal{B}^2 \cap res(x)$ with the corresponding pairs and blocks of $\mathcal{S}_x$, in particular, we will use notations like $a \cap b$, $a \cup b$, and $a \subseteq t$, where $a, b \in \mathcal{B}^1 \cap res(x)$, $t \in \mathcal{B}^2 \cap res(x)$.

The elements of $\mathcal{F}^1$ in the residue of an element from $\mathcal{F}^4 = (\mathcal{F}^T)^3$ correspond to the 22 points of the corresponding Steiner system. Clearly, for any $x \in \mathcal{B}^3$, we can define the 22 points in $\mathcal{S}_x$ locally but, to reconstruct $\mathcal{F}$, we must know when to identify points in different residues. For this it will be useful to assume the following.

(IV) (a) Any two different elements of type 1 in $\mathcal{B}$ are incident to at most one common element of type 2.

(b) If $t \in \mathcal{B}^2$, $x, y \in res(t) \cap \mathcal{B}^3$, and $a, b \in res(t) \cap \mathcal{B}^1$ then the pairs corresponding to a, b in $\mathcal{S}_x$ intersect in a point of $\mathcal{S}_x$ if and only if the same holds in $\mathcal{S}_y$.

For $t \in \mathcal{B}^2$, let $\sim_t$ be the relation defined on $\mathcal{B}^1 \cap res(t)$ by $a \sim_t b$ if $a \cap b$ is a point of $\mathcal{S}_x$ for some (and so for all by (IV)) $x \in res(t) \cap \mathcal{B}^3$. Set

$$\begin{aligned} Y_t &:= \{\{a, b\}_t \mid a, b \in \mathcal{B}^1 \cap res(t), a \sim_t b\}, \\ Y &:= \bigcup_{t \in \mathcal{B}^2} Y_t, \\ \sim &:= \bigcup_{t \in \mathcal{B}^2} \sim_t, \\ \text{and } Y_x &:= \bigcup_{t \in \mathcal{B}^2 \cap res(x)} Y_t \quad \text{for } x \in \mathcal{B}^3. \end{aligned}$$

Let Υ be the graph with vertices $V(\Upsilon) := Y$ in which two vertices $\{a, b\}_t$, $\{c, d\}_s$ are adjacent if $t, s \in res(x)$ for some $x \in \mathcal{B}^3$ and $a \cap b = c \cap d$ is the same point of $\mathcal{S}_x$. For $t \in \mathcal{B}^2$ let Υ_t be the subgraph of Υ induced on Y_t and for $x \in \mathcal{B}^3$ let Υ_x be the subgraph with vertices Y_x and edges all the pairs $\{a, b\}_t$, $\{c, d\}_s$, $t, s \in res(x)$, with $a \cap b = c \cap d$ in $\mathcal{S}_x$.

By the properties of $\mathcal{S}(3, 6, 22)$ and the diagram of $\mathcal{B}$ we have

Lemma 6.1

(i) *If $x \in \mathcal{B}^3$ then the cliques of maximal size in Υ_x are of size $\frac{21 \cdot 20}{2} = 210$, there are 22 such cliques, and they correspond bijectively to the points of the Steiner system $\mathcal{S}_x$.*

(ii) *If $t \in \mathcal{B}^2$ then the cliques of maximal size in Υ_t are of size $\frac{5 \cdot 4}{2} = 10$, there are 6 such cliques, and they correspond bijectively to the points of $\mathcal{S}_x$ contained in the block t for any $x \in res(t) \cap \mathcal{B}^3$.*

(iii) *If $x \in \mathcal{B}^3$ and $C \subseteq \Upsilon_x$ is a 210-clique corresponding to the point p of $\mathcal{S}_x$ then C is the disjoint union of 21 10-cliques $C_t \subseteq \Upsilon_t$, where $t \in res(x) \cap \mathcal{B}^2$ runs over the blocks of $\mathcal{S}_x$ containing p.*

We will call the cliques as in Lemma 6.1 *special cliques*, and for $i = 2, 3$, we denote by $\mathcal{C}^i$ the set of special cliques induced by elements of $\mathcal{B}^i$. We define maps $p : \mathcal{C}^2 \cup \mathcal{C}^3 \to \mathcal{B}^1$ and $b : \mathcal{C}^3 \to \mathcal{B}^2$ by

$$p(C) := \{a \in \mathcal{B}^1 \mid \{a, b\}_t \in V(C) \text{ for some } b \in \mathcal{B}^1, t \in \mathcal{B}^2\},$$
$$b(C) := \{t \in \mathcal{B}^2 \mid \{a, b\}_t \in V(C) \text{ for some } a, b \in \mathcal{B}^1\}.$$

Here, "p" stands for "pair" and "b" stands for "block", and the following holds.

Lemma 6.2

(i) *$|p(C)| = 5$ if $C \in \mathcal{C}^2$ and $|p(C)| = |b(C)| = 21$ if $C \in \mathcal{C}^3$.*

(ii) *If $C \in \mathcal{C}^i$ then there exists a unique $z \in \mathcal{B}^i$ such that $C \subseteq \Upsilon_z$. In particular, the element $t \in \mathcal{B}^2$ in the definition of $p(C)$ is uniquely determined if $C \in \mathcal{C}^2$.*

(iii) *If $C \in \mathcal{C}^3$ and $C_1, C_2 \in \mathcal{C}^2$ with $C_1, C_2 \subseteq C$ and $C_1 \neq C_2$ then $|p(C_1) \cap p(C_2)| = 1$.*

Proof: (i) and (iii) follow from well-known properties of $\mathcal{S}(3, 6, 22)$. If $C \in \mathcal{C}^2$ and $C \subseteq \Upsilon_t$ for some $t \in \mathcal{B}^2$ then by definition all vertices of C are of shape $\{a, b\}_t$ for suitable $a, b \in \mathcal{B}^1$. So t is uniquely determined by C and (ii) holds in this case.

Let $C \in \mathcal{C}^3$ and suppose $C \subseteq \Upsilon_x \cap \Upsilon_y$ for some $x, y \in \mathcal{B}^3$. Consider the map b. We have $|b(C)| = 21$ and $b(C) \subseteq res(x) \cap res(y)$. Further, if $a \in p(C)$ then $|b(C) \cap res(a)| = 5$ since a is contained in 5 blocks of $\mathcal{S}_x$ as well as of $\mathcal{S}_y$. So x and y are incident to more than one common element of type 2 in $res(a)$ and the structure of $res(a)$ implies $x = y$. $\square$

By (6.2)(ii) there exists a well-defined map $\gamma : \mathcal{C}^i \to \mathcal{B}^i$, $i = 2, 3$, such that $\gamma(C) := z$ where z is the unique element of $\mathcal{B}^i$ with $C \subseteq \Upsilon_z$.

