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A Determinantal Formula for Supersymmetric Schur Polynomials

E.M. MOENS ElsM.Moens@UGent.be
J. VAN DER JEUGT Joris.VanderJeugt@UGent.be
Department of Applied Mathematics and Computer Science, University of Ghent, Krijgslaan 281-S9, B-9000 Gent, Belgium

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Abstract. We derive a new formula for the supersymmetric Schur polynomial $s_\lambda(x/y)$. The origin of this formula goes back to representation theory of the Lie superalgebra $\mathfrak{gl}(m/n)$. In particular, we show how a character formula due to Kac and Wakimoto can be applied to covariant representations, leading to a new expression for $s_\lambda(x/y)$. This new expression gives rise to a determinantal formula for $s_\lambda(x/y)$. In particular, the denominator identity for $\mathfrak{gl}(m/n)$ corresponds to a determinantal identity combining Cauchy's double alternant with Vandermonde's determinant. We provide a second and independent proof of the new determinantal formula by showing that it satisfies the four characteristic properties of supersymmetric Schur polynomials. A third and more direct proof ties up our formula with that of Sergeev-Pragacz.

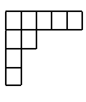
Keywords: supersymmetric Schur polynomials, Lie superalgebra $\mathfrak{gl}(m/n)$, characters, covariant tensor representations, determinantal identities

1. Introduction

This paper deals with a new formula for supersymmetric Schur polynomials $s_\lambda(x/y)$, parametrized by a partition λ, and symmetric in two sets of variables $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$. Supersymmetric Schur polynomials, or S-functions, appeared for the first time in the work of Berele and Regev [3], who also showed that these polynomials are closely related to characters of certain representations of the Lie superalgebra $\mathfrak{gl}(m/n)$ or $\mathfrak{sl}(m/n)$. More precisely, the characters of irreducible covariant tensor representations of $\mathfrak{gl}(m/n)$ are supersymmetric S-functions (just as the characters of irreducible covariant tensor representations of $\mathfrak{gl}(m)$ are ordinary Schur functions). It is in the context of $\mathfrak{gl}(m/n)$ representations that our new formula for supersymmetric S-functions was discovered, as a consequence of a certain character formula of Kac and Wakimoto [6]. In their general study of characters for classical Lie superalgebras and affine superalgebras, Kac and Wakimoto [6] developed the idea of so-called tame representations. These tame representations allow for the construction of an explicit character formula. Unfortunately, it is not easy to give a simple characterization of which representations are tame. In this paper, we show that the covariant tensor representations of $\mathfrak{gl}(m/n)$ are indeed tame. As a consequence we can apply the character formula of Kac and Wakimoto, and obtain a new expression for supersymmetric S-functions. This new expression is not particularly elegant, but after certain manipulations we show that it is equivalent to a nice determinantal formula. Once this determinantal formula for $s_\lambda(x/y)$ was found, we realized that its validity could also be proved independently. Indeed, in [10, Section I.3, Exercise 23] or [13] it is shown that the supersymmetric S-functions are characterized by four properties. Using our determinantal expression, it is possible to show that these four properties are indeed satisfied. Finally, one can also prove the determinantal formula using a double Laplace expansion and a special case of the so-called Sergeev-Pragacz formula [22].

The structure of this paper is as follows. In the first section, we fix the notation and recall some known formulas for ordinary and supersymmetric Schur functions. The simplest form of our new determinantal formula for $s_\lambda(x/y)$ is already given in (1.17). Furthermore, we point out that the case $\lambda = 0$ gives rise to an interesting determinantal identity, which can be called "the denominator identity for $\mathfrak{gl}(m/n)$". Section 2 is devoted to showing that covariant tensor representations of the Lie superalgebra $\mathfrak{gl}(m/n)$ are tame, and to applying the Kac-Wakimoto character formula. This section uses a lot of representation theory, and closes with a formula for the character, i.e. a formula for $s_\lambda(x/y)$. This formula is not in an optimal form, and in Section 3 we use a number of intricate but elementary manipulations to derive from this form our main result, Theorem 3.4, giving the determinantal expression for $s_\lambda(x/y)$. In Section 4, we show that the determinantal expression satisfies the four characteristic properties of supersymmetric S-functions, thus yielding an independent proof of the determinantal formula. Finally, in Section 5, we provide a straightforward proof, without using representation theory, based on the formula of Sergeev-Pragacz and Laplace's theorem. So in fact we provide three proofs: one in the context of representation theory of $\mathfrak{gl}(m/n)$, one by means of the characteristic properties, and the last one using the formula of Sergeev-Pragacz. A reader who is not so familiar with representation theory can easily skip Section 2 (apart from Definition 2.2). In fact, we could have presented our main result without any reference to representation theory. However, we did not want to hide the natural background of this new formula, and therefore we have chosen to present also its representation theoretic origin. This is also clear from the determinantal formula itself: in this expression, the so-called (m, n)-index k of λ is crucial; this definition of k has a natural interpretation in representation theory of $\mathfrak{gl}(m/n)$ (in terms of *atypicality*).

Let $\lambda = (\lambda_1, \ldots, \lambda_p, 0, 0, \ldots)$ be a partition of the nonnegative integer N, with $\lambda_1 \geq \cdots \geq \lambda_p > 0$ and $\sum_i \lambda_i = |\lambda| = N$. The number $p = \ell(\lambda)$ is the length of λ. The Young diagram F^λ of shape λ is the set of left-adjusted rows of squares with λ_i squares (or boxes) in the ith row (reading from top to bottom). For example, the Young diagram of (5, 2, 1, 1) is given by:

$F^\lambda =$  (1.1)

As usual, λ' denotes the conjugate to λ; e.g. if $\lambda = (5, 2, 1, 1)$ then $\lambda' = (4, 2, 1, 1, 1)$. Denote by $\mathcal{S}(x)$ the ring of symmetric functions in m independent variables $x_1, \ldots, x_m$ [10]. The Schur functions or S-functions s_λ form a $\mathbb{Z}$-basis of $\mathcal{S}(x)$. There are various ways to define the S-functions [10]. For a partition λ with $\ell(\lambda) \leq m$, there is the determinantal formula (as a quotient of two alternants):

$$s_\lambda(x) = \frac{\det\bigl(x_i^{\lambda_j+m-j}\bigr)_{1\le i,j\le m}}{\det\bigl(x_i^{m-j}\bigr)_{1\le i,j\le m}}. \tag{1.2}$$

The numerator can be rewritten as

$$\det\bigl(x_i^{\lambda_j+m-j}\bigr)_{1\le i,j\le m} = \det(x^{\lambda+\delta_m}) = \sum_{w\in S_m} \varepsilon(w)w(x^{\lambda+\delta_m}), \tag{1.3}$$

where S_m is the symmetric group acting on $x = (x_1, \dots, x_m)$, $\varepsilon(w)$ the signature of w, and

$$\delta_m = (m-1, m-2, \dots, 1, 0). \tag{1.4}$$

Thus (1.2) is essentially equivalent to Weyl's character formula for the Lie algebra $\mathfrak{gl}(m)$, where an irreducible representation of $\mathfrak{gl}(m)$ is characterized by a partition λ and its character is given by $s_\lambda(x)$. The denominator on the right hand side of (1.2) is the Vandermonde determinant, equal to the product $\prod_{1\le i<j\le m}(x_i - x_j)$; this is Weyl's denominator formula for $\mathfrak{gl}(m)$.

When $\lambda = (r)$, $s_\lambda(x)$ is the complete symmetric function $h_r(x)$, and when $\lambda = (1^r)$, $s_\lambda(x)$ is the elementary symmetric function $e_r(x)$. The Jacobi-Trudi formula and the Nägelsbach-Kostka formula give s_λ in terms of these [10]:

$$s_\lambda(x) = \det\bigl(h_{\lambda_i-i+j}(x)\bigr)_{1\le i,j\le \ell(\lambda)} = \det\bigl(e_{\lambda'_i-i+j}(x)\bigr)_{1\le i,j\le \ell(\lambda')}. \tag{1.5}$$

Other determinantal formulas for s_λ are Giambelli's formula [10] and the ribbon formula [9]. Finally, recall there is a combinatorial formula for $s_\lambda(x)$ as a sum of monomials summed over all column-strict Young tableaux of shape λ [10].

All these formulas, except (1.2), have their extensions to skew Schur functions $s_{\lambda/\mu}(x)$, where λ and μ are two partitions with $\lambda_i \ge \mu_i$ for all i.

Let us now recall some notions of *supersymmetric S-functions* [3, 7, 17]. The ring of doubly symmetric polynomials in $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ is $\mathcal{S}(x, y) = \mathcal{S}(x) \otimes_{\mathbb{Z}} \mathcal{S}(y)$. An element $p \in \mathcal{S}(x, y)$ has the *cancellation property* if it satisfies the following: when the substitution $x_1 = t$, $y_1 = -t$ is made in p, the resulting polynomial is independent of t. We denote $\mathcal{S}(x/y)$ the subring of $\mathcal{S}(x, y)$ consisting of the elements satisfying the cancellation property. The elements of $\mathcal{S}(x/y)$ are the supersymmetric polynomials [17].

The complete supersymmetric functions $h_r(x/y)$ belong to $\mathcal{S}(x/y)$, and are defined by

$$h_r(x/y) = \sum_{k=0}^{r} h_{r-k}(x)e_k(y). \tag{1.6}$$

The following gives a first formula for the supersymmetric S-functions:

$$s_\lambda(x/y) = \det\bigl(h_{\lambda_i-i+j}(x/y)\bigr)_{1\le i,j\le \ell(\lambda)}. \tag{1.7}$$

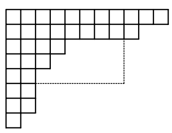
The polynomials $s_\lambda(x/y)$ are identically zero when $\lambda_{m+1} > n$. Denote by $\mathcal{H}_{m,n}$ the set of partitions with $\lambda_{m+1} \le n$, i.e. the partitions (with their Young diagram) inside the (m, n)-hook. Stembridge [17] showed that the set of $s_\lambda(x/y)$ with $\lambda \in \mathcal{H}_{m,n}$ form a $\mathbb{Z}$-basis of $\mathcal{S}(x/y)$.

