Perfect Matchings and Perfect Powers

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Abstract. In the last decade there have been many results about special families of graphs whose number of perfect matchings is given by perfect or near perfect powers (N. Elkies et al., *J. Algebraic Combin.* **1** (1992), 111–132; B.-Y. Yang, Ph.D. thesis, Department of Mathematics, MIT, Cambridge, MA, 1991; J. Propp, *New Perspectives in Geometric Combinatorics*, Cambridge University Press, 1999). In this paper we present an approach that allows proving them in a unified way. We use this approach to prove a conjecture of James Propp stating that the number of tilings of the so-called Aztec dungeon regions is a power (or twice a power) of 13. We also prove a conjecture of Matt Blum stating that the number of perfect matchings of a certain family of subgraphs of the square lattice is a power of 3 or twice a power of 3. In addition we obtain multi-parameter generalizations of previously known results, and new multi-parameter exact enumeration results. We obtain in particular a simple combinatorial proof of Bo-Yin Yang's multivariate generalization of fortresses, a result whose previously known proof was quite complicated, amounting to evaluation of the Kasteleyn matrix by explicit row reduction. We also include a new multivariate exact enumeration of Aztec diamonds, in the spirit of Stanley's multivariate version.

Keywords: perfect matchings, tilings, exact enumeration

1. Introduction

A lattice in the plane divides the plane into elementary regions. A *tile* is the union of two elementary regions that share an edge. A *region* is a connected region in the plane whose boundary consists of lattice segments. A *tiling* of a region is a way to pair the elementary regions it contains into disjoint tiles.

Tiles are allowed to carry weights, in which case the weight of a tiling is defined to be the product of the weights of the constituent tiles. The *tiling generating function* of a region R, denoted T(R), is the sum of the weights of all its tilings.

An important motivation for studying tiling generating functions is that tilings of lattice regions can be identified with perfect matchings of their dual graphs, and are therefore instances of the dimer model of statistical physics on various lattice graphs.

In the last decade there have been many results about special families of regions on various lattices whose number of tilings is given by perfect powers [5, 8, 11]. In this paper we present a unified approach that allows proving these cases as well as new results.

The uniform approach in our proofs is afforded by Theorem 2.3 (the General Complementation Theorem), a result relating the matching generating function of weighted graphs in a large class (that includes in particular all planar bipartite graphs[1]) to the matching generating function of their "complement." This result is an extension of Propp's Generalized Domino-Shuffling [9], which in turn extended the Aztec diamond case of the

Complementation Theorem of [2] to arbitrary weights by eliminating a technical assumption in the statement of the Complementation Theorem of [2] (the assumption that the cell-factors are constant along lines of cells—see Section 2 and [2]). Our proofs employ combinatorial arguments to bring the problems to a form where Theorem 2.3 applies.

This more general version can in fact be proved using the arguments employed in [2] to prove the Complementation Theorem. However, it is shorter and clearer to prove it in the spirit of [9]. We present such a proof, based on certain local graph replacement operations, in Section 2.

A crucial ingredient in our proofs is that certain matrices that encode the type of each family of regions behave well under the action of a naturally arising matrix operator d whose action is given by certain rational maps (see Section 3). More precisely, in all these examples it turns out that after applying d successively to the encoding matrix some convenient number of times—in two of our examples, twelve and respectively thirty(!) times—, one arrives at a matrix of the same type as the original one, thus obtaining simple recurrence relations that determine the tiling generating function.

In several of our examples we find in fact more general matrices, with entries indeterminates, which also have this periodicity property under d. These can be interpreted as parametrized subvarieties of the algebraic variety of matrices having some fixed order under the action of d.

On the square lattice, Elkies, Kuperberg, Larsen and Propp considered in [5] the Aztec diamond regions and proved that the region of order n has $2^{n(n+1)/2}$ tilings (for $n = 4$ this region is illustrated in figure 1).

For the square lattice with the diagonals drawn in—the lattice corresponding to the affine Coxeter group B_2—, B.-Y. Yang [11] considered regions called fortresses, and proved that the number of their tilings is always a power of 5 or twice a power of 5 (figure 17 illustrates the case $n = 4$).

On the triangular lattice, which corresponds to the affine Coxeter group A_2, MacMahon [7] showed that the number of tilings of the hexagon of sides a, b, c, a, b, c (in cyclic order) is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i + j + k - 1}{i + j + k - 2}.$$

It is interesting that there is no known family of regions on the triangular lattice whose number of perfect matchings is given non-trivially by perfect powers. The "rectangular"
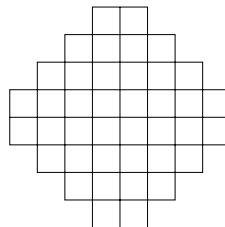

Figure 1. The Aztec diamond region of order 4.

region obtained from a hexagon with sides m, m, n, m, m, n by cutting off the vertices not incident to the sides of length n along zig-zag lattice lines has n^m tilings, but it can be split into m hexagons that can be tiled independently (see [3]).

There is only one other affine Coxeter group of order two, G_2, and its corresponding lattice is the triangular lattice with all altitudes of unit triangles drawn in. Propp considered regions on this lattice called Aztec dungeons (see figure 10 for an example), and conjectured that the number of their tilings is always a power of 13 or twice a power of 13. We prove this conjecture and present two weighted generalizations in Section 3.

B.-Y. Yang [11] extended his result on enumerating tilings of fortresses by proving, using a quite complicated argument—explicit row reduction of the Kasteleyn matrix—, a conjecture of Kuperberg and Propp stating that a certain multivariate weighting of the Aztec diamond leads to a matching generating function that factors nicely. We present a simple combinatorial proof of this general version in Section 4, which also contains a weighted form that keeps track of the number of different types of tiles used in a tiling of the fortress (for a simple proof of the original version of B-Y. Yang's result, based on the Complementation Theorem, see [2]).

Section 5 contains a couterpart of Stanley's multivariate generalization ([10]; see also [2]) of the Aztec diamond theorem of [5].

In Section 6 we present what seems to be a new result, stating that the number of tilings of a certain subgraph of the Aztec diamond is a power of 2. We also obtain a natural weighted generalization that keeps track of the horizontal and vertical edges.

In Section 7 we consider the tiling of the plane by equilateral triangles, squares and regular hexagons. Another ex-conjecture of Propp [8], first proved (but not yet published) by Ben Wieland, states that the number of tilings of a certain family of regions on this lattice, called dragons (an example is illustrated in figure 22), equals a power of two. We present a simple proof of this, and a natural weighted generalization.

Section 8 presents a proof of a conjecture of Matt Blum stating that the number of perfect matchings of a certain family of subgraphs of the square lattice is a power of 3 or twice a power of 3. The last section of the paper contains some concluding remarks.

2. The General Complementation Theorem

A *perfect matching* of a graph is a collection of vertex-disjoint edges that form a spanning subgraph. For a weighted graph G, the weight of a perfect matching is the product of the weights of the constituent edges. The *matching generating function* $M(G)$ is the sum of the weights of all perfect matchings of G.

We present in this section a generalization of the Complementation Theorem of [2]. We review the necessary definitions and notations from [2] below.

A *cellular graph* is a finite graph whose edges can be partitioned into 4-cycles—the *cells* of the graph—such that each vertex is contained in at most two cells.

Let c_0 be a cell of a graph G and consider two opposite vertices x_0 and y_0 of c_0. By definition, x_0 is contained in at most one other cell besides c_0. If such a cell c_1 exists, let x_1 be its vertex opposite x_0. Then x_1 in turn is contained in at most one cell other than c_1; if

such a cell exists denote by x_2 its vertex opposite x_1, and continue in this fashion. Repeat the procedure starting with y_0.

The set of cells that arise this way is said to be a *line* of G. If the sequence $\{x_i\}$ (and hence the analogous sequence $\{y_i\}$) defined above is finite, the line is called a *path*, and the last entry of each of the two sequences is called an *extremal vertex* of G.

A weighted graph is a graph equipped with a weight function on its edges. Unless stated otherwise, the edge-weights are considered independent indeterminates. Let $V(G)$ and $X(G)$ be the sets of vertices and extremal vertices, respectively, of G. Let $E(G)$ be the set of edges of G.

The following definition is a relaxation of the corresponding one from [2] (the original definition involved the additional requirement that H is an induced subgraph of G).

Definition 2.1   Given a weighted graph H, a weighted cellular graph G is said to be a *cellular completion* of H if

 (i) H is a subgraph of G, and all edges of $E(G)\backslash E(H)$ with both endpoints in H have weight 0 in G, and
(ii) $V(G)\backslash V(H) \subseteq X(G)$.

Therefore, any (not necessarily induced) subgraph of a weighted cellular graph has a cellular completion; see Remark 2.4. A graph may have more than one cellular completion: figure 2(c) shows a cellular completion of the graph in figure 2(a); another is obtained by shading in figure 2(b) the other class of the indicated chessboard coloring. The cellular completion of a subgraph of the square grid is not necessarily a subgraph of the square grid.

Let G be a cellular completion of the graph H. The *complement H' of H* (with respect to G) is defined as follows. First we define the underlying graph of the complement (the weight on H' is defined in the next paragraphs): it is the induced subgraph H' of G whose vertex set is determined by the equation $V(H') \triangle V(H) = X(G)$, where the triangle denotes symmetric difference of sets (an example is illustrated in figures 2(a)–(d);[2] the edges of G not contained in H are represented by dotted lines in figure 2(c)). In other words, $V(H')$ is the set obtained from $V(H)$ after performing the following operation at each end of every path of G: if the corresponding extremal vertex belongs to $V(H)$, remove it; otherwise, include it.

Starting with the weight wt on G, we define a new weight wt_1 on G as follows (the weight of the complement H' will be defined by restriction from wt_1). If the cell c has edges weighted by x, y, z and w (in cyclic order), set $\Delta(c) := xz + yw$. We call $\Delta(c)$ the *cell-factor* of c.