Let $\mathcal{C}$ be the graph with vertices $V(\mathcal{C}) := \mathcal{C}^3$ and edges

$$E(\mathcal{C}) := \{\{C_1, C_2\} \mid C_i \in \mathcal{C}^3, C_1 \cap C_2 \in \mathcal{C}^2\}$$

and let B be the graph with vertices $V(B) := \mathcal{B}^3$ and edges

$$E(B) := \{\{x, y\} \mid x, y \in \mathcal{B}^3, x, y \in res(t) \text{ for some } t \in \mathcal{B}^2\}.$$

For $a \in \mathcal{B}^1$ and $i = 2, 3$, we set

$$\begin{aligned}\mathcal{C}_a^i &:= \{C \in \mathcal{C}^i \mid a \in p(C)\}\\ &= \{C \in \mathcal{C}^i \mid \{a, b\}_t \in C \text{ for some } b \in \mathcal{B}^1, t \in \mathcal{B}^2\}.\end{aligned}$$

We define $\mathcal{C}_a$ as the subgraph of $\mathcal{C}$ with vertices $V(\mathcal{C}_a) := \mathcal{C}_a^3$ and edges

$$E(\mathcal{C}_a) := \left\{\{C_1, C_2\} \mid C_1 \cap C_2 \in \mathcal{C}_a^2\right\}$$

and B_a as the subgraph of B with $V(B_a) := res(a) \cap \mathcal{B}^3$ and

$$E(B_a) := \{\{x, y\} \mid x, y \in res(t) \text{ for some } t \in \mathcal{B}^2 \cap res(a)\}.$$

Lemma 6.3 *Every 2-path and every triangle of $\mathcal{C}$ is contained in a subgraph $\mathcal{C}_a$ for some $a \in \mathcal{B}^1$ and every triangle consists of three 210-cliques which intersect in the same 10-clique.*

Proof: Let (C_1, C_2, C_3) be a 2-path and set $D_i := C_i \cap C_{i+1}$ for $i = 1, 2$. Then $D_1, D_2 \subseteq C_2$ and by (6.2) (iii) there is some $a \in p(D_1) \cap p(D_2)$. So there are $b_i \in \mathcal{B}^1$, $t_i \in \mathcal{B}^2$ such that $\{a, b_i\}_{t_i} \in D_i$, $i = 1, 2$. But this just means that the path (C_1, C_2, C_3) is a path in the subgraph $\mathcal{C}_a$.

Now assume that also $D_3 := C_1 \cap C_3 \in \mathcal{C}^2$. Then by the same argument as before there is also some $a_1 \in p(D_1) \cap p(D_3)$. Hence $a, a_1 \in p(C_i)$ for $i = 1, 2, 3$. If $a = a_1$ then $\{a, b_3\}_{t_3} \in D_3$ for suitable b_3, t_3, and so the triangle $\{C_1, C_2, C_3\}$ is a triangle of the graph $\mathcal{C}_a$. Furthermore, $\{\gamma(C_1), \gamma(C_2), \gamma(C_3)\}$ is a triangle of B_a with edges determined by t_1, t_2, t_3. From the structure of $res(a)$ it follows that B_a is isomorphic to the Schläfli graph and by the properties of that graph we must have $t_1 = t_2 = t_3$. Then also $D_1 = D_2 = D_3$ and $\{C_1, C_2, C_3\}$ is as stated.

If $a \neq a_1$ then in each of the Steiner systems $\mathcal{S}_{\gamma(C_i)}$, a and a_1 correspond to two different pairs which intersect in the point determined by the clique C_i. Therefore, in each of the Steiner systems they must be contained in a common block, i.e., for each $i = 1, 2, 3$, there exists an element $t_i \in \mathcal{B}^2$, such that $\{a, a_1\}_{t_i} \in C_i$. Since $a \neq a_1$, by (IV) (a) we must have $t_1 = t_2 = t_3$ in which case $C_1 \cap C_2 \cap C_3$ must be the unique 10-clique determined by t_1 and so the triangle $\{C_1, C_2, C_3\}$ is a triangle in both graphs $\mathcal{C}_a$ and $\mathcal{C}_{a_1}$. □

Lemma 6.4

(i) *The map $\gamma : \mathcal{C}^2 \cup \mathcal{C}^3 \to \mathcal{B}^2 \cup \mathcal{B}^3$ induces a covering from $\mathcal{C}$ to B.*

(ii) *If $a \in \mathcal{B}^1$ then $\mathcal{C}_a$ is the disjoint union of two Schläfli graphs.*

Proof: It is straightforward from the definitions that γ maps adjacent vertices of $\mathcal{C}$ onto adjacent vertices of B and adjacent vertices of $\mathcal{C}_a$ onto adjacent vertices of B_a if $a \in \mathcal{B}^1$. So (i) holds, and if $a \in \mathcal{B}^1$, then the restriction of γ to $\mathcal{C}_a$ induces a covering onto B_a. From the structure of $res(a)$ it follows that B_a is isomorphic to the Schläfli graph.

Further, if $x \in res(a) \cap \mathcal{B}^3$ then there are precisely two special cliques $C_1, C_2 \subseteq \Upsilon_x$ with $C_1, C_2 \in \mathcal{C}_a$ and we have $C_1 \cap C_2 = \emptyset$. Now any of the 5 elements $t \in \mathcal{B}^2 \cap res(x) \cap res(a)$ determines a unique special subclique $C_{t,i} \subseteq C_i$, $i = 1, 2$, and also two neighbours y, z of x in B_a. Then $C_{t,1}, C_{t,2}$ are special subcliques of size 10 of the graphs Υ_y and Υ_z. So each of them is contained in unique special 210-clique in each of Υ_y and Υ_z and by the properties of $\mathcal{S}(3, 6, 22)$ and (6.2) (i) those special 210-cliques are all pairwise different. In particular, the 10 neighbours of x in B_a determine $2 \cdot 10 = 20$ special 210-cliques which divide into two disjoint sets of 10 neighbours of each of C_1, C_2 in $\mathcal{C}_a$.

From (6.3) it follows that every triangle or quadrangle in $\mathcal{C}_a$ maps isomorphically onto a triangle or quadrangle, respectively, in B_a. Hence there are exactly $\frac{10 \cdot 8}{5} = 16$ vertices at distance 2 from C_i in $\mathcal{C}_a$ and no vertices at distance 3. This proves the lemma. □

Lemma 6.5 *If $S_a \subseteq \mathcal{C}_a$ and $S_b \subseteq \mathcal{C}_b$ are two different Schläfli subgraphs of $\mathcal{C}$ (where $a, b \in \mathcal{B}^1$) then $|S_a \cap S_b| \in \{0, 1, 3\}$. Moreover, $|S_a \cap S_b| = 3$ holds iff $C_1 \cap C_2 \cap C_3 \in \mathcal{C}_a^2 \cap \mathcal{C}_b^2$ where $\{C_1, C_2, C_3\} = S_a \cap S_b$.*

Proof: Suppose there are $C_x, C_y \in \mathcal{C}^3$ (corresponding via γ to $x, y \in \mathcal{B}^3$) such that $C_x, C_y \in S_a \cap S_b$.

Then by (6.4) $a \neq b$ and by definition of $\mathcal{C}_a$, $\mathcal{C}_b$, we have $a, b \in p(C_x) \cap p(C_y)$. So a and b correspond to intersecting pairs in both Steiner systems $\mathcal{S}_x$ and $\mathcal{S}_y$. Notice that the point $a \cap b$ is uniquely determined inside $\mathcal{S}_x$ respectively $\mathcal{S}_y$ through the clique C_x respectively C_y. In particular, there exist $t \in res(x) \cap \mathcal{B}^2$ and $s \in res(y) \cap \mathcal{B}^2$ such that $\{a, b\}_t \in C_x$, $\{a, b\}_s \in C_y$. Since $t, s \in res(a) \cap res(b)$ and $a \neq b$, (IV) (a) yields $t = s$ and so $C_t \subseteq C_x \cap C_y$ where $C_t \in \mathcal{C}^2$ is the unique special 10-clique containing $\{a, b\}_t$. As C_t is contained in exactly three special 210-cliques which by construction must belong to S_a and S_b we have $|S_a \cap S_b| \geq 3$ and $S_a \cap S_b$ contains a triangle of both of them.

Arguing in the same way for any two vertices in the intersection and using the facts that maximal cliques in the Schläfli graph are of size 3 and that there is only one class of such cliques (namely those determined by elements of $\mathcal{B}^2$) we see that the intersection cannot be larger than 3 and the lemma is proved. □

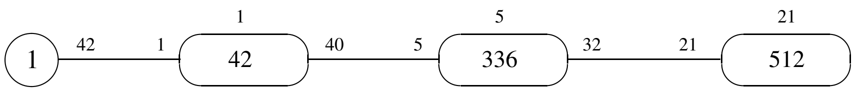
Now let $\mathcal{C}_0$ be a connected component of $\mathcal{C}$, $\mathcal{C}_0^3 := V(\mathcal{C}_0)$,

$$\mathcal{C}_0^2 := \left\{C \in \mathcal{C}^2 \mid C \subseteq C' \text{ for some } C' \in \mathcal{C}_0^3\right\}$$

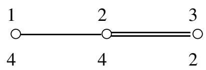
Let $\mathcal{H}_0$ be the geometry whose objects of types 3, 2, 1 are respectively the vertices, triangles, and the connected components of $\mathcal{C}_0 \cap \mathcal{C}_a$, $a \in \mathcal{B}^1$, (which are isomorphic to the Schläfli graph) with incidence defined as usual by inclusion.