There exist a number of other formulas for the supersymmetric S-functions. One is a combinatorial formula in terms of supertableaux of shape λ, see [3]. From the combinatorial formula, one finds expansions of $s_\lambda(x/y)$ in terms of ordinary S-functions:

$$s_\lambda(x/y) = \sum_\mu s_\mu(x) s_{(\lambda/\mu)'}(y) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(x) s_{\nu'}(y), \tag{1.8}$$

where $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients [10].

Just as the functions $s_\lambda(x)$ are characters of simple modules of the Lie algebra $\mathfrak{gl}(m)$, the supersymmetric S-functions are characters of (a class of) simple modules of the Lie superalgebra $\mathfrak{gl}(m/n)$ [3]. In this context, a different formula for $s_\lambda(x/y)$ was found by Sergeev (see [12]) and in [19]; the first proof of this formula was given by Pragacz [12]. To describe the so-called Sergeev-Pragacz formula, let λ be a partition with $\lambda_{m+1} \le n$. Consider the Young diagram F^λ, let F^κ be the part of F^λ that falls within the $m \times n$ rectangle, and let F^τ, resp. F^η, be the remaining part to the right, resp. underneath this rectangle; i.e. $\lambda = (\kappa + \tau) \cup \eta$. This is illustrated, for $m = 5$, $n = 8$ and $\lambda = (11, 9, 4, 3, 2, 2, 2, 1)$, as follows:

$F^\lambda =$  hence

$$\begin{aligned} \kappa &= (8,8,4,3,2) \\ \tau &= (3,1) \\ \eta &= (2,2,1) \end{aligned} \tag{1.9}$$

Then, the Sergeev-Pragacz formula for $s_\lambda(x/y)$ is given by

$$s_\lambda(x/y) = D_0^{-1} \sum_{w \in S_m \times S_n} \varepsilon(w)\, w \left(x^{\tau+\delta_m} y^{\eta'+\delta_n} \prod_{(i,j)\in F^\kappa} (x_i + y_j) \right), \tag{1.10}$$

where $(i, j) \in F^\kappa$ if and only if the box with row-index i (read from left to right) and column-index j (read from top to bottom) belongs to F^κ, and

$$D_0 = \prod_{1\le i<j\le m} (x_i - x_j) \prod_{1\le i<j\le n} (y_i - y_j). \tag{1.11}$$

This formula is useful for the computation of $s_\lambda(x/y)$, and even for the computation of Littlewood-Richardson coefficients [2, 18]. Note that for the special case that $\lambda_m \ge n$, (1.10) becomes:

$$s_\lambda(x/y) = s_\tau(x) s_{\eta'}(y) \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j). \tag{1.12}$$

This is the case of *factorization*. This formula was first derived by Berele and Regev [3, Theorem 6.20] and will referred to as the Berele-Regev formula. Furthermore, from (1.10) one easily deduces duality:

$$s_\lambda(x/y) = s_{\lambda'}(y/x). \tag{1.13}$$

Observe that D_0 in (1.10) is just Weyl's denominator for $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. So when $\lambda = 0$, (1.10) does not yield a new denominator identity related to $\mathfrak{gl}(m/n)$; it only gives the denominator identity for $\mathfrak{gl}(m)$ and $\mathfrak{gl}(n)$.

In this paper, we shall give a new formula for $s_\lambda(x/y)$. In its simplest form, this yields a new determinantal formula for supersymmetric S-functions. Furthermore, this formula yields a genuine denominator identity related to $\mathfrak{gl}(m/n)$. Let us briefly describe one of the forms of the new formula. First, we introduce some new notations. Let

$$D(x) = \prod_{1\le i<j\le m}(x_i - x_j) \quad \text{and} \quad E(x, y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j) \tag{1.14}$$

and define D by

$$D = \frac{\prod_{1\le i<j\le m}(x_i - x_j)\prod_{1\le i<j\le n}(y_i - y_j)}{\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j)} = \frac{D(x)D(y)}{E(x, -y)}. \tag{1.15}$$

Let λ be a partition with $\lambda_{m+1} \le n$, i.e. $\lambda \in \mathcal{H}_{m,n}$, and put

$$k = \min\{j \mid \lambda_j + m + 1 - j \le n\}; \tag{1.16}$$

since $\lambda_{m+1} \le n$, we have that $1 \le k \le m + 1$. Then the new formula reads

$$s_\lambda(x/y) = (-1)^{mn-m+k-1} D^{-1} \det\begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \tag{1.17}$$

where the (rectangular) blocks of the determinant are given by

$$R = \left(\frac{1}{x_i + y_j}\right)_{1\le i\le m, 1\le j\le n},$$
$$X_\lambda = \left(x_i^{\lambda_j+m-n-j}\right)_{1\le i\le m, 1\le j\le k-1}, \quad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i}\right)_{1\le i\le n-m+k-1, 1\le j\le n}.$$

For example, let $m = 3$, $n = 5$ and $\lambda = (7, 2, 2, 2, 1)$. Then $\kappa = (5, 2, 2)$, $\tau = (2)$, $\eta = (2, 1)$ and $k = 2$. Thus, according to formula (1.17),

$$s_{(7,2,2,2,1)} = -D \det \begin{pmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \frac{1}{x_1+y_3} & \frac{1}{x_1+y_4} & \frac{1}{x_1+y_5} & x_1^4 \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3} & \frac{1}{x_2+y_4} & \frac{1}{x_2+y_5} & x_2^4 \\ \frac{1}{x_3+y_1} & \frac{1}{x_3+y_2} & \frac{1}{x_3+y_3} & \frac{1}{x_3+y_4} & \frac{1}{x_3+y_5} & x_3^4 \\ y_1^6 & y_2^6 & y_3^6 & y_4^6 & y_5^6 & 0 \\ y_1^4 & y_2^4 & y_3^4 & y_4^4 & y_5^4 & 0 \\ y_1^0 & y_2^0 & y_3^0 & y_4^0 & y_5^0 & 0 \end{pmatrix}. \tag{1.18}$$

When $\lambda = 0$ it follows from (1.7) or (1.10) that $s_\lambda(x/y) = 1$. The new formula (1.17) gives rise to a denominator identity for $\mathfrak{gl}(m/n)$. Suppose $m \leq n$ ($m \geq n$ is similar); when $\lambda = 0$, it follows from (1.16) that $k = 1$. So the X_λ-block and 0-block disappear in (1.17). Changing the order of the R-block and Y_λ-block, implies

$$\det \begin{pmatrix} y_1^{n-m-1} & \cdots & y_n^{n-m-1} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_n^0 \\ \frac{1}{x_1+y_1} & \cdots & \frac{1}{x_1+y_n} \\ \vdots & & \vdots \\ \frac{1}{x_m+y_1} & \cdots & \frac{1}{x_m+y_n} \end{pmatrix} = \frac{\prod_{1\leq i<j\leq m}(x_i - x_j)\prod_{1\leq i<j\leq n}(y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j)}. \tag{1.19}$$

Clearly, when $m = n$, this is simply Cauchy's double alternant; when $m = 0$, it is just Vandermonde's determinant. When $0 < m < n$, it is a combination of the two. These type of determinants have already been encountered in a different context [1] (we found this reference in [8]); here they are for the first time related to a denominator identity.

2. Covariant modules for the Lie superalgebra $\mathfrak{gl}(m/n)$

For general theory on classical Lie superalgebras and their representations, we refer to [4, 5, 14]; for representations of the Lie superalgebra $gl(m/n)$, see [19–21].

Let $\mathfrak{g} = \mathfrak{gl}(m/n)$, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra, and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ the consistent $\mathbb{Z}$-grading. Note that $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, and put $\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ and $\mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. The dual space $\mathfrak{h}^*$ has a natural basis $\{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\}$, and the roots of $\mathfrak{g}$ can be expressed in terms of this basis. Let Δ be the set of all roots, Δ_0 the set of even roots, and Δ_1 the set of odd roots. One can choose a set of simple roots (or, equivalently, a triangular decomposition), but note that contrary to the case of simple Lie algebras not all such choices are equivalent. The so-called *distinguished choice* [4] for a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is such that $\mathfrak{g}_{+1} \subset \mathfrak{n}^+$ and $\mathfrak{g}_{-1} \subset \mathfrak{n}^-$. Then $\mathfrak{h} \oplus \mathfrak{n}^+$ is the corresponding distinguished Borel subalgebra, and Δ_+ the set of positive roots. For this choice we have explicitly:

$$\begin{aligned}\Delta_{0,+} &= \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le m\} \cup \{\delta_i - \delta_j \mid 1 \le i < j \le n\},\\ \Delta_{1,+} &= \{\beta_{ij} = \epsilon_i - \delta_j \mid 1 \le i \le m,\ 1 \le j \le n\},\end{aligned} \tag{2.1}$$

and the corresponding set of simple roots (the distinguished set) is given by

$$\Pi = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}. \tag{2.2}$$

Thus in the distinguished basis there is only one simple root which is odd. As usual, we put

$$\rho_0 = \frac{1}{2}\Bigg(\sum_{\alpha\in\Delta_{0,+}} \alpha\Bigg), \quad \rho_1 = \frac{1}{2}\Bigg(\sum_{\alpha\in\Delta_{1,+}} \alpha\Bigg), \quad \rho = \rho_0 - \rho_1. \tag{2.3}$$

There is a symmetric form $(\,,\,)$ on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$, and in the natural basis it takes the form $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{ij}$. The odd roots are isotropic: $(\alpha, \alpha) = 0$ if $\alpha \in \Delta_1$.