Consider a cell c of G. Assume first that the set of endpoints of edges of H contained in c is the full set of four vertices of c—we call such cells *whole*. In this case, for each edge e of c set $\mathrm{wt}_1(e) := \mathrm{wt}(f)/\Delta(c)$, where f is the edge opposite e in c (this is summarized in figure 3, where the left part shows the original weight on a whole cell and the right part the new weight). If c is whole it follows that $\Delta(c) \neq 0$, and this is well-defined. Note that this definition has the effect of sliding each 0-weight edge to the opposite edge in its cell, so by interpreting these back as missing edges, the graph changes—see figures 2(a) and (d), where 0-weight edges are indicated as missing.
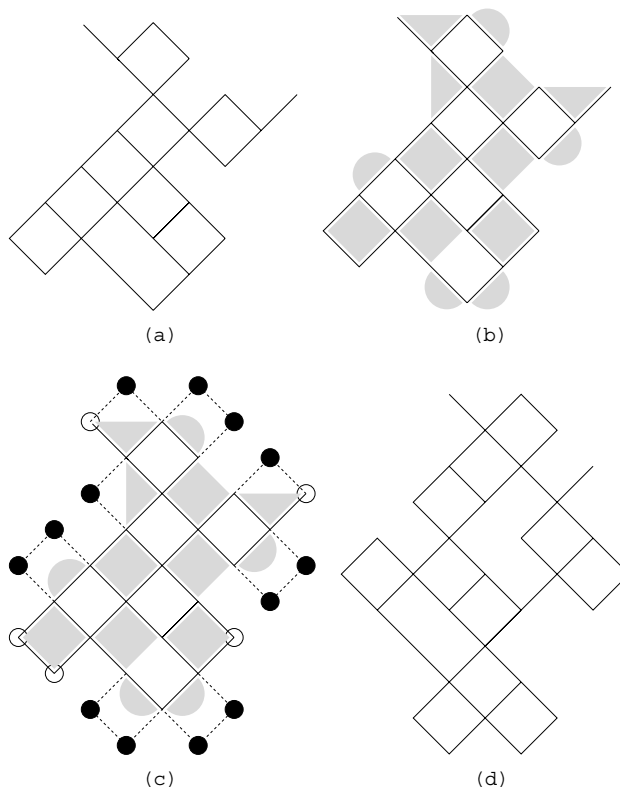
Figure 2. (a) The graph H. (b) Whole and partial cells of H. (c) A cellular completion of H. (d) The graph H'.
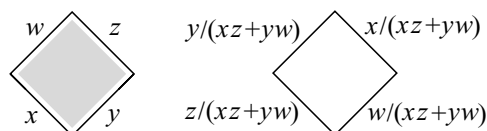

Figure 3. Weight change on whole cells.

If on the other hand the set of endpoints of edges of H contained in c has at most three elements—such a cell is called *partial*—, define wt_1 on c as indicated by figures 4 and 5. More precisely, if the aforementioned set has three elements, the partial cell c looks like the left part of figure 4. By our definition of the complement, the missing extremal point from cell c will be included in H', making this a whole cell in H'. Assign it new weights as indicated by the right part of figure 4.

In case the set of endpoints of edges of H contained in c has two elements, define wt_1 similarly, using figure 5.

We note that if a partial cell contains an extremal vertex, this cell doesn't generate a whole cell in H'; in this case just use the corresponding restriction of the right side of figure 4 or 5. Instances of this occur in figures 2(a) and (d).
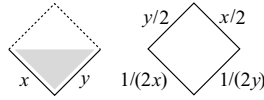
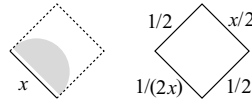*Figure 4.*   Weight change on partial cells with three vertices.

Figure 5.   Weight change on partial cells with two vertices.

Definition 2.2   Let wt be a weight on the edge set of H. The derived weight wt$'$ on the complement H' is the weight obtained by restricting the above defined weight wt$_1$ to H'.

(The above defined weight is related to, but different from the complementary weight defined in [2] and denoted there by the same symbol.)

The following result is a generalization of Theorem 2.1 of [2]. The main improvements are that here the cell-factors are not required to be constant along lines, and that in the current set-up H does not need to be an induced subgraph of its cellular completion G (see Remark 2.4).

Theorem 2.3 (General Complementation Theorem)   *Let H be a weighted graph and let G be a cellular completion of H. Let wt be a weight function on the edges of H. Then $M(H; wt)$ can be expressed in terms of the complement H' of H with respect to G, weighted by the derived weight wt$'$, as*

$$
M(H; wt) = 2^{|\mathcal{C}_0|}\left(\prod_{c \in \mathcal{C}_1} \Delta(c) \right) M(H'; wt'),
$$

where $\mathcal{C}_1$ and $\mathcal{C}_0$ are the sets of whole and partial cells, respectively.

Proof: The particular case when H is an Aztec diamond is due to Propp [9] and follows by repeated application of Lemma 2.5 below. For the general case we proceed as follows. Detach the cells from one another and introduce one new vertex and two new unit-weighted edges between any two touching cells; introduce two new vertices and two new unit-weighted edges at every vertex of a cell that doesn't belong to any other cell (the latter are precisely the extremal vertices of the cellular completion G that are contained also in H; for the graph in figure 2(a), this procedure is illustrated in figures 6(a) and (b)). By the "vertex splitting" trick (see e.g. [1]), the matching generating function of the resulting graph H_1 is the same as that of H.

In the graph H_1 there is an opportunity to apply Lemma 2.5 around each whole cell, and an opportunity to apply either part (a) or (b) of Lemma 2.6 around every partial cell. Let H_2 be the graph resulting from H_1 after performing all local replacements prescribed by these
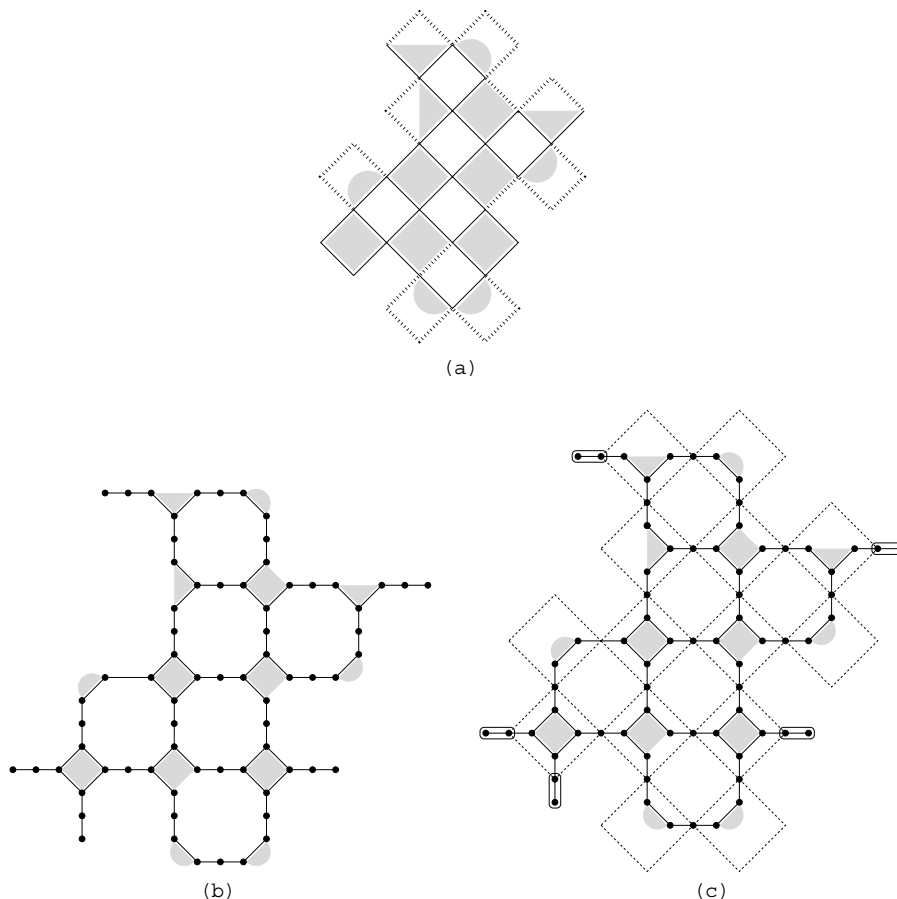
Figure 6.   Illustrating the proof of Theorem 2.3.

Lemmas (figure 6(c) illustrates H_2 for the example of figures 6(a) and (b); H_2 is obtained by including the dotted 4-cycles and removing all vertices and edges they enclose).

Corresponding to each extremal vertex of G contained in H we have a forced edge in H_2 (these are circled in figure 6(c)). Corresponding to each extremal vertex of G *not* contained in H, we have a new vertex in H_2, created by application of Lemma 2.6(a) or (b). Therefore, the graph obtained from H_2 by removing all forced edges (which by construction have weight 1) is precisely the complement H' (compare figures 6(c) and 2(d)). By Lemmas 2.5 and 2.6, the resulting weight on H' is precisely wt$'$.   □

Remark 2.4   Any subgraph H of a weighted cellular graph G has a cellular completion. Indeed, restricting G to the cells containing at least one edge of H, redefining as being 0 the weight of all edges in $E(G)\backslash E(H)$ having both endpoints in H, and finally "detaching" the remaining cells of G at vertices in $V(G)\backslash V(H)$ (so that they become extremal), we obtain
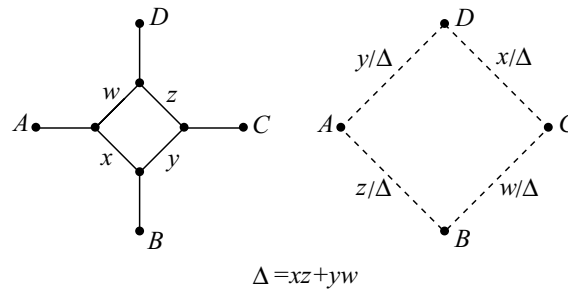
$$\Delta = xz + yw$$

Figure 7.   Urban renewal.

a cellular completion of H. Therefore, Theorem 2.3 applies to any subgraph of a weighted cellular graph.

Let G be a weighted graph containing a subgraph isomorphic to the graph K shown in the left part of figure 7 (the labels indicate weights; unlabeled edges have weight 1). Suppose in addition that the four inner vertices of K have no neighbors outside K. Let $s(G)$ be the graph obtained from G (by "surgery") by replacing K by the graph $\bar{K}$ shown on right in figure 7 (dashed lines indicate new edges, weighted as shown).