The crucial point of the identification $\mathcal{B} \cong \mathcal{F}^T$ is the following

Proposition 6.6
(i) $\mathcal{C}_0$ *has intersection array*



(ii) $\mathcal{H}_0$ *has the diagram*



and $\mathcal{H}_0$ *is isomorphic to the polar space of the unitary group* $U_6(2)$.

Proof: By definition of $\mathcal{H}_0$ the diagram of $\mathcal{H}_0$ is a string and the residue of any element of type 1 is the generalized $U_4(2)$-quadrangle. If $C \in \mathcal{H}_0^3$ then $res_{\mathcal{H}_0}(C)$ can be identified with the geometry whose objects are the sets $p(C)$ and $b(C)$ and whose incidence relation is inherited from $\mathcal{B}$. By the properties of $\mathcal{S}_{\gamma(C)}$ this geometry is isomorphic to the projective plane over $GF(4)$. So the diagram of $\mathcal{H}_0$ follows.

By (6.5) any two different elements of $\mathcal{H}_0^1$ are incident to at most one common element of $\mathcal{H}_0^2$. In terms of [12, Chapter 7] this means that the "Intersection Property" holds in $\mathcal{H}_0$ or, equivalently, that $\mathcal{H}_0$ "has a good system of lines". We can therefore use [12, (7.38), (7.39)] to deduce that $\mathcal{H}_0^2$ is a polar space. Then, by the diagram, it must be the polar space $\mathcal{P}(U_6(2))$ of the group $U_6(2)$. This implies that (i) also holds and we are finished.

For convenience of the reader who is unfamiliar with the notation in [12], in the appendix we also provide an elementary proof calculating first the intersection array of $\mathcal{C}_0^2$ and then deducing the isomorphism with $\mathcal{P}(U_6(2))$ from it. $\square$

Notice that instead of the triangles in $\mathcal{C}_0$ we could also take the elements of $\mathcal{C}_0^2$ as objects of type 2 in $\mathcal{H}_0$. So the map γ defined after (6.2) induces a map $\gamma : \mathcal{H}_0^2 \to \mathcal{B}^2$, $\gamma : \mathcal{H}_0^3 \to \mathcal{B}^3$. Let us extend γ to $\mathcal{H}_0^1$ by setting $\gamma(S) := a$ where a is the (unique) element in $\mathcal{B}^1$ such that S is a connected component of $\mathcal{C}_a$.

Lemma 6.7 *The map* $\gamma : \mathcal{H}_0 \to \mathcal{B}$ *is injective*; *in particular*, $\mathcal{B}$ *contains a class of subgeometries isomorphic to the polar space of* $U_6(2)$.

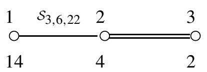
Proof: Suppose first, $\gamma|_{\mathcal{H}_0^1}$ is not injective; i.e., there exists some $a \in \mathcal{B}^1$ such that both connected components of $\mathcal{C}_a$ are contained in $\mathcal{C}_0$. Let us denote them by S_1, S_2. Clearly $S_1 \cap S_2 = \emptyset$. But by the structure of the collinearity graph of $\mathcal{H}_0$ there must be some $b \in \mathcal{B}^1$ and a connected component S of $\mathcal{C}_b$ with $S_1 \cap S$, $S_2 \cap S \in \mathcal{H}_0^2$. Now $\gamma(S_1 \cap S)$ and $\gamma(S_2 \cap S)$

must be different elements of $\mathcal{B}^2$ which are both incident to a and b and this contradicts (IV)(a). So $\gamma|_{\mathcal{H}_0^1}$ is injective and for each $a \in \mathcal{B}^1$ there is at most one connected component of $\mathcal{C}_a$ contained in $\mathcal{C}_0$.

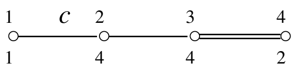
Now let $x \in \mathcal{B}^3$ and let $C_1, C_2 \in \mathcal{C}^3$ be two different 210-cliques in $\mathcal{C}$ with $\gamma(C_1) = \gamma(C_2) = x$. Then the subsets $p(C_1)$ and $p(C_2)$ of $res(x) \cap \mathcal{B}^1$ correspond via p_x to two different collections of 21 pairs of $\mathcal{S}_x$ containing a common point. So we see in $\mathcal{S}_x \cong S(3, 6, 22)$ that $|p(C_1) \cap p(C_2)| = 1$. Let $a \in p(C_1) \cap p(C_2)$. Then $C_1, C_2 \in \mathcal{C}_a$. It follows from the proof of (6.4) that the restriction of γ to a connected component of $\mathcal{C}_a$ is injective. Hence C_1 and C_2 must be in different connected components of $\mathcal{C}_a$ and only one of them can belong to $\mathcal{H}_0^1$ by the first paragraph.

So we have shown that $\gamma|_{\mathcal{H}_0^1}$ and $\gamma|_{\mathcal{H}_0^3}$ are both injective and it is easy to see that this implies injectivity of $\gamma|_{\mathcal{H}_0^2}$. Since the definition of γ also implies that it is an incidence preserving map the assertion follows. $\square$

Proposition 6.8 *Let $\mathcal{B}$ be a geometry with diagram*



and assume that $\mathcal{B}$ satisfies hypothesis (IV). *Then $\mathcal{B} \cong \mathcal{F}^T$ where $\mathcal{F}$ is the geometry with diagram*



related to the sporadic simple group Fi_{22}.

Proof: Let $\mathcal{H}$ be the rank 4 geometry whose objects of type 1 are all the images $\gamma(\mathcal{H}_0)$ (where for a connected component $\mathcal{C}_0$ of $\mathcal{C}$, $\mathcal{H}_0$ denotes as above the corresponding $U_6(2)$-geometry) and whose objects of type i, $i > 1$, are just the sets $\mathcal{H}^i := \mathcal{B}^{i-1}$. The incidence between objects of types i and j is the same as in $\mathcal{B}$ if both $i, j > 1$. An element $\gamma(\mathcal{H}_0) \in \mathcal{H}^1$ is incident to some $z \in \mathcal{H}^i$, $i > 1$, if $z \in \gamma(\mathcal{H}_0)$. Since any element $a \in \mathcal{B}^1$ corresponds to two Schläfli subgraphs in $\mathcal{C}$, by (6.6) we have $a \in \gamma(\mathcal{H}_0)$ for exactly two geometries $\mathcal{H}_0$. As by definition the diagram of $\mathcal{H}$ must be a string we get that $\mathcal{H}$ has the desired diagram. In other words, $\mathcal{H}$ is an extension of the polar space of $U_6(2)$. So we can use (3.2) to identify $\mathcal{H}$ with $\mathcal{F} = \mathcal{F}(Fi_{22})$. $\square$

7. The group-free characterizations

In this final section we establish our group-free characterizations of the geometries $\mathcal{E}$ and $\mathcal{G}$. In particular, we prove a group-free version of Theorem 1. Since we would like to apply (6.8) we need a uniform way to construct the geometries $\mathcal{G}$ and $\mathcal{E}$ from $\mathcal{F} = \mathcal{F}(Fi_{22})$ or from its truncation $\mathcal{F}^T$.

7.1. From $\mathcal{F}$ to $\mathcal{G}$

In this subsection we assume that $\mathcal{F} = \mathcal{F}(Fi_{22})$ and we reconstruct $\mathcal{G}(Fi_{22})$ from $\mathcal{F}$. Then we state and prove Theorem 2.