The Weyl group of $\mathfrak{g}$ is the Weyl group W of $\mathfrak{g}_0$, hence it is the direct product of symmetric groups $S_m \times S_n$. For $w \in W$, we denote by $\varepsilon(w)$ its signature.

Let $\Lambda \in \mathfrak{h}^*$; the *atypicality* of Λ, denoted by $\mathrm{atyp}(\Lambda)$, is the maximal number of linearly independent roots β_i such that $(\beta_i, \beta_j) = 0$ and $(\Lambda, \beta_i) = 0$ for all i and j [6]. Such a set $\{\beta_i\}$ is called a Λ-maximal isotropic subset of Δ.

Given a set of positive roots Δ_+ of Δ, and a simple odd root α, one may construct a new set of positive roots [6, 11] by

$$\Delta'_+ = (\Delta_+ \cup \{-\alpha\}) \setminus \{\alpha\}. \tag{2.4}$$

The set Δ'_+ is called a simple reflection of Δ_+. Since we use only simple reflections with respect to simple odd roots, $\Delta_{0,+}$ remains invariant, but $\Delta_{1,+}$ will change and the new ρ is given by:

$$\rho' = \rho + \alpha. \tag{2.5}$$

Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. Such modules are $\mathfrak{h}$-diagonalizable with weight decomposition $V = \oplus_\mu V(\mu)$, and the character is defined to be $\mathrm{ch}\, V = \sum_\mu \dim V(\mu)\, e^\mu$, where e^μ ($\mu \in \mathfrak{h}^*$) is the formal exponential. Consider such a module V. If we fix a set of positive roots Δ_+, we may talk about the highest weight Λ of V and about the corresponding ρ. If Δ'_+ is obtained from Δ_+ by a simple α-reflection, where α is odd, and Λ' denotes the highest weight of V with respect to Δ'_+, then [6]

$$\Lambda' = \Lambda - \alpha \text{ if } (\Lambda, \alpha) \neq 0; \quad \Lambda' = \Lambda \text{ if } (\Lambda, \alpha) = 0. \tag{2.6}$$

If α is a *simple* odd root from Δ_+ then $(\rho, \alpha) = \frac{1}{2}(\alpha, \alpha) = 0$ [6, p. 421], and therefore, following (2.5) and (2.6):

$$\begin{aligned}\Lambda' + \rho' &= \Lambda + \rho \ \text{ if } \ (\Lambda + \rho, \alpha) \neq 0,\\ \Lambda' + \rho' &= \Lambda + \rho + \alpha \ \text{ if } \ (\Lambda + \rho, \alpha) = 0.\end{aligned} \tag{2.7}$$

From this, one deduces that for the $\mathfrak{g}$-module V, $\operatorname{atyp}(\Lambda + \rho)$ is independent of the choice of Δ_+; then $\operatorname{atyp}(\Lambda + \rho)$ is referred to as the atypicality of V (if $\operatorname{atyp}(\Lambda + \rho) = 0$, V is typical, otherwise it is atypical). If one can choose a $(\Lambda + \rho)$-maximal isotropic subset S_Λ in Δ_+ such that $S_\Lambda \subset \Pi \subset \Delta_+$ (Π is the set of simple roots with respect to Δ_+), then the $\mathfrak{g}$-module V is called *tame*, and a character formula is known due to Kac and Wakimoto [6]. It reads:

$$\operatorname{ch} V = j_\Lambda^{-1} e^{-\rho} R^{-1} \sum_{w\in W} \varepsilon(w) w \left(e^{\Lambda+\rho} \prod_{\beta \in S_\Lambda} (1 + e^{-\beta})^{-1} \right), \tag{2.8}$$

where

$$R = \prod_{\alpha\in\Delta_{0,+}} (1 - e^{-\alpha}) \Big/ \prod_{\alpha\in\Delta_{1,+}} (1 + e^{-\alpha}) \tag{2.9}$$

and j_Λ is a normalization coefficient to make sure that the coefficient of e^Λ on the right hand side of (2.8) is 1.

The rest of this section is now devoted to a particular class of finite-dimensional irreducible $\mathfrak{g}$-modules, the *covariant* modules, and to showing that these modules are tame. Berele and Regev [3], and Sergeev [15], showed that the tensor product of N copies of the natural $(m + n)$-dimensional representation of $\mathfrak{g} = \mathfrak{gl}(m/n)$ is completely reducible, and that the irreducible components V_λ can be labeled by a partition λ of N such that λ is inside the (m, n)-hook, i.e. such that $\lambda_{m+1} \leq n$. These representations V_λ are known as covariant modules. Let us first consider V_λ in the distinguished basis fixed by (2.2). Then, the highest weight Λ_λ of V_λ in the standard ϵ-δ-basis is given by [19]

$$\Lambda_\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_m\epsilon_m + \nu_1'\delta_1 + \cdots + \nu_n'\delta_n, \tag{2.10}$$

where $\nu_j' = \max\{0, \lambda_j' - m\}$ for $1 \leq j \leq n$. Let us consider the atypicality of V_λ, in the distinguished basis. For this purpose, it is sufficient to compute the numbers $(\Lambda_\lambda + \rho, \beta_{ij})$, with $\beta_{ij} = \epsilon_i - \delta_j$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, and count the number of zeros. It is convenient to put the numbers $(\Lambda_\lambda + \rho, \beta_{ij})$ in a $(m \times n)$-matrix (the atypicality matrix [19, 20]), and give the matrix entries in the (m, n)-rectangle together with the Young frame of λ. This is illustrated here for example (1.9):

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
18 & 16 & 13 & 12 & 11 & 10 & 9 & 8 & & & \\
\hline
15 & 13 & 10 & 9 & 8 & 7 & 6 & 5 & & \multicolumn{2}{c}{} \\
\cline{1-9}
9 & 7 & 4 & 3 & 2 & 1 & 0 & -1 & \multicolumn{3}{c}{} \\
\cline{1-8}
7 & 5 & 2 & 1 & 0 & -1 & -2 & -3 & \multicolumn{3}{c}{} \\
\cline{1-8}
5 & 3 & 0 & -1 & -2 & -3 & -4 & -5 & \multicolumn{3}{c}{} \\
\cline{1-8}
 & & \multicolumn{9}{c}{} \\
\cline{1-2}
 & & \multicolumn{9}{c}{} \\
\cline{1-2}
 & \multicolumn{10}{c}{} \\
\cline{1-1}
\end{array} \tag{2.11}$$

So for this example, $\mathrm{atyp}(\Lambda_\lambda + \rho) = 3$, since the atypicality matrix contains three zeros. In the following, we shall sometimes refer to (i, j) as the position of β_{ij}; the position is also identified by a box in the (m, n)-rectangle. The row-index k of the zero with smallest row index will play an important role (see Definition 2.2).

The maximal isotropic subset S_{Λ_λ} consist now of those odd roots β_{ij} with $(\Lambda_\lambda + \rho, \beta_{ij}) = 0$, i.e. it corresponds to the zeros in the atypicality matrix. From the combinatorics of atypicality matrices [20], it follows that

$$S_{\Lambda_\lambda} = \{\beta_{i,\lambda_i+m+1-i} \mid 1 \le i \le m,\ 1 \le \lambda_i + m + 1 - i \le n\}.$$

That is to say, one finds the zeros in the atypicality matrix as follows: on row m in column $\lambda_m + 1$; on row $m - 1$ in column $\lambda_{m-1} + 2$; in general one has, see also (2.11):

Proposition 2.1 *The atypicality matrix has its zeros on row i and in column $\lambda_i + m - i + 1$ $(i = m, m - 1, \ldots)$ as long as these column indices are not exceeding n.*

Clearly, S_{Λ_λ} is in general not a subset of the set of simple roots Π (since Π contains only one odd root, $\epsilon_m - \delta_1$). So formula (2.8) cannot be applied. The purpose is to show that there exists a sequence of simple odd α-reflections such that for the new Δ'_+, where the module V_λ has highest weight Λ', there exists a $(\Lambda' + \rho')$-maximal isotropic subset $S_{\Lambda'}$ with $S_{\Lambda'} \subset \Pi' \subset \Delta'_+$.