The following result is a generalization due to Propp of the "urban renewal" trick first observed by Kuperberg (which corresponds to the case when all weights are 1). It follows by analyzing the restrictions of matchings of G and $s(G)$ to K and $\bar{K}$, respectively (see [9]).

Lemma 2.5 $M(G) = (xz + yw)M(s(G))$.

The next result appears to be new. Together with Lemma 2.5, it is an essential ingredient in the proof of Theorem 2.3.

Lemma 2.6

(a) *Consider the above local replacement operation when K and $\bar{K}$ are the graphs shown in figure 8, with the indicated weights (in particular, $s(G)$ has one vertex, D, that was not a vertex of G, which is incident only to A and C). We have*

$$M(G) = 2M(s(G)).$$

(b) *The statement of part (a) is also true when K and $\bar{K}$ are the graphs indicated in figure 9 (in this case $s(G)$ has two vertices, C and D, not belonging to G; they are adjacent only to one another and to B and A, respectively).*

Proof: To prove part (a), partition the perfect matchings of G into three classes: (1) those containing the edge of K weighted x and the edge of K incident to C, (2) those containing the edge of K weighted y and the edge of K incident to A, and (3) those containing the three edges of K incident to A, B or C. Partition the perfect matchings of $s(G)$ also in
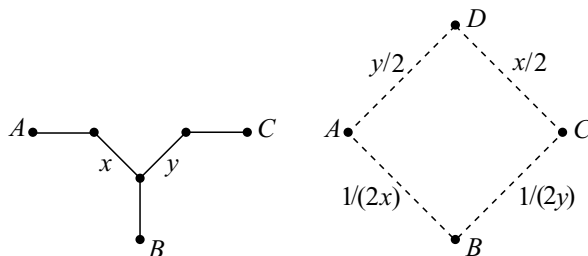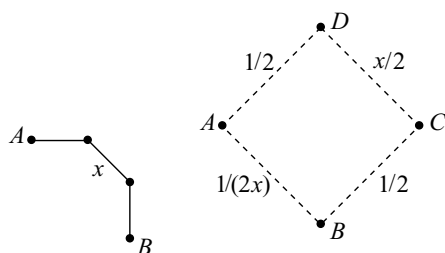
Figure 8. A local graph replacement.

Figure 9. Another local graph replacement.

three classes: (1) those containing edge CD but not AB, (2) those containing AD but not BC, and (3) those containing two opposite edges of the 4-cycle $ABCD$ (this is indeed a partition, since D is incident only to A and C).

Given a perfect matching μ of G, construct a perfect matching μ' of $s(G)$ by discarding the edges of μ with both endpoints in K, and including edges of $\bar{K}$ as indicated by the correspondence of like-numbered classes in the partitions from the previous paragraph. It is easy to check that $\mu \to \mu'$ is a bijection and that $\text{wt}(\mu) = 2\text{wt}(\mu')$. This proves part (a).

Part (b) is proved similarly. □

When H is an Aztec diamond we obtain the following important special case due to Propp (called Generalized Domino-Shuffling in [9]).

Corollary 2.7 (Reduction Theorem [9])

$$\text{M}(AD_n; \text{wt}) = \text{M}(AD_{n-1}; \text{wt}') \prod_{c \in \mathcal{C}} \Delta(c).$$

Provided all cell-factors are nonzero and keep being so after each application of the Reduction Theorem, Corollary 2.7 can be applied successively—with the weight changing at every step—until we get down to the Aztec diamond of order 0, whose matching generating function is equal to 1. This provides in particular an algorithm for finding the matching generating function of the original weighted Aztec diamond. In certain situations, the weight

pattern repeats after a small number of successive applications. In these cases the matching generating function turns out to be given by perfect (or near-perfect) powers. Several instances of this phenomenon are presented in the following sections.

3. Aztec dungeons

The plane lattice corresponding to the affine Coxeter group G_2 is obtained from the equilateral triangular lattice by drawing in all altitudes (see figure 10). Consider the family of all lattice lines parallel to a given lattice line direction ℓ, and also the family of all lattice lines perpendicular to ℓ. The union of these two families forms a sublattice isomorphic to the square lattice.

Tracing out what would correspond to an Aztec diamond region of order n on this sublattice we obtain the *Aztec dungeon* D_n of order n (see figure 10 for an example). Propp conjectured that the number of tilings of an Aztec dungeon is either a power of 13 or twice a power of 13. In this section we prove two different generalizations of this conjecture (see Theorems 3.1 and 3.8).

There are three possible shapes of tiles on the lattice G_2: an equilateral triangle, an obtuse triangle and a kite-shaped tile. It is natural to keep track of them by assigning them distinct weights. Since the total number of tiles in a tiling is determined by the order of the Aztec dungeon, we may assume without loss of generality that one of our weights is equal to 1. Assign therefore our three types of tiles weights y, x and 1, respectively. Denote by wt the weight obtained this way on D_n.

Theorem 3.1 *Let P denote the irreducible polynomial* $P := x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 2x^3 + 2xy^2 + 1$. *The tiling generating functions of the first six Aztec dungeons D_n are given by*

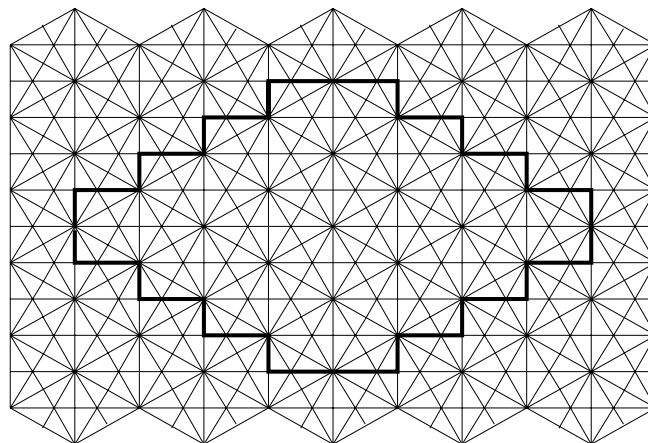$$\mathrm{T}(D_0; \mathrm{wt}) = 1$$
$$\mathrm{T}(D_1; \mathrm{wt}) = x^2 + y^2$$

Figure 10.   The lattice G_2 and the Aztec dungeon D_4.

$$T(D_2; \text{wt}) = x^2 y^2 P$$
$$T(D_3; \text{wt}) = x^6 y^6 P^3$$
$$T(D_4; \text{wt}) = x^{10} y^{14} (x^2 + y^2) P^5$$
$$T(D_5; \text{wt}) = x^{16} y^{24} P^8,$$

and for $n \geq 5$ we have the recurrence

$$T(D_{n+1}; \text{wt}) = x^{8n-16} y^{16n-20} P^{4n-8} T(D_{n-5}; \text{wt}). \tag{3.1}$$

Propp's original conjecture corresponds to the following special case.

Corollary 3.2 *The number of tilings of the Aztec dungeon D_n is given by the equalities $T(D_0) = 1$, $T(D_1) = 2$, $T(D_2) = 13$, $T(D_3) = 13^3$, $T(D_4) = 2 \cdot 13^5$, $T(D_5) = 13^8$, and for $n \geq 5$ by the recurrence*

$$T(D_{n+1}) = 13^{4n-8} T(D_{n-5}).$$

Proof: Set $x = y = 1$ in Theorem 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The proof of Theorem 3.1 will follow from the following three preliminary results. The first of them expresses $M(D_n; \text{wt})$ in terms of a certain weighted count of perfect matchings of the Aztec diamond graph AD_{2n-2}.

Let A be a given $k \times l$ matrix with k and l even. The centers of the edges of the Aztec diamond graph AD_n form a $2n \times 2n$ array. Place a copy of A in the upper left corner of this array and fill in the rest of the array periodically with period A (i.e., translate A to the right in the array l units at a time and down in the array k units at a time; if $2n$ is not a multiple of k or l some of these translates will fit only partially in the array).

Definition 3.3 Define the weight wt_A on the edges of AD_n by assigning each edge the corresponding entry of A in the array described above.

Lemma 3.4 *Let N be the matrix*

$$N = \begin{bmatrix} \frac{y}{x^2+y^2} & y & x & \frac{x}{x^2+y^2} \\ y & 0 & 1 & x \\ x & 1 & 0 & y \\ \frac{x}{x^2+y^2} & x & y & \frac{y}{x^2+y^2} \end{bmatrix} \tag{3.2}$$

We have

$$T(D_n; \text{wt}) = y^{4n} (x^2 + y^2)^{n^2} M(AD_{2n-2}; \text{wt}_N). \tag{3.3}$$

Proof: Construct the graph dual to the region D_n, i.e. the graph whose vertices are the elementary regions of D_n and whose edges connect precisely those elementary regions that
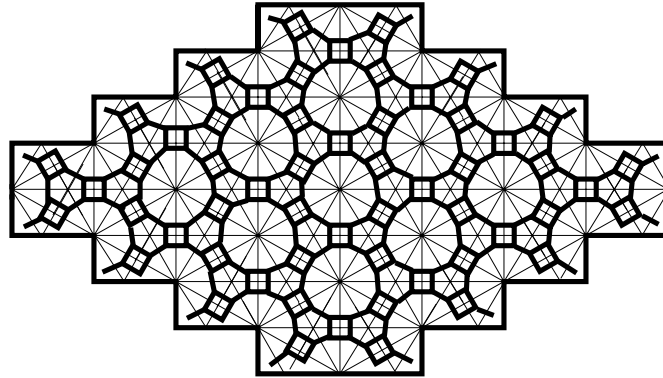
Figure 11.   The Aztec dungeon D_4 and its dual graph.

share an edge (this is illustrated in figure 11 for $n = 4$). Weight each edge by the weight of the corresponding tile.

Clearly, the tilings of the region D_n can be identified with the perfect matchings of its dual graph. Therefore, the tiling generating function $T(D_n; wt)$ is the same as the matching generating function of the dual graph of D_n.

This dual graph contains many local configurations like the one described at the beginning of Section 2, providing one with as many opportunities to apply Lemma 2.5. By their geometric orientation, these local configurations can be grouped into three families. The largest family contains n^2 members (these are indicated in figure 12). Apply Lemma 2.5 at each of these n^2 places. The resulting weighted graph is readily seen to be isomorphic with a weighted subgraph of the square lattice (shown in figure 13 for $n = 4$).