Recall that the objects of $\mathcal{F}$ are the vertices, the edges, certain 6-cliques, and the maximal cliques of the 3-transposition graph A of the group $G = Fi_{22}$. In particular, the set $\mathcal{F}^1$ of objects of type 1 in $\mathcal{F}$ can be identified with the conjugacy class of involutions in G possessing the property that the order of the product of any two of them is 1, 2, or 3.

By [1] G contains a unique conjugacy class of subgroups isomorphic to $2^6.Sp_6(2)$. Let $H \leq G$ be such a subgroup. Then H has a unique conjugacy class, say H^1, of length 126 consisting of involutions which are 3-transpositions (i.e. $H^1 \subseteq \mathcal{F}^1$). If $x \in H^1$ then $H_x = C_H(x) \cong 2^{1+4}.2^{1+4}.Sp_4(2)$. In particular, $|O_2(H) : O_2(H) \cap H_x| = 2$. Let H^2 be the set of orbits of $O_2(H)$ on H^1. Then $|H^2| = 63$, $|O| = 2$ for each $O \in H^2$, H acts transitively on H^2 with kernel $O_2(H)$ and stabilizer $H_O \cong 2^6.2^{1+4}.Sp_4(2)$, and the elements of H^2 correspond bijectively to the points of the 6-dimensional symplectic polar space or, equivalently, to the transvections in the symplectic group $Sp_6(2) \cong H/O_2(H)$. If $g \in O_2(H)$ and $\{x, x^g\} \in H^2$ then $1 \neq xx^g = [x, g] \in O_2(H)$. So the order of xx^g is 2 and $\{x, x^g\}$ is an edge of A. Hence $H^2 \subseteq \mathcal{F}^2 = (\mathcal{F}^T)^1$.

Let $\mathcal{H}$ be the graph on H^2 in which two orbits $O_1, O_2 \in H^2$ are adjacent if the corresponding transvections in $Sp_6(2)$ commute, i.e., if the corresponding points of the symplectic polar space are perpendicular. In this case, $O_1 \cup O_2$ is contained in a 6-clique of A which is an object of type 3 in $\mathcal{F}$ (of type 2 in $\mathcal{F}^T$). So $\mathcal{H}$ is a subgraph of the graph, say A^T, which we define as the graph on the set of objects of $\mathcal{F}^2 = (\mathcal{F}^T)^1$ in which two such objects are adjacent if they are incident to a common element of $\mathcal{F}^3 = (\mathcal{F}^T)^2$. Since two transvections of $Sp_6(2)$ which correspond to non-perpendicular points generate a subgroup isomorphic to Σ_3 we see that any two orbits $O_1, O_2 \in H^2$ whose union is contained in a clique of A must correspond to perpendicular points. Hence $\mathcal{H}$ is the induced subgraph of A^T on H^2.

Let $\mathcal{H}^v$ be the set of vertices of $\mathcal{H}$, $\mathcal{H}^l$ the set of triangles corresponding to lines, and $\mathcal{H}^p$ the set of 7-cliques corresponding to planes of the symplectic polar space. Then the cliques in $\mathcal{H}^p$ are the cliques of maximal size in $\mathcal{H}$ and the triangles in $\mathcal{H}^l$ can be distinguished from other triangles in $\mathcal{H}$ by the fact that the transvections corresponding to a triangle in $\mathcal{H}^l$ generate a fours group while other triangles will generate an elementary abelian subgroup of order 8 in $Sp_6(2)$. The 3-transpositions of G in the corresponding orbits of $O_2(H)$ on A will generate subgroups of H isomorphic to 2^{2+2}, 2^{3+3}, respectively.

Let $\mathcal{G}^1$ be the set of all subgraphs like $\mathcal{H}$ which can be obtained varying H over the conjugacy class of $2^6 Sp_6(2)$-subgroups of G, set

$$\mathcal{G}^2 := \bigcup_{\mathcal{H} \in \mathcal{G}^1} \mathcal{H}^p,$$

$$\mathcal{G}^3 := \bigcup_{\mathcal{H} \in \mathcal{G}^1} \mathcal{H}^l,$$

$$\mathcal{G}^4 := \bigcup_{\mathcal{H} \in \mathcal{G}^1} \mathcal{H}^v = (\mathcal{F}^T)^1 = \mathcal{F}^2,$$

and define an incidence relation on $\mathcal{G}$ by inclusion. It follows from the remark preceeding the definition of $\mathcal{G}$ that if $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{G}^1$ and $C \in \mathcal{G}^i$, $i > 1$, such that $C \subseteq \mathcal{H}_1 \cap \mathcal{H}_2$ then $C \in H_i^x$ for $i = 1, 2$ and a suitable $x \in \{v, l, p\}$. This implies that the diagram of $\mathcal{G}$ is a string, that the residue of any element $\mathcal{H} \in \mathcal{G}^1$ is isomorphic to 6-dimensional symplectic polar space, and that the stabilizer H of $\mathcal{H}$ acts flag-transitively on it.

Let $C \in \mathcal{H}^p$ be a clique of size 7 in $\mathcal{H}$. Then C is an element of type 2 in the residue of $\mathcal{H}$ in $\mathcal{G}$ and $H_C \cong 2^6.2^6.L_3(2)$. Let $C_0 := \bigcup_{O \in C} O$. Then C_0 is a clique of size 14 in A. So it is contained in a unique maximal clique C_A in A, i.e., it corresponds to a unique maximal element of $\mathcal{F}^4 = (\mathcal{F}^T)^3$, and therefore $H_C \leq G_{C_A}$. Recall that $G_{C_A} \cong 2^{10}.M_{22}$. Since H_C does not possess a trivial composition factor on $O_2(H_C)$ we see that $|H_C \cap O_2(G_{C_A})| = 2^9$. Furthermore, the 3-transpositions in C_A generate the elementary abelian group $O_2(G_{C_A})$, so $O_2(G_{C_A}) \leq G_C$. On the other hand, $O_2(G_{C_A})$ is an irreducible G_{C_A}-module, so $G_{C_A} \not\leq G_C$. Since $2^3.L_3(2)$ is a maximal subgroup of M_{22} this implies $G_C = O_2(G_{C_A})H_C$ and $|G_C : H_C| = 2$. Let $g \in G_C \backslash H_C$ and let $\mathcal{H}^g$ be the subgraph corresponding to H^g. Then C is also in $(\mathcal{H}^g)^p$. Furthermore, if $C \subseteq \mathcal{H}^f$ for some $f \in G$ then by the above we must have $H_C^f \leq G_C$ and so $H^f = H$ or H^g. Hence C is incident to exactly two objects from $\mathcal{G}^1$ and these two objects are conjugate by an element in G_C.

Notice that together with the flag-transitivity of stabilizers of objects from $\mathcal{G}^1$ on the corresponding residues this implies flag-transitivity of G on $\mathcal{G}$.