Definition 2.2 For $\lambda \in \mathcal{H}_{m,n}$, the (m, n)-index of λ is the number

$$k = \min\{i \mid \lambda_i + m + 1 - i \le n\}, \quad (1 \le k \le m + 1). \tag{2.12}$$

In what follows, k will always denote this number; it will be a crucial entity for our developments. By Proposition 2.1, $m - k + 1$ is the atypicality of V_λ. If $k = m + 1$, then $S_{\Lambda_\lambda} = \emptyset$ and V_λ is typical and trivially tame. Thus in the following, we shall assume that $k \le m$. To begin with, Δ_+ corresponds to the distinguished choice, and Π is the distinguished set of simple roots (2.2). The highest weight of V_λ is given by Λ_λ. Denote $\Lambda^{(1)} = \Lambda_\lambda$, $\rho^{(1)} = \rho$ and $\Pi^{(1)} = \Pi$. Now we perform a sequence of simple odd $\alpha^{(i)}$-reflections; each of these reflections preserve $\Delta_{0,+}$ but may change $\Lambda^{(i)} + \rho^{(i)}$ and $\Pi^{(i)}$. Denote the sequence of reflections by:

$$\Lambda^{(1)} + \rho^{(1)},\ \Pi^{(1)} \xrightarrow{\alpha^{(1)}} \Lambda^{(2)} + \rho^{(2)},\ \Pi^{(2)} \xrightarrow{\alpha^{(2)}} \cdots \xrightarrow{\alpha^{(f)}} \Lambda' + \rho',\ \Pi' \tag{2.13}$$

where, at each stage, $\alpha^{(i)}$ is an odd root from $\Pi^{(i)}$. For given λ, consider the following sequence of odd roots (with positions on row m, row $m - 1, \ldots$, row k):

$$\begin{array}{ll} \text{row } m: & \beta_{m,1}, \beta_{m,2}, \ldots, \beta_{m,\lambda_k-k+m} \\ \text{row } m-1: & \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,\lambda_k-k+m-1} \\ \vdots & \vdots \\ \text{row } k: & \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,\lambda_k} \end{array} \tag{2.14}$$

in this particular order (i.e. starting with $\beta_{m,1}$ and ending with β_{k,λ_k}). Then we have:

Lemma 2.3 *The sequence* (2.14) *is a proper sequence of simple odd reflections for* Λ_λ, *i.e.* $\alpha^{(i)}$ *is a simple odd root from* $\Pi^{(i)}$. *At the end of the sequence, one finds*:

$$\begin{aligned}\Pi' = \big\{&\epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \ldots, \epsilon_{k-2}-\epsilon_{k-1}, \epsilon_{k-1}-\delta_1, \delta_1-\delta_2, \delta_2-\delta_3, \ldots, \delta_{\lambda_k-1}-\delta_{\lambda_k},\\ &\delta_{\lambda_k}-\epsilon_k, \epsilon_k-\delta_{\lambda_k+1}, \delta_{\lambda_k+1}-\epsilon_{k+1}, \epsilon_{k+1}-\delta_{\lambda_k+2}, \ldots, \delta_{\lambda_k+m-k}-\epsilon_m,\\ &\epsilon_m-\delta_{\lambda_k+m+1-k}, \delta_{\lambda_k+m+1-k}-\delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1}-\delta_n\big\}.\end{aligned} \tag{2.15}$$

Furthermore,

$$\Lambda' + \rho' = \Lambda_\lambda + \rho + \sum_{i=k+1}^{m} \sum_{j=\lambda_i+1}^{\lambda_k-k+i} \beta_{i,j}. \tag{2.16}$$

Proof: Let us consider, in the first stage, the reflections with respect to the roots in row m. Clearly, $\alpha^{(1)} = \beta_{m,1} = \epsilon_m - \delta_1$ is an odd root from $\Pi^{(1)} = \Pi$. Performing the reflection with respect to $\beta_{m,1}$ implies that $\Pi^{(2)}$ contains $\epsilon_{m-1}-\delta_1, \delta_1-\epsilon_m, \epsilon_m-\delta_2$ as simple odd roots. Thus $\Pi^{(2)}$ contains $\alpha^{(2)} = \beta_{m,2}$. A reflection with respect to β_{m-2} implies that $\Pi^{(3)}$ contains $\epsilon_{m-1}-\delta_1, \delta_2-\epsilon_m, \epsilon_m-\delta_3$ as simple odd roots. So this process continues, and after $\lambda_k - k + m$ such reflections (i.e. at the end of row m), we have

$$\begin{aligned}\Pi^{(\lambda_k-k+m+1)} = \big\{&\epsilon_1-\epsilon_2, \ldots, \epsilon_{m-2}-\epsilon_{m-1}, \epsilon_{m-1}-\delta_1,\\ &\delta_1-\delta_2, \ldots, \delta_{\lambda_k-k+m-1}-\delta_{\lambda_k-k+m}, \delta_{\lambda_k-k+m}-\epsilon_m,\\ &\epsilon_m-\delta_{\lambda_k-k+m+1}, \delta_{\lambda_k+m+1-k}-\delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1}-\delta_n\big\}.\end{aligned} \tag{2.17}$$

Observe that this process can continue since $\lambda_k - k + m < n$ by definition of the (m, n)-index k of λ. So after the first stage (i.e. after the reflections with respect to odd roots of row m) there are three odd roots in $\Pi^{(\lambda_k-k+m+1)}$, and the set is ready to continue the reflections with respect to the elements of row $m-1$, since $\beta_{m-1,1}$ belongs to $\Pi^{(\lambda_k-k+m+1)}$. Since at each stage $\alpha^{(i)}$ is a simple odd root, (2.7) implies

$$\begin{aligned}\Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} \ \text{ if } \ \big(\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}\big) \neq 0,\\ \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} + \alpha^{(i)} \ \text{ if } \ \big(\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}\big) = 0.\end{aligned}$$

Examining this explicitly for the elements of row m yields

$$\Lambda^{(\lambda_k-k+m+1)} + \rho^{(\lambda_k-k+m+1)} = \Lambda_\lambda + \rho + \sum_{j=\lambda_m+1}^{\lambda_k-k+m} \beta_{m,j}. \tag{2.18}$$

If $k = m$ the lemma follows. If $k < m$ the process continues; suppose this is the case. But now we are in a situation where the elements of row $m-1$ play completely the same role as those of row m in the first stage. This means that at the end of the second stage, the new set of simple roots is given by

$$\begin{aligned}\Pi^{(i)} = \big\{&\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-3} - \epsilon_{m-2}, \epsilon_{m-2} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{\lambda_k-k+m-2} - \delta_{\lambda_k-k+m-1},\\ &\delta_{\lambda_k-k+m-1} - \epsilon_{m-1}, \epsilon_{m-1} - \delta_{\lambda_k-k+m}, \delta_{\lambda_k-k+m} - \epsilon_m, \epsilon_m - \delta_{\lambda_k-k+m+1},\\ &\delta_{\lambda_k+m+1-k} - \delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1} - \delta_n\big\},\end{aligned} \tag{2.19}$$

and the new $\Lambda^{(i)} + \rho^{(i)}$ by

$$\Lambda^{(i)} + \rho^{(i)} = \Lambda_\lambda + \rho + \sum_{j=\lambda_m+1}^{\lambda_k-k+m} \beta_{m,j} + \sum_{j=\lambda_{m-1}+1}^{\lambda_k-k+m-1} \beta_{m-1,j} \tag{2.20}$$

(the last addition follows by inspecting the atypicality matrix). Continuing with the remaining stages (i.e. rows in (2.14)) leads to (2.15) and (2.16). □

Corollary 2.4 *The covariant module V_λ is tame.*

Proof: Having performed the simple odd reflections (2.14), one can compute the atypicality matrix for $\Lambda' + \rho'$ using (2.16). This gives:

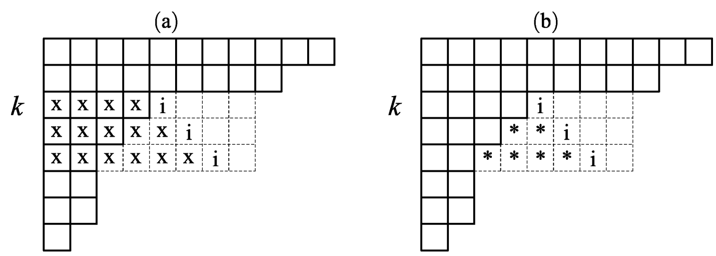
$$(\Lambda' + \rho', \beta_{ij}) = 0 \quad \text{for all } (i, j) \text{ with } k \le i \le m, \lambda_k + 1 \le j \le \lambda_k + m + 1 - k. \tag{2.21}$$

Therefore the set

$$S_{\Lambda'} = \big\{\epsilon_k - \delta_{\lambda_k+1}, \epsilon_{k+1} - \delta_{\lambda_k+2}, \ldots, \epsilon_m - \delta_{\lambda_k+m+1-k}\big\} \tag{2.22}$$

is a $(\Lambda' + \rho')$-maximal isotropic subset. Furthermore, $S_{\Lambda'} \subset \Pi'$, see (2.15). □

Let us illustrate some of these notions for Example (1.9).



(2.23)

In (2.23)(a) the positions marked with "i" refer to the $(\Lambda' + \rho')$-maximal isotropic set (2.22). The first element, with row index k, is simply the position of the box just to the right of the Young diagram F^λ in row k. For the remaining positions one continues in the direction of the diagonal until one reaches row m. For convenience, let us refer to these positions as "the isotropic diagonal." The positions of the odd roots that have been used for the sequence of reflections to go from Λ_λ and Π to Λ' and Π' are marked by "x" in (2.23)(a). So, they are simply all positions to the left of the isotropic diagonal. Finally, (2.23)(b) shows the positions of those β_{ij} that appear on the right hand side of (2.16); they are marked by "*". These are all positions to the left of the isotropic diagonal that are not inside F^λ. One can see from these examples that the (m, n)-index k determines all other necessary ingredients.