Furthermore, because of the vertices of degree one, the resulting subgraph of the square lattice will have some edges around the boundary that are forced to be contained in all of its perfect matchings (the forced edges are shown in thick lines in figure 13). It is easy to see that all these forced edges except $4n$ of them correspond to kite-shaped tiles, and have
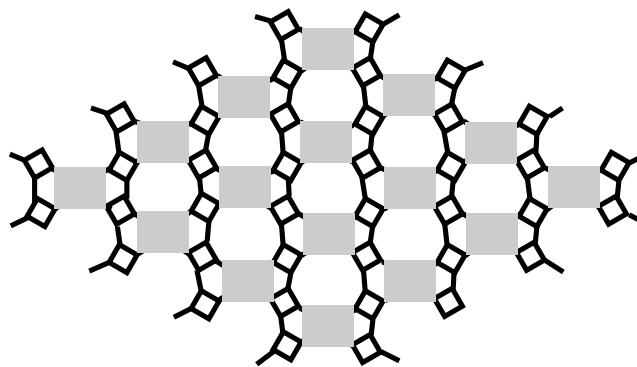

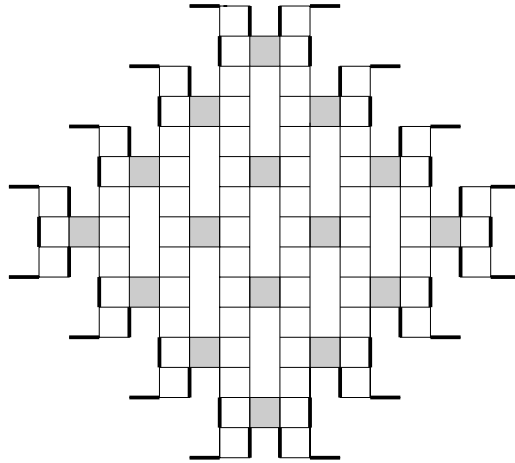*Figure 12.*   Applying Lemma 2.5 to the graph dual to D_4.

Figure 13. A graph embedded in $\mathbb{Z}^2$.

therefore weight 1. The remaining $4n$ edges correspond to equilateral triangle tiles, thus having weight y.

Removing all vertices (together with all incident edges) connected by forced edges we are left with a weighted spanning subgraph of the Aztec diamond of order $2n - 2$ (figure 14 illustrates the case $n = 4$. This can be regarded as the entire graph AD_{2n-2} by weighting all missing edges by 0. By Lemma 2.5, the obtained weight comes out to be exactly wt_N, with the matrix N given by (3.2). Taking into account the contribution of the $4n$ forced edges of weight y and applying n^2 times Lemma 2.5 we obtain (3.3). $\qquad\square$

For a $k \times l$ matrix A with k and l even define a new $k \times l$ matrix $d(A)$ as follows. Divide matrix A into 2×2 blocks

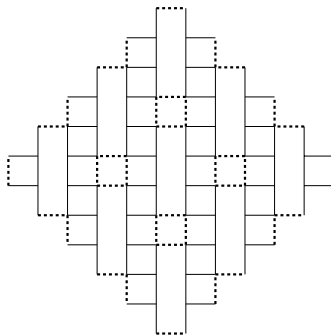$$\begin{bmatrix} x & w \\ y & z \end{bmatrix}$$

Figure 14. A weight on AD_6.

and assume $xz + yw \neq 0$ for all such blocks. Replace each such block by

$$\begin{bmatrix} z/(xz + yw) & y/(xz + yw) \\ w/(xz + yw) & x/(xz + yw) \end{bmatrix}$$

and denote the resulting $k \times l$ matrix by B. Define $\mathrm{d}(A)$ to be the $k \times l$ matrix obtained from B by cyclically shifting its columns one unit up and cyclically shifting the rows of the resulting matrix one unit left.

The next simple observation provides a very convenient way of keeping track of the evolution of periodic weights on Aztec diamonds when one applies the Reduction Theorem.

Lemma 3.5 *Let A be a $k \times l$ matrix with k and l even and consider the weight wt_A it determines on AD_n according to Definition 3.3. We have*

$$\mathrm{wt}'_A = \mathrm{wt}_{\mathrm{d}(A)},$$

where the weight on the left hand side is defined as in Definition 2.2.

Proof: This follows from Definition 2.2 and the above construction of $\mathrm{d}(A)$. The reason we need the cyclic shifts in the construction of $\mathrm{d}(A)$ is because the derived weight of Definition 2.2 is defined by viewing the Aztec diamond of order $n - 1$ as embedded concentrically into AD_n. $\qquad\square$

Due to the location of the 0's in the matrix N given by (3.2), all the edges in an Aztec diamond that are weighted 0 under wt_N are parallel among themselves. By Definition 2.2, this property is preserved by the derived weight. In particular, in any cell at least one pair of opposite edges is assigned nonzero weights. Therefore all cell-factors are nonzero and the Reduction Theorem can be applied successively.

By Lemma 3.5, the successive weights that occur are the weights corresponding via Definition 3.3 to the iterates $\mathrm{d}^{(i)}(N)$, for $i = 1, 2, 3, \ldots$. Therefore, if one of these iterates would be the same as N—or indeed the same up to a scalar multiplicative factor—then we would get a recurrence for the matching generating function on the right hand side of (3.3). By (3.3) this would then translate into a recurrence for $\mathrm{M}(D_n; \mathrm{wt})$ and would solve the problem of computing the latter.

The computation of the iterates $\mathrm{d}^{(i)}(N)$ can be done very easily with a computer algebra package like Maple. The first few iterates don't look very promising, but perseverance pays off: the twelfth(!) iterate turns out to be, up to a scalar multiple, exactly the same as N. More precisely, one obtains

$$\mathrm{d}^{(12)}(N) = k_0 N, \tag{3.4}$$

with

$$k_0 = \frac{y^4(x^3 + xy^2 + 1)^4(x^4 + 2x^2y^2 + y^4 + x)^4}{(x^2 + y^2)^4(x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 2x^3 + 2xy^2 + 1)^4}. \tag{3.5}$$

Since at each application of the Reduction Theorem the order of the resulting Aztec diamond decreases one unit, it follows that

$$M(AD_{2n}; \mathrm{wt}_N) = k_1 M(AD_{2n-12}; \mathrm{wt}_N), \tag{3.6}$$

where the constant k_1 is the product of all the cell-factors arising in the twelve applications of the Reduction Theorem, multiplied by $k_0^{(2n-12)(2n-11)}$ (the latter factor is due to (3.4) and to the fact that each perfect matching of AD_n contains $n(n+1)$ edges). To carry out the computation of k_1 by hand is a fairly strenuous enterprise, but with the assistance of Maple it is quite easy. We obtain by (3.6) the following result.

Proposition 3.6   *For $n \geq 6$ we have*

$$M(AD_{2n}; \mathrm{wt}_N) = x^{8n-16} y^{16n-44} (x^2 + y^2)^{24-12n} P^{4n-8} M(AD_{2n-12}; \mathrm{wt}_N), \tag{3.7}$$

where $P = x^6 + 3x^4 y^2 + 3x^2 y^4 + y^6 + 2x^3 + 2xy^2 + 1$.

We are now ready to present the proof of the main result of this section.

Proof of Theorem 3.1:   By Lemma 3.4 we have

$$T(D_{n+1}; \mathrm{wt}) = y^{4n+4} (x^2 + y^2)^{(n+1)^2} M(AD_{2n}; \mathrm{wt}_N)$$

and

$$T(D_{n-5}; \mathrm{wt}) = y^{4n-20} (x^2 + y^2)^{(n-5)^2} M(AD_{2n-12}; \mathrm{wt}_N).$$

Together with (3.7) these two equalities imply (3.1). The stated values for the tiling generating functions of the first six Aztec dungeons are easily checked using (3.3) and applying repeatedly the Reduction Theorem to evaluate the weighted perfect matching counts of the resulting Aztec diamonds. $\qquad\square$

Remark 3.7   In fact, it turns out that we can further distinguish between the two possible orientations of the equilateral triangle tile by assigning them distinct weights in our enumeration. Indeed, keeping the weight y for the up-pointing equilateral triangle tiles and assigning weight z to the down-pointing ones, the same calculations as above lead to the recurrence

$$T(D_{n+1}; \mathrm{wt}) = x^{8n-16} (yz)^{8n-10} (x^6 + 3x^4 yz + 3x^2 y^2 z^2 + y^3 z^3 + 2x^3$$
$$+ 2xyz + 1)^{4n-8} \times T(D_{n-5}; \mathrm{wt}),$$

for $n \geq 5$ (where wt denotes now the new weight). Similarly, we obtain the values

$$T(D_0; \mathrm{wt}) = 1$$
$$T(D_1; \mathrm{wt}) = x^2 + yz$$

$$\mathrm{T}(D_2; \mathrm{wt}) = x^2 yz Q$$
$$\mathrm{T}(D_3; \mathrm{wt}) = x^6 (yz)^3 Q^3$$
$$\mathrm{T}(D_4; \mathrm{wt}) = x^{10} (yz)^7 (x^2 + yz) Q^5$$
$$\mathrm{T}(D_5; \mathrm{wt}) = x^{16} (yz)^{12} Q^8,$$

where $Q := x^6 + 3x^4 yz + 3x^2 y^2 z^2 + y^3 z^3 + 2x^3 + 2xyz + 1$.

One observes that all above polynomials have the property that the exponents of y and z in each of their monomials agree. This implies that the same property holds for each monomial in the expansion of $\mathrm{T}(D_n; \mathrm{wt})$, and therefore each tiling of D_n contains the same number of up-pointing and down-pointing equilateral triangles.

However, the ultimate such generalization that would keep track also of the orientations of the other two kinds of tiles (six possibilities for each) does not seem to lead to a weighted count that is expressible by a simple product formula. Indeed, as the Reduction Theorem is applied successively larger and larger irreducible polynomials occur in the factorization of the resulting Δ-factors and their form does not seem to fit a simple pattern.

The key fact that allowed our proof of Theorem 3.1 is the fact that the weight function obtained by successive applications of the Reduction Theorem repeated—up to a scalar multiple—after a finite number of applications (in our case twelve).