Let $O \in \mathcal{H}^v \subseteq \mathcal{G}^4 = \mathcal{F}^2$. Then $H_O \cong 2^6.2^{1+4}.Sp_4(2)$ and $G_O \cong (2 \times 2^{1+8}.U_4(2))2$. From the action of H_O on $O_2(H_O)$ and $O_2(G_O)$ we see that $O_2(G_O) \leq H_O$ and $H_O/O_2(G_O) \cong 2 \times Sp_4(2)$. Considering the isomorphism $U_4(2) \cong \Omega_6^-(2)$ this means that the objects of $\mathcal{G}^1 \cap res_{\mathcal{G}}(O)$ correspond to the nonsingular points in the 6-dimensional orthogonal $GF(2)$-space of minus type on which $G_O/O_2(G_O)$ acts. Let $C \in \mathcal{H}^v$ be such that $O \in C$, i.e., $C \in res_{\mathcal{G}}(O)$. Then $H_O \cap H_C \cong 2^6.[2^8].\Sigma_3$, $(H_O \cap H_C)/O_2(G_O) \cong 2 \times (2 \times \Sigma_4)$, and the images of the 3-transpositions in the orbits in C generate the fours group $Z((H_O \cap H_C)/O_2(G_O))$. Since a subgroup $2 \times (2 \times \Sigma_4)$ of $O_6^-(2) \cong \Omega_6^-(2).2$ is contained in a subgroup $2 \times 2^{1+4}.\Sigma_3$ we see that $(G_O \cap G_C)/O_2(G_O) \cong 2 \times 2^{1+4}.\Sigma_3$. So $|G_O \cap G_C : H_O \cap H_C| = 4$ and there are exactly 4 elements in $res_{\mathcal{G}}(C) \cap \mathcal{G}^1$. Since $|res(C) \cap res(\mathcal{H}) \cap \mathcal{G}^2| = 3$ any of these 4 elements are incident to a common element of type 2 in $res(C)$. By transitivity the same should hold for any element of type 1 and so the diagram of $\mathcal{G}$ follows.

Now we can state and prove Theorem 2.

Theorem 2 *Let $\mathcal{G}$ be a $c.C_3^*$-geometry with $c.C_2$-residues belonging to $U_4(2)$ and satisfying condition (II). Assume that the graph B defined from $\mathcal{G}$ as in Section 5 satisfies (III). Then $\mathcal{G}$ is either the $c.C_3^*$-geometry belonging to Fi_{22} or its triple cover belonging to $3 \cdot Fi_{22}$.*

Proof: By the results of Sections 5 and assumption we can construct a geometry $\mathcal{B}$ from $\mathcal{G}$ such that $\mathcal{B}$ satisfies (IV). By Proposition 6.8 we have that $\mathcal{B} \cong \mathcal{F}^T$ for $\mathcal{F} = \mathcal{F}(Fi_{22})$. In particular, there exists a flag-transitive action of the group $G = Fi_{22}$ on $\mathcal{B}$.

Construct a $c.C_3^*$-geometry $\tilde{\mathcal{G}}$ from $\mathcal{B}$ as described above. Then $\tilde{\mathcal{G}} \cong \mathcal{G}(Fi_{22})$ and G acts flag-transitively on $\tilde{\mathcal{G}}$. Furthermore, by definition of $\tilde{\mathcal{G}}$ the objects of type 4 in $\tilde{\mathcal{G}}$ are just the objects of type 1 in $\mathcal{B}$, i.e., the Schläfli subgraphs in the graph B. Recall from Section 5 the graph Π and its relation with B. If all connected components of Π are isomorphic to Π_{22} then each Schläfli subgraph of B corresponds to a unique object in $\mathcal{G}^4$ while in the case of $3\Pi_{22}$ by (5.15) there are three such objects. In any case, there exists a well-defined surjection

$$\begin{aligned} \varphi : \mathcal{G}^4 &\to \mathcal{B}^1 \to \tilde{\mathcal{G}}^4 \\ q &\to B_q \to \tilde{q} \end{aligned}$$

which is bijective in the case of Π_{22} and has fibers of size three otherwise. Let us show that φ extends to a morphism of geometries $\varphi : \mathcal{G} \to \tilde{\mathcal{G}}$.

If $p, q \in \mathcal{G}^4$, $p \neq q$, are incident to a common element $t \in \mathcal{G}^3$ then $B_t := B_p \cap B_q$ is a short triangle in B. So B_p and B_q are incident to a common element of type 2 in $\mathcal{B}$. Hence $\{\tilde{p}, \tilde{q}\} \in E(D)$ (where $D \cong A^T$ denotes the graph on $\mathcal{B}^1$ defined as A^T) and so $\tilde{p}$, $\tilde{q}$ are incident to a common element $\tilde{t} \in \tilde{\mathcal{G}}^3$. By (II)(c) an element $t \in \mathcal{G}^3$ is uniquely determined by any pair of elements in $\mathcal{G}^4 \cap res_{\mathcal{G}}(t)$ and by flag-transitivity the same holds in $\tilde{\mathcal{G}}$. So the above implies that φ extends to a map

$$\varphi : \mathcal{G}^3 \to \tilde{\mathcal{G}}^3$$

which preserves the incidence relation between objects of types 3 and 4.

Since the $c.C_3$-diagram implies that objects of types 1 or 2 are uniquely determined by the sets of objects of types 3 and 4 in their residues we can identify any $x \in \mathcal{G}^1 \cup \mathcal{G}^2$ with the set $res_{\mathcal{G}}(x) \cap (\mathcal{G}^3 \cup \mathcal{G}^4)$ and in this way extend φ to an incidence preserving map from $\mathcal{G}$ onto $\tilde{\mathcal{G}}$ whose restriction to any residue will be an isomorphism. So φ is indeed a morphism of geometries. Now $|\mathcal{G}^4| = 3^j |\tilde{\mathcal{G}}^4|$ with $j \in \{0, 1\}$ and by (II)

$$|\mathcal{G}^4| \cdot |res_{\mathcal{G}}(x) \cap \mathcal{G}^i| = |\mathcal{G}^i| \cdot |res_{\mathcal{G}}(y) \cap \mathcal{G}^4|$$

for $i = 1, 2, 3$. So calculating the number of objects of each type in $\mathcal{G}$ and $\tilde{\mathcal{G}}$ we see that φ is either an isomorphism or has fibers of size 3. Since by (3.3) the universal cover of $\mathcal{G}(Fi_{22})$ is $\mathcal{G}(3 \cdot Fi_{22})$ this proves the theorem. □

7.2. *From $\mathcal{G}$ to $\mathcal{E}$*

In this subsection we assume that $\mathcal{G} = \mathcal{G}(Fi_{22})$ or $\mathcal{G}(3 \cdot Fi_{22})$ and we reconstruct the geometry $\mathcal{E}(Fi_{22})$ respectively $\mathcal{E}(3 \cdot Fi_{22})$ from $\mathcal{G}$. Then we prove Theorem 3.

Let Φ be the collinearity graph of $\mathcal{G}$, i.e., the graph with vertices $V(\Phi) = \mathcal{G}^1$ and edges $E(\Phi) = \mathcal{G}^2$. Then the elements of types 4 and 5 in $\mathcal{E}$ are just the edges and vertices of Φ; in other words

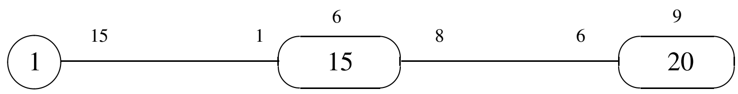
$$\mathcal{E}^5 = \mathcal{G}^1 \quad \text{and} \quad \mathcal{E}^4 = \mathcal{G}^2.$$

For $x \in \mathcal{G}^3 \cup \mathcal{G}^4$ let Φ_x be the subgraph of Φ with $V(\Phi_x) = res(x) \cap \mathcal{G}^1$ and $E(\Phi_x) = res(x) \cap \mathcal{G}^2$.