We are now in a position to evaluate the character formula (2.8),

$$\operatorname{ch} V_\lambda = j_{\Lambda'}^{-1}\, e^{-\rho'} R'^{-1} \sum_{w \in W} \varepsilon(w) w \left(e^{\Lambda'+\rho'} \prod_{\beta \in S_{\Lambda'}} (1 + e^{-\beta})^{-1} \right), \tag{2.24}$$

where R' is given by (2.9) with $\Delta_{1,+}$ replaced by $\Delta'_{1,+}$ ($\Delta_{0,+}$ remains unchanged). The mn elements of $\Delta'_{1,+}$ are $\pm\beta_{ij}$, where one must take the minus-sign if β_{ij} appears in the list (2.14) (i.e. if its position is marked by "x" in (2.23)(a)) and the plus-sign otherwise. However, all this information is not necessary here, since by definition of ρ and R

$$e^{-\rho'} R'^{-1} = e^{-\rho} R^{-1}.$$

Putting, as usual in this context,

$$x_i = e^{\epsilon_i}, \quad y_j = e^{\delta_j} \quad (1 \le i \le m, 1 \le j \le n), \tag{2.25}$$

one has

$$e^{-\rho} R^{-1} = D^{-1} \prod_{i=1}^{m} x_i^{(m-n-1)/2} \prod_{j=1}^{n} y_j^{(n-m-1)/2},$$

with D given in (1.15). Using (2.16) and (2.22), one finds

$$\begin{aligned} e^{\Lambda'+\rho'} \prod_{\beta \in S_{\Lambda'}} (1 + e^{-\beta})^{-1} &= \prod_{i=1}^{m} x_i^{(1-m+n)/2} \prod_{j=1}^{n} y_j^{(1+m-n)/2} \prod_{i=1}^{k-1} x_i^{\lambda_i+m-i-n} \\ &\quad \times \prod_{i=k}^{m} x_i^{\lambda_k+m+1-k-n} \prod_{j=1}^{\lambda_k} y_j^{\lambda'_j+n-j-m} \prod_{j=\lambda_k+1}^{\lambda_k+m+1-k} y_j^{n-\lambda_k-m-1+k} \\ &\quad \times \prod_{j=\lambda_k+m+2-k}^{n} y_j^{n-j} \Big/ \prod_{i=k}^{m} (x_i + y_{\lambda_k+i+1-k}). \end{aligned} \tag{2.26}$$

In order to rewrite this in a more appropriate form, let us introduce some further notation related to the partition $\lambda \in \mathcal{H}_{m,n}$. Clearly, the (m, n)-index k defined in (2.12) plays again an essential role. Related to this, let us also put

$$l = \lambda_k + 1, \quad r = n - m + k - l. \tag{2.27}$$

Now we have

$$\operatorname{ch} V_\lambda = j_{\Lambda'}^{-1} D^{-1} \sum_{w \in W} \varepsilon(w) w(t_\lambda),$$

with

$$t_\lambda = \prod_{i=1}^{k-1} x_i^{\lambda_i+m-i-n} \prod_{j=1}^{l-1} y_j^{\lambda'_j+n-j-m} \prod_{i=k}^{m} \frac{y_{l+i-k}^r}{x_i^r(x_i + y_{l+i-k})} \prod_{j=l+m+1-k}^{n} y_j^{n-j}. \tag{2.28}$$

This form also allows us to deduce $j_{\Lambda'} = j_{\Lambda_\lambda}$. Indeed, consider in $W = S_m \times S_n$ the subgroup H of elements $w = \sigma_x \times \sigma_y$, where σ_x is a permutation of $(x_k, x_{k+1}, \ldots, x_m)$ and σ_y is the same permutation of $(y_l, y_{l+1}, \ldots, y_{l+m-k})$. Each element of H leaves t_λ invariant. Furthermore, $\varepsilon(w) = 1$ for $w \in H$. Since H is isomorphic to S_{m-k+1},

$$\sum_{w\in H} \varepsilon(w)w(t_\lambda) = (m-k+1)!\, t_\lambda.$$

So, in order to have multiplicity one for the highest weight term, one must take $j_{\Lambda'} = (m-k+1)!$.

We can now conclude this section by the following two alternative formulas for the computation of $\operatorname{ch} V_\lambda$ (i.e. of $s_\lambda(x/y)$):

$$\operatorname{ch} V_\lambda = \frac{D^{-1}}{(m-k+1)!} \sum_{w\in S_m\times S_n} \varepsilon(w)w(t_\lambda) \tag{2.29}$$

$$= D^{-1} \sum_{w\in(S_m\times S_n)/S_{m-k+1}} \varepsilon(w)w(t_\lambda), \tag{2.30}$$

where the second sum is over the cosets of $H = S_{m-k+1}$ in $S_m \times S_n$.

3. A determinantal formula for $s_\lambda(x/y)$

Let $\lambda \in \mathcal{H}_{m,n}$; the important quantities related to λ are given by the (m,n)-index k (see (2.12)) and the related numbers l and r (see (2.27)). Since $\operatorname{ch} V_\lambda = s_\lambda(x/y)$, (2.29) or (2.30), together with the expression (2.28) for t_λ, yield a (new) expression for the supersymmetric S-function. In this section we shall rewrite (2.29) in a nicer form; in particular we shall show that it is equivalent to a determinantal form for $s_\lambda(x/y)$.

The first step in this process is the following:

Lemma 3.1 *Let t_λ be given by* (2.28). *Then*

$$\frac{1}{(m-k+1)!} \sum_{w\in S_m\times S_n} \varepsilon(w)w(t_\lambda) = (-1)^{(k-1)(l-1)} \det(C), \tag{3.1}$$

where C is the following square matrix of order $n+k-1$:

$$C = \begin{pmatrix} 0 & Y_\lambda^{(1)} \\ X_\lambda & R^{(r)} \\ 0 & Y^{(r)} \end{pmatrix} \tag{3.2}$$

with

$$X_\lambda = \left(x_i^{\lambda_j+m-n-j}\right)_{1\le i\le m,\ 1\le j\le k-1}, \quad R^{(r)} = \left(\frac{y_j^r}{x_i^r(x_i+y_j)}\right)_{1\le i\le m,\ 1\le j\le n} \tag{3.3}$$

$$Y_\lambda^{(1)} = \left(y_j^{\lambda'_i+n-m-i}\right)_{1\le i\le l-1,\ 1\le j\le n}, \quad Y^{(r)} = \left(y_j^{r-i}\right)_{1\le i\le r,\ 1\le j\le n} \tag{3.4}$$

Proof: Applying Laplace's theorem [16, Section 1.8] for the expansion of det(C) with respect to columns $1, 2, \ldots, k-1$, one finds

$$\det(C) = (-1)^{\frac{k(k-1)}{2}} \sum_{1\le i'_1<\cdots<i'_{k-1}\le n+k-1} (-1)^{i'_1+\cdots+i'_{k-1}} C^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1} \bar{C}^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}, \tag{3.5}$$

where $C^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}$ is a minor of C of order $k-1$ (i.e. the determinant of the submatrix of C obtained by keeping only rows $i'_1, \ldots, i'_{k-1}$ and columns $1, \ldots, k-1$), and $\bar{C}^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}$ is the complementary minor (i.e. the determinant of the submatrix of C obtained by deleting rows $i'_1, \ldots, i'_{k-1}$ and columns $1, \ldots, k-1$). Because of the zero blocks in C, the only row indices that yield a nonzero minor are those with $l \le i'_1 < \cdots < i'_{k-1} \le l+m-1$. So it is convenient to put $i_\kappa = i'_\kappa - l + 1$ $(1 \le \kappa \le k-1)$. Then,

$$\det(C) = (-1)^{\frac{k(k-1)}{2}} \sum_{1\le i_1<\cdots<i_{k-1}\le m} (-1)^{i_1+\cdots+i_{k-1}+(k-1)(l-1)} C^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1} \bar{C}^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}, \tag{3.6}$$

where

$$C^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1} = \det\begin{pmatrix} x_{i_1}^{\lambda_1+(m-1)-n} & \cdots & x_{i_1}^{\lambda_{k-1}+(m-k+1)-n} \\ \vdots & & \vdots \\ x_{i_{k-1}}^{\lambda_1+(m-1)-n} & \cdots & x_{i_{k-1}}^{\lambda_{k-1}+(m-k+1)-n} \end{pmatrix},$$

$$\bar{C}^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1} = \det\begin{pmatrix} y_1^{\lambda'_1+(n-1)-m} & \cdots & y_n^{\lambda'_1+(n-1)-m} \\ \vdots & & \vdots \\ y_1^{\lambda'_{l-1}+(n-l+1)-m} & \cdots & y_n^{\lambda'_{l-1}+(n-l+1)-m} \\ \frac{y_1^r}{x_{i_k}^r(x_{i_k}+y_1)} & \cdots & \frac{y_n^r}{x_{i_k}^r(x_{i_k}+y_n)} \\ \vdots & & \vdots \\ \frac{y_1^r}{x_{i_m}^r(x_{i_m}+y_1)} & \cdots & \frac{y_n^r}{x_{i_m}^r(x_{i_m}+y_n)} \\ y_1^{r-1} & \cdots & y_n^{r-1} \\ \vdots & & \vdots \\ y_1^1 & \cdots & y_n^1 \\ y_1^0 & \cdots & y_n^0 \end{pmatrix}.$$

The number of terms on the right hand side of (3.6) is $\binom{m}{k-1}(k-1)!n! = m!n!/(m-k+1)!$, so this is the same as the number of distinct terms on the lhs of (3.1). For $(i_1, \ldots, i_{k-1}) = (1, \ldots, k-1)$, and the diagonal term in the minor and in the complementary minor, the contribution on the right hand side of (3.6) is now easily seen to be $(-1)^{k(k-1)+(k-1)(l-1)}t_\lambda = (-1)^{(k-1)(l-1)}t_\lambda$. But by definition of the determinant, every term on the right hand side of (3.6) is (up to the overall sign factor $(-1)^{(k-1)(l-1)}$) of the form $\varepsilon(w)w(t_\lambda)$ with $w \in S_m \times S_n$. Conversely, every term of the form $\varepsilon(w)w(t_\lambda)$ appears as a term on the right hand side of (3.6). It follows that (3.1) holds. □

Lemma 3.2 *With k, l and r defined as in* (2.12) *and* (2.27), *one has*

(i) $\lambda_{k-1} - \lambda_k \geq r \geq 0$,
(ii) $\lambda'_{l-1+i} = k-1$ *for* $1 \leq i \leq r$.