As mentioned in Remark 3.7, this does not seem to happen if we weight each tile orientation by a different weight. In the following result we present a weight function on Aztec diamonds that depends on eight free parameters which does have the property that the weight resulting at the twelfth iteration is a constant times the original weight. The result originally conjectured by Propp corresponds, via the argument that proved (3.3), to setting parameters a and b equal to $1/2$ and the rest equal to 1.

Consider the 4×4 matrix

$$M = \begin{bmatrix} a & d & d & a \\ e & 0 & g & e \\ f & h & 0 & f \\ b & c & c & b \end{bmatrix}.$$

and the weight wt_M it determines on AD_n by Definition 3.3.

Theorem 3.8   *Let R denote the irreducible polynomial $R := abgh + acfg + bdeh + 2cdef$. The matching generating functions of the first twelve Aztec diamonds weighted by wt_M are given by*

$$\mathrm{M}(AD_0; \mathrm{wt}_M) = 1$$
$$\mathrm{M}(AD_1; \mathrm{wt}_M) = de$$
$$\mathrm{M}(AD_2; \mathrm{wt}_M) = abR$$
$$\mathrm{M}(AD_3; \mathrm{wt}_M) = (de)^2 abgh R$$

$$\mathrm{M}(AD_4; \mathrm{wt}_M) = 2(ab)^2 cdef R^3$$
$$\mathrm{M}(AD_5; \mathrm{wt}_M) = 2(de)^4 (abgh)^2 cf R^3$$
$$\mathrm{M}(AD_6; \mathrm{wt}_M) = 2^3 (ab)^4 (cdef)^3 gh R^5$$
$$\mathrm{M}(AD_7; \mathrm{wt}_M) = 2^3 (de)^7 (abgh)^4 (cf)^3 R^5$$
$$\mathrm{M}(AD_8; \mathrm{wt}_M) = 2^5 (ab)^7 (cdef)^5 (gh)^3 R^8$$
$$\mathrm{M}(AD_9; \mathrm{wt}_M) = 2^5 (de)^{10} (abgh)^7 (cf)^5 R^8$$
$$\mathrm{M}(AD_{10}; \mathrm{wt}_M) = 2^8 (ab)^{10} (cdef)^8 (gh)^5 R^{12}$$
$$\mathrm{M}(AD_{11}; \mathrm{wt}_M) = 2^8 (de)^{14} (abgh)^{10} (cf)^8 R^{12},$$

and for higher orders we have the recurrences

$$\mathrm{M}(AD_{2n}; \mathrm{wt}_M) = 2^{4n-12}(ab)^{4n-10}(cdef)^{4n-12}(gh)^{4n-16} R^{4n-8} \mathrm{M}(AD_{2n-12}; \mathrm{wt}_M)$$

(3.8)

and

$$\mathrm{M}(AD_{2n-1}; \mathrm{wt}_M) = 2^{4n-16}(de)^{4n-10}(abgh)^{4n-14}(cf)^{4n-16} R^{4n-12} \mathrm{M}(AD_{2n-13}; \mathrm{wt}_M).$$

(3.9)

Proof: As we noted in the proof of Proposition 3.6, Lemma 3.5 implies that the successive weights one obtains when applying the Reduction Theorem repeatedly are the weights corresponding to the iterates $\mathrm{d}^{(i)}(M)$, for $i = 1, 2, 3, \ldots$. The twelfth iterate is again a constant multiple of M. More precisely, we obtain

$$\mathrm{d}^{(12)}(M) = k_2 M,$$

(3.10)

with

$$k_2 = \frac{1}{16} \frac{(ag + de)^2 (bh + cf)^2 (ag + 2de)^2 (bh + 2cf)^2}{(abgh + acfg + bdeh + 2cdef)^4}.$$

(3.11)

Since at each application of the Reduction Theorem the order of the resulting Aztec diamond decreases one unit, it follows that $\mathrm{M}(AD_n; \mathrm{wt}_M)$ differs from $\mathrm{M}(AD_{n-12}; \mathrm{wt}_M)$ just by a multiplicative constant, equal to the product of all the cell-factors arising in the twelve applications of the Reduction Theorem, multiplied by $k_2^{(n-12)(n-11)}$ (since AD_{n-12} has $(n-12)(n-11)$ edges in each perfect matching). The explicit value of this multiplicative constant depends on the parity of n. (In Proposition 3.6 we were only interested in even orders because those are the ones that correspond to Aztec dungeons.)

With the assistance of Maple the multiplicative constant is easily determined. It comes out to be the one shown in (3.8) for n even, and the one in (3.9) for n odd. $\square$

Remark 3.9 Recurrence (3.1) explains why there are no new prime factors (in the polynomial ring $\mathbb{Z}[a, y]$) entering the factorizations of $\mathrm{T}(D_n; \mathrm{wt})$ for $n \geq 6$ besides the ones

occurring for the first six values of n. However, we do not have an explanation for the aesthetically pleasing and somewhat mysterious fact that the only prime factors occurring for $n \leq 5$ are the weight indeterminates themselves and the irreducible polynomial P. Indeed, (3.5) illustrates two other irreducible polynomials that occur as divisors of cell-factors arising when the Reduction Theorem is applied successively—they specialize to 3 and 5, respectively, for $x = y = 1$ (and they are the only other prime divisors of cell-factors arising in this process). However, as indicated by the initial values stated in Theorem 3.1, both these prime factors cancel out in the expressions giving $T(D_n; \text{wt})$ for $n = 0, 1, \ldots, 5$—and hence the only prime factor of the number of tilings of D_n is 13, except for the occurrence of a single factor of 2 for $n = 1 \pmod 3$.

The same remark applies to recurrences (3.8) and (3.9) and the initial values stated in Theorem 3.8, as (3.11) indicates four additional potential prime factors—$ag + de$, $bh + cf$, $ag + 2de$ and $bh + 2cf$ (again, they are the only other prime divisors of cell-factors in this case)—which, however, cancel out in the expressions of $M(AD_n; \text{wt}_M)$ for $n \leq 11$.

There is another way to place the Aztec diamond contour on the sublattice of rectangles described at the beginning of this section. Indeed, one can place it so that the resulting region has no forced tiles. Denote the region obtained this way by E_n; we will still call it an Aztec dungeon. Figure 15 illustrates the case $n = 4$.

Enumerating the tilings of E_n turns out to be a distinct problem from the case of D_n. However, as indicated in the next result, the resulting number of tilings is strikingly similar to that corresponding to D_n. The enumeration of the tilings of E_n follows by a suitable specialization of Theorem 3.8, which is thus seen to be a common generalization of the enumeration of the two kinds of Aztec dungeons.

Theorem 3.10 *The number of tilings of the Aztec dungeon E_n is given by the equalities* $T(E_0) = 1$, $T(E_1) = 2 \cdot 13$, $T(E_2) = 13^3$, $T(E_3) = 13^5$, $T(E_4) = 2 \cdot 13^8$, $T(E_5) = 13^{12}$, *and for $n \geq 5$ by the recurrence*
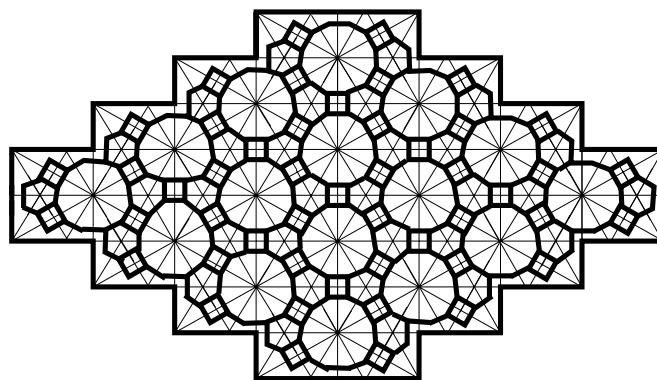
$$T(E_n) = 13^{4n-8} T(E_{n-6}).$$

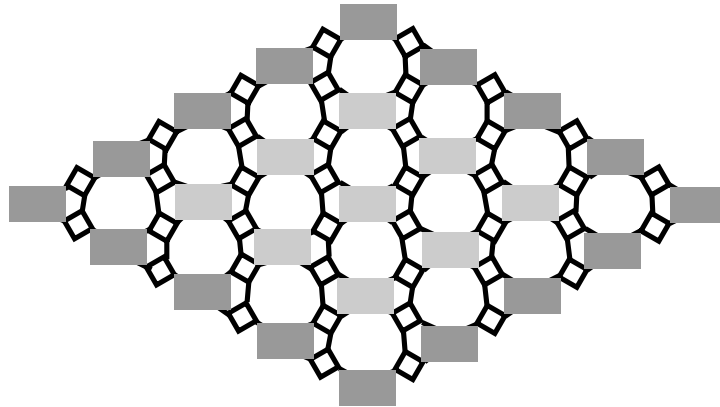*Figure 15.* The Aztec dungeon E_4 and its dual graph.

Figure 16. Applying Lemmas 2.5 and 2.6 to the graph dual to E_4.

Proof: Consider the dual graph of E_n (figure 15 illustrates the case $n = 4$). Apply Lemma 2.5 at the $(n-1)^2$ places indicated by light shading in figure 16, and Lemma 2.6 at the $4n$ places indicated in the same figure by a darker shade. By these lemmas, with each such application the matching generating function changes by a factor of two. The resulting graph is isomorphic with AD_{2n+1}, and the resulting weight is wt_B where

$$B = \begin{bmatrix} 1/2 & 1/2 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Therefore we obtain that

$$\mathrm{T}(E_n) = 2^{(n+1)^2} \mathrm{M}(AD_{2n+1}; \mathrm{wt}_B). \tag{3.12}$$

The Reduction Theorem and Lemma 3.5 yield, after combining all cell-factors of the thus weighted AD_{2n+1}, that

$$\mathrm{M}(AD_{2n+1}; \mathrm{wt}_B) = 2^{n^2-1} \mathrm{M}(AD_{2n}; \mathrm{wt}_{\mathrm{d}(B)}), \tag{3.13}$$

where

$$\mathrm{d}(B) = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 \\ 1/2 & 0 & 1 & 1/2 \\ 1/2 & 1 & 0 & 1/2 \\ 1 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

By specializing $a = b = g = h = 1$ and $c = d = e = f = 1/2$, (3.8) becomes

$$\mathrm{M}(AD_{2n}; \mathrm{wt}_{\mathrm{d}(B)}) = 2^{60-24n} 13^{4n-8} \mathrm{M}(AD_{2n-12}; \mathrm{wt}_{\mathrm{d}(B)}). \tag{3.14}$$

Using (3.12)–(3.14), $T(E_n)$ is expressed as a constant times $M(AD_{2n-12}; wt_{d(B)})$. By (3.13), the latter can be expressed in terms of $M(AD_{2n-11}; wt_B)$, which is in turn, by (3.12), a constant multiple of $T(E_{n-6})$. The multiplicative constant works out to be 13^{4n-8} and we obtain the recurrence in the statement of the Theorem. The initial cases follow by specialization, using (3.12) and (3.13), from the initial cases of Theorem 3.8. □

4. Fortresses

B.-Y. Yang [11] considered regions called fortresses on the lattice obtained from the grid lattice by drawing in all diagonals (figure 17 illustrates the fortress of order 4), and proved, using the permanent-determinant method [6], that the number of their tilings is either a power of 5 or 2 times a power of 5.