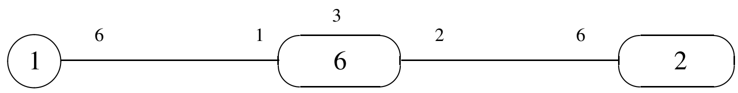
If $x \in \mathcal{G}^3$ then Φ_x is the complete graph on 4 vertices and so Φ_x contains exactly 4 triangles. For $p \in res(x) \cap \mathcal{G}^1$ let $\Phi_{x,p}$ denote the triangle in Φ_x which does not contain the vertex p. Then we define

$$\mathcal{E}^3 := \{\Phi_{x,p} \mid x \in \mathcal{G}^3,\ p \in res(x) \cap \mathcal{G}^1\}.$$

If $x \in \mathcal{G}^4$ then (as $res(x)$ belongs to $U_4(2)$) Φ_x has the intersection array



If $p, q \in res(x) \cap \mathcal{G}^1$ and p, q are at distance two in Φ_x then p and q have exactly 6 common neighbours in Φ_x and there is exactly one other vertex $r \in \Phi_x$ which is at distance two from both p and q and adjacent to all their common neighbours. Let $\Phi_{x,p,q,r}$ be the subgraph of Φ_x induced on p, q, r and their 6 common neighbours in Φ_x. Then $\Phi_{x,p,q,r}$ is isomorphic to the complete 3-partite graph $K_{3,3,3}$ which has intersection array



Let

$$\mathcal{E}^2 := \{\Phi_{x,p,q,r} \mid x \in \mathcal{G}^4,\ p, q, r \in res(x) \cap \mathcal{G}^1,$$
$$p, q, r \text{ are pairwise non-adjacent vertices of } \Phi_x\}.$$

Finally, the objects in $\mathcal{E}^1$ are certain connected components on the subgraphs of Φ fixed by an outer involution of $Fi_{22}.2$ and their stabilizer in G is isomorphic to $\Omega_8^+(2) : \Sigma_3$ of G and the incidence relation on $\mathcal{E}$ is again defined by inclusion.

We leave it to the reader to verify for himself that $\mathcal{E}$ is as desired and that G acts flag-transively on $\mathcal{E}$ and we turn to the proof of Theorem 3.

Theorem 3 *Let $\mathcal{E}$ be a $c.F_4(1)$-geometry satisfying (I). Suppose that the geometry $\mathcal{G}$ constructed from $\mathcal{E}$ as in Section 4 satisfies the conditions of Theorem 2. Then $\mathcal{E} \cong \mathcal{E}(Fi_{22})$ or $\mathcal{E}(3 \cdot Fi_{22})$.*

Proof: By Theorem 2 we have that $\mathcal{G} \cong \mathcal{G}(Fi_{22})$ or $\mathcal{G}(3 \cdot Fi_{22})$. In particular, there exists a flag-transitive action of the group G on $\mathcal{G}$ where $G = Fi_{22}$ or $3 \cdot Fi_{22}$.

Construct a $c.F_4(1)$-geometry $\tilde{\mathcal{E}}$ from $\mathcal{G}$ as described above. Then $\tilde{\mathcal{E}} \cong \mathcal{E}(Fi_{22})$ or $\mathcal{E}(3 \cdot Fi_{22})$ and G acts flag-transitively on $\tilde{\mathcal{E}}$. Furthermore, by definition of $\tilde{\mathcal{E}}$ the objects of types 4 and 5 in $\tilde{\mathcal{E}}$ are just the objects of types 2 and 1 in $\mathcal{G}$, i.e., the edges and the vertices of the graph Φ, and the incidence relation between them is the one inherited from $\mathcal{G}$. Since the same holds also holds for $\mathcal{E}$ and $\mathcal{G}$ there exists a well-defined incidence preserving bijection

$$\varphi : \mathcal{E}^4 \to \mathcal{G}^2 \to \tilde{\mathcal{E}}^4$$
$$\varphi : \mathcal{E}^5 \to \mathcal{G}^1 \to \tilde{\mathcal{E}}^5.$$

Let us show that φ extends to an isomorphism of geometries $\varphi : \mathcal{E} \to \tilde{\mathcal{E}}$.

Recall from (1.1) and the remark thereafter that the objects of $\mathcal{E}$ can be identified with certain cliques of size $1, 2, 4, 8$, and 36 in the collinearity graph $\Gamma = \Gamma(\mathcal{E})$. From the diagram of $\mathcal{E}$ and condition (I) it follows that the cliques corresponding to objects of types 1, 2, or 3 are uniquely determined as the intersections of the cliques corresponding to the objects of types 4 and 5 in their residue. Hence there is a well-defined way to extend φ to an incidence preserving map from $\mathcal{E}$ onto $\tilde{\mathcal{E}}$ whose restriction to any residue is an isomorphism. So φ is indeed a morphism of geometries. Now $|\mathcal{E}^i| = |\tilde{\mathcal{E}}^i|$ for $i \in \{4, 5\}$ and by the same counting argument as in the proof of Theorem 2 this also holds for $i \in \{1, 2, 3\}$. This implies that φ is an isomorphism and the theorem is shown. □

Finally we can prove Theorem 1.

Proof of Theorem 1: We want to deduce Theorem 1 from Theorem 3; so we have to show that the existence of a flag-transitive action on $\mathcal{E}$ implies that the graph B defined in Section 5 satisfies the condition (III).

Let E be a flag-transitive automorphism group of $\mathcal{E}$. First we show

(1) *E acts flag-transively on the $c.C_3^*$-geometry $\mathcal{G}$.*

Recall the definition of $\mathcal{G}$ from Section 4. As $\mathcal{G}^1 = \mathcal{E}^5$ and $\mathcal{G}^2 = \mathcal{E}^4$ and the incidence relation between objects of type 1 and 2 in $\mathcal{G}$ is the one inherited from $\mathcal{E}$, E acts flag-transitively on the truncation of $\mathcal{G}$ consisting only of the objects of types 1 and 2. Furthermore, flag-transitivity of E on $\mathcal{E}$ implies that E acts vertex- and edge-transitively on the graphs $\tilde{\Xi}$ and $\tilde{\Delta}$ defined at the beginning of Section 4 and so it permutes transitively their connected components, i.e., the sets of objects of types 3 and 4 in $\mathcal{G}$. Finally, any maximal flag of $\mathcal{G}$ is of the shape $\{x_1, x_2, \tilde{\Delta}^{x_3}, \tilde{\Xi}^{x_4}\}$ where $x_i \in \mathcal{E}^{6-i}$ and $\{x_1, x_2, x_3, x_4\}$ is a flag in $\mathcal{E}$. So flag-transitivity of E on $\mathcal{E}$ implies that E is transitive on the set of maximal flags in $\mathcal{G}$. One can also easily see from this that any flag of $\mathcal{G}$ is contained in a maximal one; so (1) follows.

Now flag-transitivity of E on $\mathcal{G}$ implies that E acts transitively on the set of connected components of the graph Π defined in Section 5.1 and that the (setwise) stabilizer E_X of a connected component X of Π acts flag-transitively on the associated P-geometry $\mathcal{P}(X)$. Hence by (5.4) and [6] E_X induces an action containing M_{22} or $3{\cdot}M_{22}$ on $\mathcal{P}(X)$ and on X. Since this group acts transitively on the set $\mathcal{H}(\mathcal{P}(X))$ of subgeometries of $\mathcal{P}(X)$ defined in Section 5.2 and because by (5.8) the short triangles of the graph B correspond to those subgeometries, we see that E acts transitively on the sets of vertices and short triangles of B and E_X acts transitively on the set of short triangles through X. Furthermore, the Schläfli subgraphs of B correspond to the elements in $\mathcal{G}^4$. So E is also transitive on the set $\mathcal{S}$ of Schläfli subgraphs of B and the (setwise) stabilizer E_S of a Schläfli graph $S \in \mathcal{S}$ is transitive on the sets of vertices and triangles contained in S. Let K_S be the kernel of the action of E_S on S. Then by [14] this implies

(2) $U_4(2) \leq E_S/K_S$.