Proof: By (2.27), $r = n-m+k-l = n-m+k-\lambda_k-1$, so by definition of k it follows that $r \geq 0$. Suppose that $\lambda_{k-1} - \lambda_k < r$; this would imply that $\lambda_{k-1} + m - (k-1) + 1 \leq n$, contradicting the definition of k as the minimal index for which $\lambda_j + m - j + 1 \leq n$ holds. So (i) holds. To prove (ii), observe that by definition of the conjugate partition,

$$\lambda'_j = k-1 \quad \text{for } \lambda_k + 1 \leq j \leq \lambda_{k-1}.$$

So in particular, using (i), $\lambda'_j = k-1$ for $\lambda_k + 1 \leq j \leq \lambda_k + r$, i.e. for $l \leq j \leq l+r-1$. □

Lemma 3.3 *Let C be defined as in Lemma* 3.1. *Then*

$$\det(C) = (-1)^{m(n-m)+l(k-1)} \det\begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \tag{3.7}$$

where X_λ has been defined in (3.3), *and*

$$R = \left(\frac{1}{x_i + y_j}\right)_{1\leq i\leq m, 1\leq j\leq n}, \quad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i}\right)_{1\leq i\leq n-m+k-1, 1\leq j\leq n}.$$

Proof: If $r = 0$ the result is easy; so suppose that $r > 0$. Consider C, and let R_i denote its row i. For $1 \leq i \leq m$, multiply R_{l-1+i} by x_i^r yielding the matrix C' with rows R'_i. Clearly, $\det(C) = \prod_{i=1}^m x_i^{-r} \det(C')$, and the element on position $(l-1+i, k-1+j)$ $(1 \leq i \leq m;$ $1 \leq j \leq n)$ in C' is $\frac{y_j^r}{x_i+y_j}$. Now consider the following row operations on C' (for all i with $1 \leq i \leq m$):

$$R'_{l-1+i} \to R'_{l-1+i} - \sum_{s=0}^{r-1}(-1)^s x_i^s R'_{l+m+s}.$$

This means the matrix element $\frac{y_j^r}{x_i+y_j}$ at position $(l-1+i, k-1+j)$ becomes

$$\frac{y_j^r}{x_i+y_j} - \sum_{s=0}^{r-1}(-1)^s x_i^s y_j^{r-s-1} = \frac{y_j^r}{x_i+y_j} - \frac{y_j^r + (-1)^{r-1}x_i^r}{x_i+y_j} = \frac{(-1)^r x_i^r}{x_i+y_j}.$$

Dividing these rows by $(-1)^r x_i^r$ (and then the first $k-1$ columns by $(-1)^r$) finally leads to

$$\det(C) = (-1)^{mr+r(k-1)} \det \begin{pmatrix} 0 & Y_\lambda^{(1)} \\ X_\lambda & R \\ 0 & Y^{(r)} \end{pmatrix}$$
$$= (-1)^{r(m+k-1)+m(l-1)+n(k-1)} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda^{(1)} & 0 \\ Y^{(r)} & 0 \end{pmatrix}.$$

Using Lemma 3.2(ii), it follows that $r - i = \lambda'_{l-1+i} + n - (l-1+i) - m$, thus

$$\begin{pmatrix} Y_\lambda^{(1)} \\ Y^{(r)} \end{pmatrix} = Y_\lambda. \tag{3.8}$$

Simplifying the sign leads to (3.7). □

Combining Lemmas 3.1 and 3.3 leads to the main result:

Theorem 3.4 *Let $\lambda \in \mathcal{H}_{m,n}$ and k be the (m,n)-index of λ. Then*

$$s_\lambda(x/y) = (-1)^{mn-m+k-1} D^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix}. \tag{3.9}$$

In Section 1 we have already pointed out that for $\lambda = 0$ and $m \le n$, (3.9) gives rise to the determinantal identity (1.19) combining Cauchy's double alternant with Vandermonde's determinant. It is easy to verify that for $\lambda = 0$ and $m > n$ we have that $k = n - m + 1$, and then (3.9) gives rise to an identity equivalent to (1.19).

Finally, let us consider the case with $k = m+1$ (corresponding to a typical representation V_λ in terms of the previous section). Then the blocks X_λ and Y_λ are square matrices, and one finds:

$$s_\lambda(x/y) = D^{-1} \det\left(x_i^{\lambda_j+m-n-j}\right)_{1\le i,j\le m} \det\left(y_j^{\lambda'_i+n-m-i}\right)_{1\le i,j\le n}$$
$$= s_\tau(x)s_{\eta'}(y) \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j) = s_\tau(x)s_{\eta'}(y)E(x,-y), \tag{3.10}$$

where τ and η are the parts of λ defined in (1.9). This is in agreement with (1.12).

4. Four characterizing properties of supersymmetric Schur polynomials

In [10, Section I.3, Exercise 23], Macdonald shows that the supersymmetric S-polynomials satisfy four properties (see also [13]) which also characterize these polynomials.[1] Here we shall show that the polynomials defined by means of the right hand side of (3.9) do indeed satisfy these four properties. This gives an independent proof of the determinantal expression (3.9).

So let $x^{(m)} = (x_1, \ldots, x_m)$, $y^{(n)} = (y_1, \ldots, y_n)$, $\lambda \in \mathcal{H}_{m,n}$, denote by k the (m,n)-index of λ, and define:

$$s_\lambda\big(x^{(m)}/y^{(n)}\big) = s_\lambda(x/y) = (-1)^{mn-m+k-1} D_{m,n}^{-1} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \tag{4.1}$$

where

$$R = \left(\frac{1}{x_i + y_j}\right)_{\substack{1\le i\le m\\ 1\le j\le n}}, \quad X_\lambda = \big(x_i^{\lambda_j+m-n-j}\big)_{\substack{1\le i\le m\\ 1\le j\le k-1}}, \quad Y_\lambda = \big(y_j^{\lambda'_i+n-m-i}\big)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}},$$

and $D_{m,n}$ is an obvious notation for (1.15). If $\lambda \notin \mathcal{H}_{m,n}$, then $s_\lambda(x^{(m)}/y^{(n)})$ is defined to be zero.

Before showing that (4.1) satisfies the four characteristic properties, we need some preliminary results. First of all, it will sometimes be convenient to use an alternative formula to (4.1):

Lemma 4.1 *Let $\lambda \in \mathcal{H}_{m,n}$, k the (m,n)-index of λ, $l = \lambda_k + 1$, $r = n - m + k - l$, and*

$$R^{(r)} = \left(\frac{y_j^r}{x_i^r(x_i + y_j)}\right)_{\substack{1\le i\le m\\ 1\le j\le n}}$$

then

$$s_\lambda\big(x^{(m)}/y^{(n)}\big) = (-1)^{(m-k+1)(l-1)+n(k-1)} D_{m,n}^{-1} \det \begin{pmatrix} R^{(r)} & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}. \tag{4.2}$$

Proof: The equivalence of the two determinants follows from (3.7). □

Lemma 4.2 *Let $\lambda \in \mathcal{H}_{m,n}$ and k the (m,n)-index of λ. Then $\lambda' \in \mathcal{H}_{n,m}$ and the (n,m)-index of λ' is given by $n - m + k$.*

Proof: For any partition λ and any two positive integers u and v, we have that

$$\lambda_u \le v \Rightarrow \lambda'_{v+1} \le u - 1. \tag{4.3}$$

Applying this for $(u, v) = (m+1, n)$ implies that $\lambda' \in \mathcal{H}_{n,m}$. Put $k' = n - m + k$. Since k is the (m, n)-index of λ, one has by definition that $\lambda_k \leq k' - 1$. Applying (4.3) gives $\lambda'_{k'} \leq k - 1 = m - n + k' - 1$, implying that the (n, m)-index of λ' is less than or equal to k'. On the other hand, suppose that the (n, m)-index of λ' is strictly less than k'. Then $\lambda'_{k'-1} \leq m - n + k' - 2 = k - 2$. But then (4.3) implies $\lambda_{k-1} \leq k' - 2 = n - m + k - 2$, contradicting the fact that k is (m, n)-index of λ. □

Corollary 4.3

$$s_{\lambda'}\left(y^{(n)}/x^{(m)}\right) = s_\lambda\left(x^{(m)}/y^{(n)}\right). \tag{4.4}$$

Proof: Using (4.1), Lemma 4.2, and the fact that transposing a matrix leaves the determinant invariant, the result follows. □

Now we are in a position to prove the validity of the four characteristic properties.