By applying Lemma 2.5 to the dual graph, enumerating the tilings of a fortress amounts to finding the matching generating function of an Aztec diamond weighted in a certain pattern with weights 1 and 1/2. B.-Y. Yang proved in fact a more general result [11], included here as Corollary 4.3.

However, Yang's proof is quite complicated and does not explain why a nice factorization exists. This makes it inviting to look for a combinatorial proof. A combinatorial argument for the weight relevant to fortress tilings can be found in [2]. We present below a combinatorial proof of the general case of Yang's theorem. Our proof also explains why such a nice factorization is obtained.

Consider the square matrix

$$
A = \begin{bmatrix}
x_1 & y_1 & 1/x_1 & 1/y_1 & x_1 & y_1 & 1/x_1 & 1/y_1 & \cdot & \cdot & \cdot \\
x_2 & y_2 & 1/x_2 & 1/y_2 & x_2 & y_2 & 1/x_2 & 1/y_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
x_{2n} & y_{2n} & 1/x_{2n} & 1/y_{2n} & x_{2n} & y_{2n} & 1/x_{2n} & 1/y_{2n} & \cdot & \cdot & \cdot
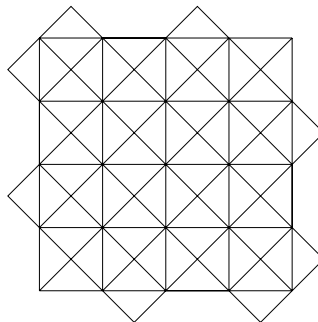\end{bmatrix}.
$$

Figure 17.   The fortress of order 4.

The weight wt_A it defines on AD_n according to Definition 3.3 is determined by its submatrix

$$
T = \begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
x_{2n} & y_{2n}
\end{bmatrix}.
$$

For simplicity of notation, rename the weight wt_A as y_T. Let $S_i := (x_{2i-1} y_{2i-1} x_{2i} y_{2i})^{1/2}$, $i = 1, \ldots, n$. Define r to be the operator sending T to the $(2n-2) \times 2$ matrix

$$
\text{r}(T) = \begin{bmatrix}
x_1/S_1 & S_1/y_1 \\
x_4/S_2 & S_2/y_4 \\
x_3/S_2 & S_2/y_3 \\
x_6/S_3 & S_3/y_6 \\
x_5/S_3 & S_3/y_5 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
x_{2n-2}/S_{n-1} & S_{n-1}/y_{2n-2} \\
x_{2n-3}/S_{n-1} & S_{n-1}/y_{2n-3} \\
x_{2n}/S_n & S_n/y_{2n}
\end{bmatrix}.
$$

Yang's theorem (together with an explanation for why a nice factorization exists) will follow from our next two results.

Theorem 4.1 *For $n \geq 1$ we have*

$$
\text{M}(AD_n; y_T) = \text{M}(AD_{n-1}; y_{\text{r}(T)}) \prod_{i=1}^{n} \Delta_i / S_i, \tag{4.1}
$$

for n even, and

$$
\text{M}(AD_n; y_T) = \text{M}(AD_{n-1}; y_{\text{r}(T)}) \prod_{i=1}^{n} \Delta_i, \tag{4.2}
$$

for odd n, where $\Delta_i = x_{2i-1} y_{2i} + x_{2i} y_{2i-1}$, for $i = 1, \ldots, n$.

Using the above result one can easily prove Yang's formula by induction, provided the formula has been conjectured explicitly (it is not difficult to conjecture it based on computer calculations of the first few cases). However, a conceptual way to obtain the formula is by using the following observation.

Lemma 4.2   *For any $2n \times 2$ matrix T having rows $[x_i, y_i]$, $i = 1, \ldots, 2n$, we have*

$$
\mathrm{r}^{(4)}(T) = \begin{bmatrix} x_5 & y_5 \\ x_6 & y_6 \\ x_7 & y_7 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_{2n-4} & y_{2n-4} \end{bmatrix}.
$$

Proof: This is checked directly from the definition of the operator r (assistance from a computer algebra package like Maple makes this verification quite easy).   □

By Lemma 4.2 we are guaranteed that successive application of Theorem 4.1 will only generate a small number of types of cell-factors and quantities S_i (all types appear within four consecutive applications of Theorem 4.1). We obtain the following result.

Corollary 4.3 (B.-Y. Yang)   *The matching generating function $\mathrm{M}(AD_n; \mathrm{y}_T)$ can be expressed as a simple product in the variables x_i and y_i, $i = 1, \ldots, 2n$.*

In our proof of Theorem 4.1 we will employ the following simple observation.

Lemma 4.4   *Partition the set of edges of AD_n into $n + 1$ classes as follows. View the set of edges as forming a $2n \times 2n$ array. For each $i = 1, \ldots, n - 1$ include the edges in rows $2i$ and $2i + 1$ in a class. Let the edges in the first row and those in the $2n$-th row form two more classes. Then every perfect matching of AD_n contains precisely n edges in each class.*

Proof: Let μ be a perfect matching of AD_n. For each class, consider the set of n vertices of AD_n incident only to edges in that class. These guarantee there will be at least n edges of μ in each class. The statement follows from the fact that the number of edges in μ is $n(n + 1)$.   □

Proof of Theorem 4.1:   Suppose n is even. For notational simplicity, we illustrate in detail the case $n = 4$. The arguments generalize with no difficulty to general even n.

The original weight y_T on the edges of our Aztec diamond AD_4 can be written out explicitly as wt_B, where

$$B = \begin{bmatrix} x_1 & y_1 & 1/x_1 & 1/y_1 & x_1 & y_1 & 1/x_1 & 1/y_1 \\ x_2 & y_2 & 1/x_2 & 1/y_2 & x_2 & y_2 & 1/x_2 & 1/y_2 \\ x_3 & y_3 & 1/x_3 & 1/y_3 & x_3 & y_3 & 1/x_3 & 1/y_3 \\ x_4 & y_4 & 1/x_4 & 1/y_4 & x_4 & y_4 & 1/x_4 & 1/y_4 \\ x_5 & y_5 & 1/x_5 & 1/y_5 & x_5 & y_5 & 1/x_5 & 1/y_5 \\ x_6 & y_6 & 1/x_6 & 1/y_6 & x_6 & y_6 & 1/x_6 & 1/y_6 \\ x_7 & y_7 & 1/x_7 & 1/y_7 & x_7 & y_7 & 1/x_7 & 1/y_7 \\ x_8 & y_8 & 1/x_8 & 1/y_8 & x_8 & y_8 & 1/x_8 & 1/y_8 \end{bmatrix}.$$

Apply the Reduction Theorem. The derived weight is wt_C, with

$$C = \begin{bmatrix} x_1/\Delta_1 & x_1x_2y_2/\Delta_1 & x_2y_1y_2/\Delta_1 & y_1/\Delta_1 & x_1/\Delta_1 & x_1x_2y_2/\Delta_1 \\ x_4/\Delta_2 & x_3x_4y_3/\Delta_2 & x_3y_3y_4/\Delta_2 & y_4/\Delta_2 & x_4/\Delta_2 & x_3x_4y_3/\Delta_2 \\ x_3/\Delta_2 & x_3x_4y_4/\Delta_2 & x_4y_3y_4/\Delta_2 & y_3/\Delta_2 & x_3/\Delta_2 & x_3x_4y_4/\Delta_2 \\ x_6/\Delta_3 & x_5x_6y_5/\Delta_3 & x_5y_5y_6/\Delta_3 & y_6/\Delta_3 & x_6/\Delta_3 & x_5x_6y_5/\Delta_3 \\ x_5/\Delta_3 & x_5x_6y_6/\Delta_3 & x_6y_5y_6/\Delta_3 & y_5/\Delta_3 & x_5/\Delta_3 & x_5x_6y_6/\Delta_3 \\ x_8/\Delta_4 & x_7x_8y_7/\Delta_4 & x_7y_7y_8/\Delta_4 & y_8/\Delta_4 & x_8/\Delta_4 & x_7x_8y_7/\Delta_4 \end{bmatrix}.$$

(where as defined in the statement of the theorem, $\Delta_i = x_{2i-1}y_{2i} + x_{2i}y_{2i-1}$, for $i = 1, \ldots, 4$). Taking into account the cell-factors, the Reduction Theorem yields

$$\mathrm{M}(AD_4; \mathrm{wt}_B) = \prod_{i=1}^{4} (x_{2i-1}y_{2i} + x_{2i}y_{2i-1})^2 \left(\frac{1}{x_{2i-1}y_{2i}} + \frac{1}{x_{2i}y_{2i-1}} \right)^2 \mathrm{M}(AD_3; \mathrm{wt}_C).$$

(4.3)

Group the rows of C into 4 blocks corresponding to the partition of edges of AD_3 from Lemma 4.4. By Lemma 4.4 we can scale the blocks to obtain

$$\mathrm{M}(AD_3; \mathrm{wt}_C) = \prod_{i=1}^{4} \Delta_i^{-3} \mathrm{M}(AD_3; \mathrm{wt}_D),$$