Now let $S_1, S_2 \in \mathcal{S}$ be two different Schläfli graphs and X, Y two different connected components of Π such that $X, Y \in S_1 \cap S_2$. Set $E_1 := E_{S_1}$, $K_1 := K_{S_1}$, let $\mathcal{T}$ be the set of short triangles of B through X, and denote by $K_X \leq E_X$ the kernel of the action E_X on $\mathcal{T}$. Then independently from the isomorphism type of X we have

(3) *$M_{22} \leq E_X/K_X \leq M_{22}.2$ and the action of E_X on $\mathcal{T}$ is similar to the action of M_{22} on the set of cosets of a subgroup $2^4.A_6$ (respectively $M_{22}.2$ on $2^4.\Sigma_6$).*

Set

$$\mathcal{U} := \{(T, Z) \mid T \in \mathcal{T}, X \neq Z \in T\},$$
$$\mathcal{T}_1 := \{T \in \mathcal{T} \mid T \subseteq S_1\},$$
$$\mathcal{U}_1 := \{(T, Z) \mid (T, Z) \in \mathcal{U}, T \in \mathcal{T}_1\}.$$

Then $|\mathcal{U}| = 77 \cdot 2 = 154$, $|\mathcal{T}_1| = 5$, and $|\mathcal{U}_1| = 10$. Furthermore, from (2) and the action of $U_4(2)$ on the Schläfli graph we can deduce

(4) *$2^4.A_5 \leq E_1 \cap E_X/K_1$, the action of $E_1 \cap E_X$ on $\mathcal{T}_1$ involves an action similar to that of $2^4.A_5$ on the set of cosets of a subgroup $2^4.A_4$, and the action of $E_1 \cap E_X$ on $\mathcal{U}_1$ involves an action similar to that of $2^4.A_5$ on the set of cosets of a subgroup $2^3.A_4$.*

Notice that (4) and transitivity of E_X on $\mathcal{T}$ implies transitivity of E_X on $\mathcal{U}$. Next we show

(5) $K_X \neq 1$.

Assume $K_X = 1$. Then we deduce from (3) and the fact that $2^4.A_6$ does not possess a subgroup of index 2 that we must have $E_X \cong M_{22}.2$ (so the action of E_X on $\mathcal{T}$ is similar to the action of $M_{22}.2$ on the set of cosets of a subgroup $2^4.\Sigma_6$) and that the action of E_X on $\mathcal{U}$ must be similar to the action of $M_{22}.2$ on the set of cosets of a subgroup $2^4.A_6$. But then $E_1 \cap E_X \cong 2^4(\mathbb{Z}_2 \times \Sigma_5)$ and the action of $E_1 \cap E_X$ on $\mathcal{U}_1$ must be similar to the action of $2^4.(\mathbb{Z}_2 \times \Sigma_5)$ on the set of cosets of a subgroup $2^4.\Sigma_4$ where the irreducible 4-dimensional submodule in $O_2(E_1 \cap E_X)$ acts trivially on $\mathcal{U}_1$. This contradicts the statement in (4) about the action of a subgroup $2^4.A_5 \leq E_1 \cap E_X$. So (5) holds.

The proof of (5) shows even more. Let $(T, Z) \in \mathcal{U}$, let E_{XTZ} be its stabilizer in E_X, and denote by E_0 the full preimage in $E_1 \cap E_X$ of the subgroup $2^4.A_5 \leq E_1 \cap E_X/K_1$ and by K_0 the full preimage in E_0 of $O_2(E_0/K_1)$. Then

(6) (i) *$E_{XTZ}K_X/K_X \cong 2^4.A_6$ or $2^4.\Sigma_6$ (depending on $E_X/K_X \cong M_{22}$ or $M_{22}.2$), $|K_X : K_X \cap E_{XTZ}| = 2$, and there exists $k \in K_X$ such that $T = \{X, Z, Z^k\}$.*
(ii) *$K_0 = K_1 K_X$.*

Notice that since K_X stabilizes each triangle in $\mathcal{T}$ it must preserve each Schläfli graph through X as a set; in particular, $K_X \leq E_1 \cap E_{S_2}$. Hence we deduce from (6)(i)

(7) *If there exists some $T \in \mathcal{T}_1$ such that $(T, Y) \in \mathcal{U}_1$ then $T \subseteq S_1 \cap S_2$.*

By (7) and the fact that maximal cliques in the Schläfli graph are just triangles it suffices to show that X and Y cannot be at distance 2 in S_1. But if this is the case then by the action of $U_4(2)$ on the Schläfli graph there are 16 conjugates of Y under $E_1 \cap E_X$ in S_1 and these form an orbit under K_0. So by (6)(ii) they are all conjugate to Y in K_X and hence they all belong to S_2. Repeating the argument for other pairs of vertices in $S_1 \cap S_2$ we now easily get the contradiction $S_1 = S_2$.

This proves that (III) holds for B and so Theorem 1 follows from Theorem 3. □

Appendix A: Some remarks on the flag-transitive case

Here we establish flag-transitive versions for some of the results of Sections 5.1 and 5.2. Our assumptions are as in Section 5. Furthermore, we assume that $E \leq Aut(\mathcal{G})$ acts flag-transitively on $\mathcal{G}$. However, we do not use any information about the precise structure of E. In particular, we do not use (3.3). For any object x of $\mathcal{G}$ we denote by E_x the stabilizer of x in E and by K_x the kernel of the action of E_x on $res_{\mathcal{G}}(x)$.

First of all, flag-transitivity of E on $\mathcal{G}$ implies that E acts vertex- and edge-transitively on the graph Π, that it permutes the connected components of Π transitively, and that the (setwise) stabilizer in E of a connected component Π^x, $x \in \mathcal{G}^2$, acts vertex- and edge-transitively on Π^x. Notice also that the stabilizer E_q of any $q \in \mathcal{G}^4$, involves $U_4(2)$ on $res_{\mathcal{G}}(q)$ by [7, (6.1)] and that the stabilizer E_a of any $a \in \mathcal{G}^1$ acts as $Sp_6(2)$ on $res_{\mathcal{G}}(a)$ by [14]. From this we get

Lemma A.1 (5.2) *If Π^x, $x \in \mathcal{G}^2$, is a connected component of Π and $q \in \mathcal{G}^4$ then either $\Pi^x \cap \Pi_q$ is connected or Π is connected.*

Proof: Set $X := \Pi^x$ and denote by E_X the setwise stabilizer of X in E. Suppose $X \cap \Pi_q$ contains two connected components X_1, X_2. Since the stabilizer in $U_4(2)$ of a Petersen graph is a maximal subgroup of $U_4(2)$ (isomorphic to $2^4.A_5$) and the stabilizers of two different Petersen graphs are different, we then get

$$E_q = K_q\big\langle E_q \cap E_{X_1}, E_q \cap E_{X_2}\big\rangle \leq K_q E_{X \cap \Pi_q} \leq E_X.$$

On the other hand, we also have $E_x \leq E_X$ and as we may assume $x \in res_{\mathcal{G}}(q)$ we deduce that $E_X \geq \langle E_x, E_q\rangle$ acts flag-transitively on $\mathcal{G}$. Hence $\Pi = X$. □

Lemma A.2 (5.3) *If $x, y \in \mathcal{G}^2$, $x \neq y$, such that $\Pi^x = \Pi^y$ then either $x \cap y = \emptyset$ or Π is connected.*

Proof: Set $X := \Pi^x$. Suppose $a \in x \cap y$, $a \in \mathcal{G}^1$. Then in $res_{\mathcal{G}}(a)$ the objects x and y correspond to two maximal totally isotropic subspaces in the symplectic space related to $Sp_6(2)$. Since the stabilizer of such a subspace is maximal subgroup of $Sp_6(2)$ (isomorphic to $2^6.L_3(2)$) and the stabilizers of two different subspaces are different we can argue as before:

$$E_a = K_a\langle E_a \cap E_x, E_a \cap E_y\rangle \leq E_X.$$

Hence $E_X \geq \langle E_a, E_x\rangle$ is flag-transitive on $\mathcal{G}$ and $\Pi = X$. □

As already mentioned in the proof of Theorem 1, the definition of the P-geometry $\mathcal{P}^x$ associated with $X := \Pi^x$ implies that E_X acts flag-transitively on $\mathcal{P}^x$ and so (5.4) and [6] imply that E_X involves M_{22} or $3{\cdot}M_{22}$. Since none of them contains a subgroup isomorphic to $U_4(2)$ and $E_x \cap E_q$ acts non-trivially on $X \cap \Pi_q$ if $X \cap \Pi_q \neq \emptyset$ we see that Π is disconnected and so the first alternatives of (A.1) and (A.2) hold.