Proposition 4.4 (Homogeneity) *$s_\lambda(x^{(m)}/y^{(n)})$ is a homogeneous polynomial of degree $|\lambda|$.*

Proof: The factor $D_{m,n}^{-1}$ on the right hand side of (4.1) stands for the multiplication by all $(x_i + y_j)$ and for the division by all $(x_i - x_j)$ and $(y_i - y_j)$ $(i < j)$. Clearly, the determinant on the right hand side of (4.1) is divisible by all $(x_i - x_j)$ and $(y_i - y_j)$ $(i < j)$. Hence $s_\lambda(x^{(m)}/y^{(n)})$ is a polynomial, and by definition of the determinant it is also homogeneous. From this determinant, one finds

$$\begin{aligned}\deg s_\lambda\left(x^{(m)}/y^{(n)}\right) = mn - m(m-1)/2 - n(n-1)/2 + \Bigg(\sum_{j=1}^{k-1}(\lambda_j + m - n - j) \\ + \sum_{i=1}^{n-m+k-1}(\lambda'_i + n - m - i) - (m - k + 1)\Bigg) = |\lambda|,\end{aligned}$$

since $\sum_{j=1}^{k-1}\lambda_j + \sum_{i=1}^{n-m+k-1}\lambda'_i = |\lambda| + (n - m + k - 1)(k - 1)$. □

Proposition 4.5 (Factorization) *If the partition λ satisfies $\lambda_m \geq n \geq \lambda_{m+1}$, so that λ can be written in the form $\lambda = ((n^m) + \tau) \cup \eta$, with τ (resp. η) a partition of length $\leq m$ (resp. $\leq n$), then*

$$s_\lambda\left(x^{(m)}/y^{(n)}\right) = s_\tau\left(x^{(m)}\right)s_{\eta'}\left(y^{(n)}\right)\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j).$$

Proof: If $n \geq \lambda_{m+1}$, then $k = m + 1$, and then the right hand side of (4.1) reduces to the right hand side of (3.10). □

Proposition 4.6 (Cancellation) *Let $m, n \geq 1$. The result of substituting $x_m = t$ and $y_n = -t$ in $s_\lambda(x^{(m)}/y^{(n)})$ is the polynomial $s_\lambda(x^{(m-1)}/y^{(n-1)})$.*

Proof: If k, the (m, n)-index of λ, is equal to $m+1$, then it follows from Proposition 4.5 that the result of this substitution is 0; but also $s_\lambda(x^{(m-1)}/y^{(n-1)}) = 0$ since then $\lambda \notin \mathcal{H}_{m-1,n-1}$. So let λ be such that $k \leq m$. Then $\lambda \in \mathcal{H}_{m-1,n-1}$ and the $(m-1, n-1)$-index of λ is also k. Consider the right hand side of (4.1); divide $D_{m,n}^{-1}$ by $(x_m + y_n)$ and multiply row m of the matrix by $(x_m + y_n)$. Now make the substitution $x_m = t$ and $y_n = -t$; one obtains $D_{m-1,n-1}^{-1}$. In the determinant, all entries of row m are zero except the entry in column n, which is 1. So, apart from introducing an extra factor $(-1)^{m+n}$, we can delete row m and column n. But the remaining determinant is exactly the one in the expression of $s_\lambda(x^{(m-1)}/y^{(n-1)})$, and because of the factor $(-1)^{m+n}$ also the sign works out correctly. □

The only property that is more tedious to prove is the last one:

Proposition 4.7 (Restriction) *Let $m \geq 1$ (resp. $n \geq 1$). The result of setting $x_m = 0$ (resp. $y_n = 0$) in $s_\lambda(x^{(m)}/y^{(n)})$ is the polynomial $s_\lambda(x^{(m-1)}/y^{(n)})$ (resp. $s_\lambda(x^{(m)}/y^{(n-1)})$).*

Proof: By (4.4), it is sufficient to prove only the case $y_n = 0$. For $D_{m,n}$, we have

$$D_{m,n}^{-1}\big|_{y_n=0} = \frac{\prod_{i=1}^{m} x_i}{\prod_{j=1}^{n-1} y_j} D_{m,n-1}^{-1}. \tag{4.5}$$

There are now two cases to consider: the $(m, n-1)$-index of λ is k, or the $(m, n-1)$-index of λ is $k+1$ (no other values are possible).

First case. We shall use expression (4.2). Suppose that the $(m, n-1)$-index of λ is also k. This means that $\lambda_k + m + 1 - k < n$, or $r > 0$. By (3.8), this also means that the last row of the matrix in (4.2) consists of n ones followed by $k-1$ zeros. But substituting $y_n = 0$ in the matrix of (4.2), means that in column n all entries except the last one (which is 1) become zero. Now expand the determinant with respect to column n. The result is a new determinant of order $(n-1) + k - 1$. But when going from $(x^{(m)}, y^{(n)})$ to $(x^{(m)}, y^{(n-1)})$, k remains the same, n reduces to $n-1$ and r reduces to $r-1$. So if we multiply row i $(1 \leq i \leq m)$ of this new determinant by x_i, and divide column j $(1 \leq j \leq n-1)$ of this new determinant by y_j, the resulting determinant is exactly that of $s_\lambda(x^{(x)}/y^{(n-1)})$ in the form (4.2). Furthermore, the factors appearing in (4.5) have been taken care of, and one can verify that also the sign is correct. So in this case the lemma holds.

Second case. Suppose that the $(m, n-1)$-index of λ is $k+1$. This means that $\lambda_k + m + 1 - k = n$, or $r = 0$. Consider the matrix in (4.1). It is easy to see that all powers of y_j are strictly positive (on the last row of the matrix, the power of y_j is $\lambda'_{n-m+k-1} - k + 1 = \lambda'_{\lambda_k} - k + 1 \geq 1$). So substituting $y_n = 0$ in (4.1) yields a matrix of which column n consists of $(\frac{1}{x_1}, \ldots, \frac{1}{x_m}, 0, \ldots, 0)$. Let us push this column to the end, so that it becomes column $n + k - 1$. Up to a sign, the determinant remains the same. Next, we use the factors in (4.5) to multiply in the determinant row i by x_i $(1 \leq i \leq m)$ and divide column j by y_j

$(1 \leq j \leq n-1)$. The resulting matrix is of the following type:

$$\begin{pmatrix} \frac{x_1}{y_1(x_1+y_1)} & \cdots & \frac{x_1}{y_{n-1}(x_1+y_{n-1})} & x_1^{\lambda_1+m-n} & x_1^{\lambda_2+m-n-1} & \cdots & x_1^{\lambda_{k-1}+m-n-k} & 1 \\ \vdots & & \vdots & \vdots & & & & \vdots \\ \frac{x_m}{y_1(x_m+y_1)} & \cdots & \frac{x_m}{y_{n-1}(x_m+y_{n-1})} & x_m^{\lambda_1+m-n} & x_m^{\lambda_2+m-n-1} & \cdots & x_m^{\lambda_{k-1}+m-n-k} & 1 \\ y_1^{\lambda_1'+n-m-2} & \cdots & y_{n-1}^{\lambda_1'+n-m-2} & 0 & & \cdots & & 0 \\ \vdots & & \vdots & \vdots & & & & \vdots \\ y_1^{\lambda_{n-m+k-1}'-k} & \cdots & y_{n-1}^{\lambda_{n-m+k-1}'-k} & 0 & & \cdots & & 0 \end{pmatrix} \tag{4.6}$$

Now we can do the following: for every column j $(1 \leq j \leq n-1)$, subtract from this column $1/y_j$ times the last column of the matrix, i.e.

$$C_j \to C_j - \frac{1}{y_j} C_{n+k-1}.$$

This implies that the matrix element on position (i, j), where $1 \leq i \leq m$ and $1 \leq j \leq n-1$, becomes

$$\frac{x_i}{y_j(x_i+y_j)} - \frac{1}{y_j} = \frac{-1}{x_i+y_j}.$$

After dealing with sign changes in rows and columns, we get, up to a sign:

$$D_{m,n-1}^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n-1}} & \left(x_i^{\lambda_j+m-n+1-j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq k}} \\ \left(y_j^{\lambda_i'+n+1-m-i}\right)_{\substack{1\leq i\leq n-m+k-1 \\ 1\leq j\leq n-1}} & 0 \end{pmatrix}, \tag{4.7}$$

where we have used again that $\lambda_k + m - n - 1 - k = 0$ to see that the powers of x_i agree in the last column. But (4.7) is simply the same as the right hand side of (4.1) with n replaced by $n-1$ and k by $k+1$, so it is (up to a sign) $s_\lambda(x^{(m)}/y^{(n-1)})$. To verify that also the sign agrees is left as an exercise. □

Finally, we have

Theorem 4.8 *The polynomials* $s_\lambda(x^{m)}/y^{(n)})$, *defined by* (4.1), *are the supersymmetric Schur polynomials.*

Proof: This follows immediately from Propositions 4.4-4.7, and the result that these four properties characterize the supersymmetric Schur polynomials [10, 13]. □

5. A proof of coincidence with the Sergeev-Pragacz formula

In this final section, we provide a direct proof [22], based on the Berele-Regev formula (1.12) and Laplace's theorem [16, Section 1.8], that our determinantal formula coincides with the Sergeev-Pragacz formula.

Let us write $x = x' + x''$ for a decomposition of $x = (x_1, x_2, \ldots, x_m)$ into two disjoint subsets of fixed size, say $|x'| = p$ and $|x''| = q$ with $p + q = m$. Recall the definition of $E(x', x'')$ in (1.14).

Lemma 5.1 *For $m = p + q$, let $\mu = (\mu_1, \ldots, \mu_p)$, $\nu = (\nu_1, \ldots, \nu_q)$ be two partitions and $\lambda = (\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q)$. Then*

$$\sum_{x'+x''} \frac{s_{\mu+(q^p)}(x')s_\nu(x'')}{E(x', x'')} = s_\lambda(x) \tag{5.1}$$

where the sum is over all possible decompositions $x = x' + x''$ with the size of x' equal to p and the size of x'' equal to q.