(4.4)

where

$$D = \begin{bmatrix} x_1 & x_1x_2y_2 & x_2y_1y_2 & y_1 & x_1 & x_1x_2y_2 \\ x_4 & x_3x_4y_3 & x_3y_3y_4 & y_4 & x_4 & x_3x_4y_3 \\ x_3 & x_3x_4y_4 & x_4y_3y_4 & y_3 & x_3 & x_3x_4y_4 \\ x_6 & x_5x_6y_5 & x_5y_5y_6 & y_6 & x_6 & x_5x_6y_5 \\ x_5 & x_5x_6y_6 & x_6y_5y_6 & y_5 & x_5 & x_5x_6y_6 \\ x_8 & x_7x_8y_7 & x_7y_7y_8 & y_8 & x_8 & x_7x_8y_7 \end{bmatrix}.$$

This matrix does not yet have the form of B, but can be brought to that form by further scaling of the blocks defined above. Indeed, what characterizes a matrix of type $B = (b_{ij})$

is that the product $b_{ij}b_{i,j+2}$ is equal to 1 for all choices of indices for which both entries are defined. Matrix $D = (d_{ij})$ does not have this property, but it does have the property that the products $d_{ij}d_{i,j+2}$ are constant—equal to $x_{2i-1}y_{2i-1}x_{2i}y_{2i}$—within each block. Therefore, Lemma 4.4 can be used to bring our matrix to the form of matrix B. With $S_i = (x_{2i-1}y_{2i-1}x_{2i}y_{2i})^{1/2}$, $i = 1, \ldots, 4$, we obtain by Lemma 4.4

$$\mathrm{M}(AD_3; \mathrm{wt}_D) = \prod_{i=1}^{4} S_i^3 \mathrm{M}(AD_3; \mathrm{wt}_E), \qquad (4.5)$$

where

$$E = \begin{bmatrix} x_1/S_1 & S_1/y_1 & S_1/x_1 & y_1/S_1 & x_1/S_1 & S_1/y_1 \\ x_4/S_2 & S_2/y_4 & S_2/x_4 & y_4/S_2 & x_4/S_2 & S_2/y_4 \\ x_3/S_2 & S_2/y_3 & S_2/x_3 & y_3/S_2 & x_3/S_2 & S_2/y_3 \\ x_6/S_3 & S_3/y_6 & S_3/x_6 & y_6/S_3 & x_6/S_3 & S_3/y_6 \\ x_5/S_3 & S_3/y_5 & S_3/x_5 & y_5/S_3 & x_5/S_3 & S_3/y_5 \\ x_8/S_4 & S_4/y_8 & S_4/x_8 & y_8/S_4 & x_8/S_4 & S_4/y_8 \end{bmatrix}.$$

By (4.3)–(4.5) we obtain that

$$\mathrm{M}(AD_4; \mathrm{wt}_B) = \prod_{i=1}^{4} \frac{\Delta_i}{S_i} \mathrm{M}(AD_3; \mathrm{wt}_E).$$

For general even n, the same argument proves (4.1).

The case of odd n is treated similarly. The only difference is that now the exponents in the product in (4.3) are $(n+1)/2$ and $(n-1)/2$, as opposed to both being $n/2$ when n is even. Relations (4.4) and (4.5) hold with no change for odd n. One obtains the recurrence (4.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

On the square lattice with all square diagonals drawn in there are two types of tiles: a square tile and a triangular tile. The latter has four different possible orientations.

As it was the case for Aztec dungeons, if we assign different weights to the five oriented tiles the resulting tiling generating function does not seem to be expressible as a simple product of low-degree polynomials in the weights. However, if the two triangular tiles whose hypothenuses face northeast and northwest have the same weight a, and also the two remaining types have the same weight b, the weighted count is a simple product.

As noted before the statement of Theorem 3.1, there is no loss of generality in assuming that the weight of the square tiles is 1.

Corollary 4.5 *The tiling generating function of a fortress with tiles weighted as described above is a simple product of quadratic polynomials in a and b.*

Proof: The described tiling generating function is clearly the same as the matching generating function of the dual graph of the fortress, with weights on edges induced by the
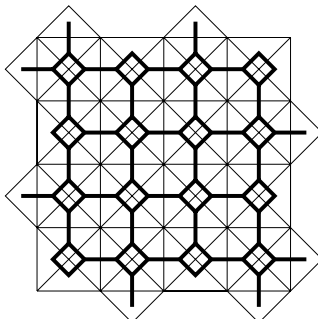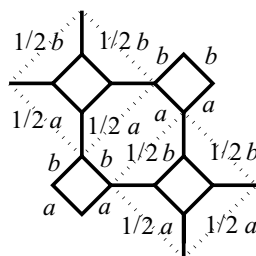
Figure 18. The dual graph of a fortress.

Figure 19. The natural weight of a fortress.

corresponding tiles (figures 17 and 18 show the fortess of order 4 and its dual graph). By choosing convenient local graph replacements of the type described in Lemma 2.5, computing it amounts to finding the matching generating function of an Aztec diamond weighted with the periodic weight indicated in figure 19. By scaling this weight matrix by $\sqrt{2}$ (with the effect of multiplying the matching generating function by a power of $\sqrt{2}$), it becomes of the type addressed in Theorem 4.1. Our statement follows then from Corollary 4.3. $\quad\square$

5. Aztec diamonds with yet another weight

We have seen in the previous section a periodic weighting of the Aztec diamond that factors into small factors. Another such factorization has been found by Stanley ([10]; see also [2]) for the weight pattern

$$\begin{bmatrix} x_1 & y_1 & x_1 & y_1 & x_1 & y_1 & x_1 & y_1 & \cdot & \cdot & \cdot \\ x_2 & y_2 & x_2 & y_2 & x_2 & y_2 & x_2 & y_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ x_{2n} & y_{2n} & x_{2n} & y_{2n} & x_{2n} & y_{2n} & x_{2n} & y_{2n} & \cdot & \cdot & \cdot \end{bmatrix}.$$

We show in this section that a third weight pattern, namely the one given by the matrix

$$
N = \begin{bmatrix}
x_1 & y_1 & y_1 & x_1 & x_1 & y_1 & y_1 & x_1 & \cdot & \cdot & \cdot \\
x_2 & y_2 & y_2 & x_2 & x_2 & y_2 & y_2 & x_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
x_{2n} & y_{2n} & y_{2n} & x_{2n} & x_{2n} & y_{2n} & y_{2n} & x_{2n} & \cdot & \cdot & \cdot
\end{bmatrix},
$$

also leads to a simple product formula. A matrix having the form above is determined by its submatrix

$$
T = \begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
x_{2n} & y_{2n}
\end{bmatrix}.
$$

consisting of its first two columns. For notational simplicity, denote the weight wt_N by z_T.

Let t be the operator sending T to the $(2n-4) \times 2$ matrix

$$
\mathrm{t}(T) = \begin{bmatrix}
1/x_4 & 1/y_4 \\
1/x_3 & 1/y_3 \\
1/x_6 & 1/y_6 \\
1/x_5 & 1/y_5 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
1/x_{2n-2} & 1/y_{2n-2} \\
1/x_{2n-3} & 1/y_{2n-3}
\end{bmatrix}.
$$

Let $\Delta_i = x_{2i-1} y_{2i} + x_{2i} y_{2i-1}$, for $i = 1, \dots, n$

Theorem 5.1 *For $n \geq 1$ we have*

$$
\mathrm{M}(AD_n; z_T) = 2^{n-1} \prod_{i=1}^{n} \Delta_i \prod_{\substack{i=1 \\ i \neq 2, 2n-1}}^{2n} x_i^{n/2} y_i^{n/2-1} \mathrm{M}(AD_{n-2}; z_{\mathrm{t}(T)}), \tag{5.1}
$$

for n even, and

$$\mathrm{M}(AD_n; z_T) = 2^{n-1} \prod_{i=1}^{n} \Delta_i \prod_{\substack{i=1 \\ i \neq 2, 2n-1}}^{2n} x_i^{(n-1)/2} y_i^{(n-1)/2} \mathrm{M}(AD_{n-2}; z_{t(T)}), \tag{5.2}$$

for odd n.

Proof: Suppose first that n is even. As for Theorem 4.1, we illustrate the proof for $n = 4$. Our arguments generalize with no difficulty to general even n. The weights on the edges of AD_4 form the pattern

$$B = \begin{bmatrix} x_1 & y_1 & y_1 & x_1 & x_1 & y_1 & y_1 & x_1 \\ x_2 & y_2 & y_2 & x_2 & x_2 & y_2 & y_2 & x_2 \\ x_3 & y_3 & y_3 & x_3 & x_3 & y_3 & y_3 & x_3 \\ x_4 & y_4 & y_4 & x_4 & x_4 & y_4 & y_4 & x_4 \\ x_5 & y_5 & y_5 & x_5 & x_5 & y_5 & y_5 & x_5 \\ x_6 & y_6 & y_6 & x_6 & x_6 & y_6 & y_6 & x_6 \\ x_7 & y_7 & y_7 & x_7 & x_7 & y_7 & y_7 & x_7 \\ x_8 & y_8 & y_8 & x_8 & x_8 & y_8 & y_8 & x_8 \end{bmatrix}.$$

Apply the Reduction Theorem. The derived weight is wt_C, with

$$C = \begin{bmatrix} x_1/\Delta_1 & x_1/\Delta_1 & y_1/\Delta_1 & y_1/\Delta_1 & x_1/\Delta_1 & x_1/\Delta_1 \\ x_4/\Delta_2 & x_4/\Delta_2 & y_4/\Delta_2 & y_4/\Delta_2 & x_4/\Delta_2 & x_4/\Delta_2 \\ x_3/\Delta_2 & x_3/\Delta_2 & y_3/\Delta_2 & y_3/\Delta_2 & x_3/\Delta_2 & x_3/\Delta_2 \\ x_6/\Delta_3 & x_6/\Delta_3 & y_6/\Delta_3 & y_6/\Delta_3 & x_6/\Delta_3 & x_6/\Delta_3 \\ x_5/\Delta_3 & x_5/\Delta_3 & y_5/\Delta_3 & y_5/\Delta_3 & x_5/\Delta_3 & x_5/\Delta_3 \\ x_8/\Delta_4 & x_8/\Delta_4 & y_8/\Delta_4 & y_8/\Delta_4 & x_8/\Delta_4 & x_8/\Delta_4 \end{bmatrix}.$$