Lemma A.3 (5.6) *The map $\beta : \mathcal{P}^x \to \mathcal{G}$ is injective.*

Proof: This can be proved similarly as (5.6) but using (A.1) and (A.2) instead of (5.2) and (5.3). $\square$

Lemma A.4 (5.7) *If* $\{\Pi^1, \Pi^2\} \in E(B)$ *then* $\Pi^1 \neq \Pi^2$.

Proof: By definition of the adjacency in B we have $\Pi^1 = \Pi^x$, $\Pi^2 = \Pi^y$ for some $x, y \in \mathcal{G}^2$ with $|x \cap y| = 1$. Hence the assertion follows from (A.2). $\square$

Appendix B: An elementary proof of Proposition 6.6

Here we prove the explicit calculation of the graph $\mathcal{C}_0$. The notation is as in Section 6.

Proof of Proposition 6.6: For $C \in \mathcal{C}_0^3 = V(\mathcal{C}_0)$ and $i \geq 0$ we denote by $D_i(C)$ the set of vertices at distance i from C in $\mathcal{C}_0$. We fix some $C_0 \in \mathcal{C}_0^3$, and we set for short, $D_i := D_i(C_0)$; so $D_0 = \{C_0\}$. We will determine the sizes of D_i, $i \geq 1$, and the intersection array in a series of steps.

(a) $|D_1| = 42$ and $|D_1 \cap D_1(C)| = 1$ for any $C \in D_1$.

By (6.1)(ii) there are exactly 21 cliques from $\mathcal{C}_0^2$ contained in C_0 and by the diagram of $\mathcal{B}$ each of them is contained in exactly two other cliques as C_0 from $\mathcal{C}^3$ which by connectedness of $\mathcal{C}_0$ must be in $\mathcal{C}_0^3$. This gives $|D_1| \leq 42$ and $|D_1 \cap D_1(C)| \geq 1$ for any $C \in D_1$.

On the other hand, let $C_1, C_2 \in D_1$ with $C_1 \cap C_0 \neq C_2 \cap C_0$. Then by (6.2)(iii) there is exactly one element $a \in \mathcal{B}^1$ such that $a \in p(C_1) \cap p(C_2) \cap p(C_0)$. By (6.4) we have that $(\gamma(C_1), \gamma(C_0), \gamma(C_2))$ is a path in the Schläfli graph B_a. But $\gamma(C_1 \cap C_0) \neq \gamma(C_2 \cap C_0)$ as $C_1 \cap C_0 \neq C_2 \cap C_0$. So $\gamma(C_1)$ and $\gamma(C_2)$ are distinct and non-adjacent vertices of B_a. This implies that also $C_1 \neq C_2$ and $\{C_1, C_2\} \notin E(\mathcal{C}_0)$. Hence we have equality in both cases.

(b) $|D_2| = 336$ and $|D_1 \cap D_1(C)| = |D_2 \cap D_1(C)| = 5$ for any $C \in D_2$.

If $C \in D_2$ then as in (a) there are $a \in \mathcal{B}^1$ and $C_1 \in D_1$ such that (C_0, C_1, C) is a path of length two in one of the Schläfli graphs in $\mathcal{C}_a$. This implies that $|D_i \cap D_1(C)| \geq 5$ for $i = 1, 2$. On the other hand, if $C_2 \in D_1(C) \cap D_1$ and (C_0, C_2, C) is the corresponding 2-path in some Schläfli graph $S_b \subseteq \mathcal{C}_b$ then (6.5) implies that $a = b$ because C_0 and C are contained in a triangle. So C_2 must be one of the already discovered five neighbours and $|D_1 \cap D_1(C)| = 5$ and $|D_2| = \frac{42 \cdot 40}{5} = 336$ holds.

(c) If $a \in p(C_0)$ and $C \in \mathcal{C}_0$, $C \notin D_0 \cup D_1 \cup D_2$, then there exists some $\tilde{C} \in \mathcal{C}_0 \cap \mathcal{C}_a$ such that $\tilde{C}$ and C_0 are in the same connected component of $\mathcal{C}_a$ and $\tilde{C} \in D_1(C)$.

Let $C_1, C_2, \ldots, C_k := C$ be a shortest path in $\mathcal{C}_0$ between C and a vertex C_1 which is in the same connected component of $\mathcal{C}_a$ as C_0. Suppose $k \geq 3$.

As before there exists $b \in \mathcal{B}^1$ such that (C_1, C_2, C_3) is a 2-path in $\mathcal{C}_b$. Then $a, b \in p(C_1)$; hence $\{a, b\}_t$ is a vertex of C_1 for a suitable $t \in \mathcal{B}^2$ and t determines a unique special 10-clique $C_t \subseteq C_1$ with $\{a, b\}_t \in C_t$. Now C_t determines a triangle containing C_1 in $\mathcal{C}_b$ and the structure of the Schläfli graph shows that one of the other two vertices in this triangle must be at distance one from C_3. In other terms, there exists some $\tilde{C} \in \mathcal{C}_0 \cap \mathcal{C}_b$ with $\tilde{C} \cap C_1 = \bar{C}_1$ and $\tilde{C} \in D_1(C_3)$. But as $a \in p(C_t)$ we see that $\tilde{C} \in \mathcal{C}_a$ and, in fact, $\tilde{C}$ lies in the same connected component as C_1, hence also as C_0. So we can replace our path by the shorter path $\tilde{C}, C_3, \ldots, C_k$ and have a contradicton.

(d) $D_4 = \emptyset$, $|D_3| = 512$, and $|D_2 \cap D_1(C)| = |D_3 \cap D_1(C)| = 21$ for any $C \in D_3$.

The first statement follows from (c) and the fact that the Schläfli graph has diameter 2.

Now let $a \in p(C_0)$ and let $C_1, C_2 \in D_2 \cap \mathcal{C}_a$ such that C_1 and C_2 are also at distance two from C_0 in $\mathcal{C}_a$ and suppose $C_1, C_2 \in D_1(C)$ for some $C \in D_3$. Then there is $b \in \mathcal{B}^1$ such that C_1, C, C_2 is path in $\mathcal{C}_b$ and again by (6.5) either $a = b$ or C_1, C_2 are contained in a triangle. The first case is not possible since $C \in D_3$ and the diameter of the Schläfli graph is 2. In the second case (6.5) also implies that, on the one hand, the triangle through C_1 and C_2 must be in $\mathcal{C}_a$, on the other hand that the third vertex of it is C. So $C \in D_3$ yields again a contradiction and we have shown that any two different vertices at distance two from C_0 in $\mathcal{C}_a$ cannot be adjacent to a common vertex in D_3. Since there are exactly 16 such vertices we get by (b) and (c) that $|D_3| = 16 \cdot 32 = 512$, $|D_2 \cap D_1(c)| = \frac{336 \cdot 32}{512} = 21$, and $|D_3 \cap D_1(c)| = |D_1(C)| - |D_2 \cap D_1(c)| = 21$. This completes the proof of (i).

Now let $\mathcal{D}$ be the graph on the set of planes of the 6-dimensional unitary polar space $\mathcal{P}(U_6(2))$ in which two planes are adjacent if they intersect in a line. Then $\mathcal{D}$ has the same intersection array as $\mathcal{C}_0$. Furthermore, each point or line $\mathcal{P}(U_6(2))$ is uniquely determined as the intersection of the planes containing it, i.e., as the subgraph of $\mathcal{D}$ induced on its residue (which is either a Schläfli graph or a triangle). Since the objects of $\mathcal{H}_0$ are just the vertices, triangles, and Schläfli subgraphs of $\mathcal{C}_0$ it is now easy to deduce the isomorphism $\mathcal{H}_0 \cong \mathcal{P}(U_6(2))$ from (i) and the diagram. □

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