Proof: We can rewrite the lhs of expression (5.1) using the determinantal formula for S-functions and the equality $D(x) = (-1)^{\frac{p(p+1)}{2}+r_1+\cdots+r_p} D(x')D(x'')E(x', x'')$ with the elements of x' denoted by $x_{r_1}, \ldots, x_{r_p}$ and those of x'' by $x_{s_1}, \ldots, x_{s_q}$:

$$\sum_{x'+x''} \frac{s_{\mu+(q^p)}(x')s_\nu(x'')}{E(x', x'')} = \frac{(-1)^{\frac{p(p+1)}{2}}}{D(x)} \sum_{x'+x''} (-1)^{r_1+\cdots+r_p} \left|x_{r_i}^{\mu_j+q+p-j}\right| \left|x_{s_i}^{\nu_j+q-j}\right|. \tag{5.2}$$

The numerator of this sum is the Laplace expansion of the following determinant with respect to columns $1, 2, \ldots, p$:

$$\left|x^{\mu+(q^p)+\delta_p} \quad x^{\nu+\delta_q}\right| = \left|x^{\lambda+\delta_m}\right| \tag{5.3}$$

with $\lambda = (\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q)$ and δ_l given in (1.4). The result follows. □

Remark 5.2 The previous lemma and its proof can easily be extended for an arbitrary p-tuple μ over $\mathbb{Z}$ and an arbitrary q-tuple ν over $\mathbb{Z}$. In this case, $s_\alpha(x)$ (α an arbitrary m-tuple) is defined according to formula (1.5). We can extend this generalization to the supersymmetric case where $s_\alpha(x/y)$ is defined as

$$s_\alpha(x/y) = \det\big(h_{\alpha_i-i+j}(x/y)\big)_{1\le i,j\le \ell(\alpha)} \quad (\alpha \textit{ a } t\textit{-tuple with } \alpha_i \in \mathbb{Z}). \tag{5.4}$$

Note that for a t-tuple over $\mathbb{Z}$, $\alpha + \delta_t$ must consist of nonnegative distinct integers for s_α to be nonzero.

Lemma 5.3 *Suppose* $|y| = n$, $y' = (y_1, \ldots, y_{n-1})$ *and* η *an arbitrary* t*-tuple*, *then*

$$s_\eta(x/y) = \sum_\zeta s_\zeta(x/y')(y_n)^d, \tag{5.5}$$

where the sum is taken over all 2^t t*-tuples* ζ *such that* $(\eta - \zeta)_i \in \{0, 1\}$ *and* $d = |\eta - \zeta|$.

Proof: First suppose that η is a partition. If we sum only over all ζ such that $(\eta - \zeta)$ is a vertical strip, the summation follows from the expression of $s_\eta(x/y)$ by means of supertableaux [3]. All the other terms vanish as there exists an index i such that $\zeta_{i+1} = \zeta_i + 1$.

Suppose now that η is not a partition. Consider $\eta + \delta_t$. If $\eta + \delta_t$ has a negative component, then $s_\eta(x/y) = 0$; but then also each $s_\zeta(x/y')$ is zero since every $\zeta + \delta_t$ has a negative component. If $\eta + \delta_t$ has two equal parts, then $s_\eta(x/y) = 0$; then the right hand side of (5.5) consists of zero terms and terms cancelling each other two by two. Finally, if $\eta + \delta_t$ consists of distinct nonnegative parts, then $s_\eta(x/y) = \pm s_\lambda(x/y)$, with λ a partition and $\sigma(\eta + \delta_t) = \lambda + \delta_t$, for some permutation σ. Now apply (5.5) to $s_\lambda(x/y)$; applying σ^{-1} to each t-tuple in both sides of this equation yields the result. $\square$

Lemma 5.4 *Let* $m = p + q$, $\mu = (\mu_1, \ldots, \mu_p)$ *an arbitrary* p*-tuple and* $\nu = (\nu_1, \nu_2, \ldots)$ *an arbitrary* t*-tuple. Denote* $\lambda = (\mu_1, \ldots, \mu_p, \nu_1, \nu_2, \ldots)$. *Then*,

$$s_\lambda(x/-y) = \sum_{x'+x''} \frac{s_{\mu+(q^p)}(x'/-y)s_\nu(x''/-y)}{E(x', x'')} \tag{5.6}$$

where the sum is over all possible decompositions $x = x' + x''$ *with the size of* x' *equal to* p *and the size of* x'' *equal to* q.

Proof: One can use, e.g., induction on n, i.e. the number of variables $y = (y_1, \ldots, y_n)$. If $n = 0$ then we are reduced to the symmetric case and the result follows from Lemma 5.1 and Remark 5.2. Otherwise, one separates the variable y_n. We can apply Lemma 5.3 twice on the right hand side of (5.6); with $(\mu - \alpha)_i \in \{0, 1\}$, $a = |\mu - \alpha|$ and $(\nu - \beta)_i \in \{0, 1\}$, $b = |\nu - \beta|$:

$$\begin{aligned}
&\sum_{x'+x''} \frac{s_{\mu+(q^p)}(x'/-y)s_\nu(x''/-y)}{E(x', x'')} \\
&\quad = \sum_{x'+x''} \frac{1}{E(x', x'')} \left(\sum_\alpha s_{\alpha+(q^p)}(x'/-y')(-y_n)^a \right) \left(\sum_\beta s_\beta(x''/-y')(-y_n)^b \right) \\
&\quad = \sum_\alpha \sum_\beta \left(\sum_{x'+x''} \frac{s_{\alpha+(q^p)}(x'/-y')s_\beta(x''/-y')}{E(x', x'')} \right) (-y_n)^{a+b}
\end{aligned}$$

Using induction, this expression reduces to $\sum_\gamma s_\gamma(x'/-y')(-y_n)^c$ with $c = a + b$, $\gamma = (\alpha_1, \ldots, \alpha_p, \beta_1, \beta_2, \ldots)$ and $(\alpha - \gamma)_i \in \{0, 1\}$. Now it is easy to see that this expression is equal to $s_\lambda(x/-y)$ applying (5.5). $\square$

We can now give a direct proof of the determinantal formula.

Theorem 5.5 *Let $\lambda \in \mathcal{H}_{m,n}$ and k be the (m, n)-index of λ. Then*

$$\frac{E(x,y)}{D(x)D(y)} \det \begin{pmatrix} \frac{1}{x-y} & X_\lambda \\ Y_\lambda & 0 \end{pmatrix} = \pm s_\lambda(x/-y), \tag{5.7}$$

where the (rectangular) blocks of the determinant are given by

$$\frac{1}{x-y} = \left(\frac{1}{x_i - y_j}\right)_{1\le i\le m, 1\le j\le n},$$
$$X_\lambda = \left(x_i^{\lambda_j+m-n-j}\right)_{1\le i\le m, 1\le j\le k-1}, \quad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i}\right)_{1\le i\le n-m+k-1, 1\le j\le n}.$$

Proof: Suppose $|x'| = k-1 \equiv p$ and $|y'| = n-m+k-1 \equiv q$. We will indicate the indices of the elements of x' by i_t $(t = 1, \ldots, p)$ and those of y' by j_t $(t = 1, \ldots, q)$. The determinant in (5.7) has a double Laplace expansion, with partitions $\alpha = (\lambda_1, \ldots, \lambda_{k-1}) - (q^p)$ and $\beta = (\lambda'_1, \ldots, \lambda'_q) - (p^q)$ determined by X_λ and Y_λ, so the lhs of (5.7) equals:

$$\frac{E(x,y)}{D(x)D(y)} \sum_{x=x'+x''} \sum_{y=y'+y''} (-1)^P D(x')D(y')s_\alpha(x')s_\beta(y') \det\left(\frac{1}{x''-y''}\right)$$
$$\left(\text{with } P = \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \sum_{t=1}^{p} i_t + \sum_{t=1}^{q} j_t\right)$$
$$= \frac{E(x,y)}{D(x)D(y)} \sum_{x'+x''} \sum_{y'+y''} (-1)^P D(x')D(y')s_\alpha(x')s_\beta(y')D(x'')D(y'')/E(x'',y'')$$
$$= \sum_{x'+x''} \sum_{y'+y''} s_\alpha(x')s_\beta(y') \frac{E(x,y')E(x',y'')}{E(x',x'')E(y',y'')}$$
$$= \sum_{y'+y''} \left(\sum_{x'+x''} \frac{s_\alpha(x')E(x',y'')}{E(x',x'')}\right) \frac{s_\beta(y')(-1)^{mq}E(y',x)}{E(y',y'')}.$$

Now we can apply the Berele-Regev formula (1.12) twice. Putting $\eta = \alpha + ((m-k+1)^p)$ and $\chi = \beta + (m^q) = (\lambda'_1, \ldots, \lambda'_q) + ((m-k+1)^q)$, there comes

$$\pm \sum_{y'+y''} \left(\sum_{x'+x''} \frac{s_\eta(x'/-y'')}{E(x',x'')}\right) \frac{s_\chi(y'/-x)}{E(y',y'')}$$
$$= \pm \sum_{y'+y''} \left(\sum_{x'+x''} \frac{s_\eta(x'/-y'')}{E(x',x'')}\right) \frac{s_\chi(-y'/x)}{E(y',y'')}.$$

Finally, we use Lemma 5.4, and duality (1.13); the last expression becomes:

$$\pm \sum_{y'+y''} s_\alpha(x/-y'')s_\chi(-y'/x)\frac{1}{E(y',y'')} = \pm(-1)^{|\alpha|} \sum_{y'+y''} \frac{s_{\alpha'}(-y''/x)s_\chi(-y'/x)}{E(y',y'')}.$$

As $\alpha' = (\lambda'_{q+1}, \lambda'_{q+2}, \ldots)$ and $\chi = (\lambda'_1, \ldots, \lambda'_q) + ((m-k+1)^q)$, this is equal to $\pm s_{\lambda'}(-y/x) = \pm s_\lambda(x/-y)$, using once more Lemma 5.4 and duality. □

Thus, the proof uses essentially a double Laplace expansion, twice the application of the Berele-Regev formula, and twice Lemma 5.4. Finally, observe that the sign in Theorem 5.5 depends on the partition λ, in particular it is equal to

$$(-1)^{\sum_{i=1}^{n-m+k-1} \lambda'_i + \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - 1}.$$

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Note

1. Our convention for the $s_\lambda(x/y)$ is slightly different from that of Macdonald's $s_\lambda^{\mathrm{Mac}}(x/y)$: $s_\lambda(x/y) = s_\lambda^{\mathrm{Mac}}(x/-y)$.

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