The Reduction Theorem gives

$$\mathrm{M}(AD_4; \mathrm{wt}_B) = \prod_{i=1}^{4} \Delta_i^4 \mathrm{M}(AD_3; \mathrm{wt}_C). \tag{5.3}$$

By Lemma 4.4, we can factor out the denominators along rows to obtain

$$\mathrm{M}(AD_3; \mathrm{wt}_C) = \prod_{i=1}^{4} \Delta_i^{-3} \mathrm{M}(AD_3; \mathrm{wt}_D), \tag{5.4}$$

where

$$
D = \begin{bmatrix}
x_1 & x_1 & y_1 & y_1 & x_1 & x_1 \\
x_4 & x_4 & y_4 & y_4 & x_4 & x_4 \\
x_3 & x_3 & y_3 & y_3 & x_3 & x_3 \\
x_6 & x_6 & y_6 & y_6 & x_6 & x_6 \\
x_5 & x_5 & y_5 & y_5 & x_5 & x_5 \\
x_8 & x_8 & y_8 & y_8 & x_8 & x_8
\end{bmatrix}.
$$

Apply the Reduction Theorem again. All new cell-factors are monomials in the variables. We obtain

$$
M(AD_3; \mathrm{wt}_D) = 2^{3^2} \prod_{\substack{i=1 \\ i \neq 2,7}}^{4} x_i^2 y_i M(AD_3; \mathrm{wt}_E), \tag{5.5}
$$

with

$$
E = \begin{bmatrix}
1/(2x_4) & 1/(2y_4) & 1/(2y_4) & 1/(2x_4) \\
1/(2x_3) & 1/(2y_3) & 1/(2y_3) & 1/(2x_3) \\
1/(2x_6) & 1/(2y_6) & 1/(2y_6) & 1/(2x_6) \\
1/(2x_5) & 1/(2y_5) & 1/(2y_5) & 1/(2x_5)
\end{bmatrix}.
$$

By Lemma 4.4, the factors of $1/2$ can be factored out, with an overall multiplicative contribution of $1/2^6$. By (5.3)–(5.5) we obtain

$$
M(AD_4; z_T) = 2^3 \prod_{i=1}^{4} \Delta_i \prod_{\substack{i=1 \\ i \neq 2,7}}^{8} x_i^2 y_i M\big(AD_2; z_{\mathrm{t}(T)}\big),
$$

where T is the submatrix of B consisting of its first two columns. For general even n, the same arguments prove (5.1).

The case when n is odd can be treated similarly. The only difference occurs in (5.5), where the exponents of x_i and y_i are now both $(n-1)/2$, as opposed to $n/2$ and $n/2 - 1$, respectively, in the case of even n. Therefore, (5.3) and (5.4) imply (5.2). $\qquad\square$

Lemma 5.2 *For any $2n \times 2$ matrix T having rows $[x_i, y_i]$, $i = 1, \dots, 2n$, we have*

$$
t^{(2)}(T) =
\begin{bmatrix}
x_5 & y_5 \\
x_6 & y_6 \\
x_7 & y_7 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
x_{2n-4} & y_{2n-4}
\end{bmatrix}.
$$

Proof: This is readily checked by the definition of the operator t. □

Corollary 5.3 *The matching generating function $M(AD_n; z_T)$ can be expressed as a simple product in the variables x_i and y_i, $i = 1, \dots, 2n$.*

Proof: By Lemma 5.2 and application of Theorem 5.1 two successive times we obtain that, for $n \geq 4$, $M(AD_n; z_T)$ is a simple product times $M(AD_{n-4}; z_{\bar{T}})$, where $\bar{T}$ is the matrix obtained from T by discarding the first four and last four rows. Repeated application of this proves the statement of the Corollary. □

6. Squares and hexagons

Consider the lattice of squares and hexagons shown in figure 20, and regard it as an infinite graph H. This graph can naturally be embedded in the square grid, as indicated in figure 21. Draw an Aztec diamond on the grid (see figure 21; dotted lines indicate portions of the boundary lying along missing edges).

A natural question to look at is to find the number of perfect matchings of the portion of the embedding of H that is a subgraph of our Aztec diamond (these subgraphs of H have a natural definition also on the original squares and hexagons lattice; the example shown in figure 21 is isomorphic, after removing forced edges, to the subgraph of H indicated in
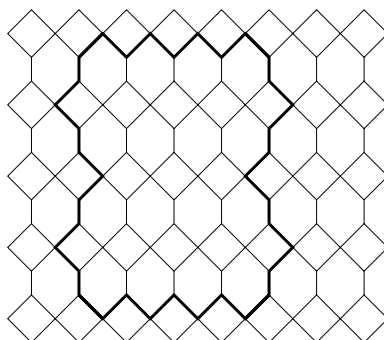

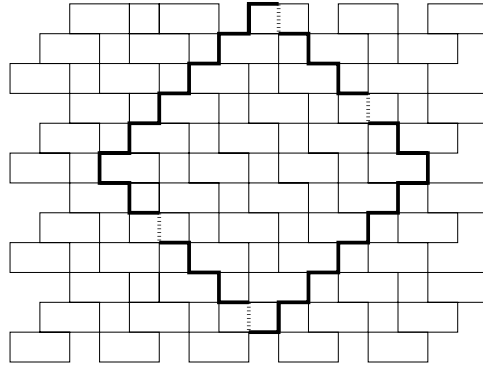*Figure 20.* A lattice of squares and hexagons.

Figure 21.   Embedding the lattice of figure 20 into a square grid.

figure 20). This is similar to a question posed by Propp and answered by Chris Douglas in [4].

We present in this section the solution to a weighted generalization of the problem posed above that keeps track of the orientation of the edges in the perfect matching. As in the previous sections, the proof will follow by repeated applications of the Reduction Theorem.

For definiteness, let us draw the Aztec diamond contour so that its westernmost edge belongs to a 4-cycle of H lying outside the contour (this is the way the contour was drawn in figure 21). Assign weight 1 to the horizontal edges, and weight a to the vertical edges. It is easy to see that the weight on AD_n arising this way is precisely wt_A, with period

$$A = \begin{bmatrix} 1 & 0 & 1 & a & 1 & a \\ a & 1 & a & 1 & 0 & 1 \end{bmatrix}.$$

Theorem 6.1   *For all $k \geq 1$, $\mathrm{M}(AD_{3k}; \mathrm{wt}_A) = \mathrm{M}(AD_{3k+1}; \mathrm{wt}_A) = (1 + a^2)^{2k(k+1)}$, and $\mathrm{M}(AD_{3k+2}; \mathrm{wt}_A = (1 + a^2)^{2(k+1)^2}$.*

In our proof we will employ the following counterpart of Lemma 4.4.

Lemma 6.2   *Partition the set of edges of AD_n into n classes as follows. View the set of edges as forming a $2n \times 2n$ array. For each $i = 1, \ldots, n$ include the edges in columns $2i - 1$ and $2i$ in a class. Then every perfect matching of AD_n contains precisely $n + 1$ edges in each class.*

Proof: Let μ be a perfect matching of AD_n. For each class, consider the set of $n + 1$ vertices of AD_n incident only to edges in that class. These guarantee there will be at least $n + 1$ edges of μ in each class. Since the number of edges in μ is $n(n + 1)$, the statement follows.   □

Proof of Theorem 6.1:   Apply the Reduction Theorem. Since all the cell-factors are either 1 or $1 + a^2$, we obtain

$$\mathrm{M}(AD_n; \mathrm{wt}_A) = (1 + a^2)^{k_1} \mathrm{M}(AD_{n-1}; \mathrm{wt}_{\mathrm{d}(A)}), \tag{6.1}$$

where k_1 is a positive integer and

$$d(A) = \begin{bmatrix} 1 & a/(1+a^2) & 1/(1+a^2) & a & 1 & 0 \\ a & 1/(1+a^2) & a/(1+a^2) & 1 & 0 & 1 \end{bmatrix}.$$

Apply the Reduction Theorem again. All the cell-factors are now equal to 1, so we obtain

$$M\big(AD_{n-1}; \mathrm{wt}_{d(A)}\big) = M\big(AD_{n-2}; \mathrm{wt}_{d^{(2)}(A)}\big), \tag{6.2}$$

where

$$d^{(2)}(A) = \begin{bmatrix} 1 & a & 1/(1+a^2) & 0 & 1 & a/(1+a^2) \\ a & 1 & a/(1+a^2) & 1 & 0 & 1/(1+a^2) \end{bmatrix}.$$

The cell-factors for $\mathrm{wt}_{d^{(2)}(A)}$ are either $1 + a^2$ or $(1+a^2)^{-1}$. One more application of the Reduction Theorem yields

$$M\big(AD_{n-2}; \mathrm{wt}_{d^{(2)}(A)}\big) = (1+a^2)^{k_2} M(AD_{n-3}; \mathrm{wt}_{d^{(3)}(A)}), \tag{6.3}$$

where k_2 is an integer and

$$d^{(3)}(A) = \begin{bmatrix} 1/(1+a^2) & 0 & 1 & a & 1+a^2 & a/(1+a^2) \\ a/(1+a^2) & 1+a^2 & a & 1 & 0 & 1/(1+a^2) \end{bmatrix}.$$

Lemma 4.4 clearly holds also when partitioning the edges of an Aztec diamond into vertical blocks. Apply this version of Lemma 4.4 to the weight pattern $\mathrm{wt}_{d^{(3)}(A)}$ we have on AD_{n-3}. Factor out from every third vertical block, starting with the leftmost one, a factor of $(1+a^2)^{-2}$; leave all other blocks unchanged. By Lemma 4.4 we obtain

$$M\big(AD_{n-3}; \mathrm{wt}_{d^{(3)}(A)}\big) = (1+a^2)^{k_3} M(AD_{n-3}; \mathrm{wt}_B), \tag{6.4}$$

where k_3 is an integer and

$$B = \begin{bmatrix} 1+a^2 & 0 & 1 & a & 1+a^2 & a(1+a^2) \\ a(1+a^2) & 1+a^2 & a & 1 & 0 & 1+a^2 \end{bmatrix}.$$

View B as consisting of three 2×2 blocks. By Lemma 6.2, scaling the entries within a block has a simple multiplicative effect on the matching generating function. Factoring out the constants $1 + a^2$, 1 and $1 + a^2$, respectively, the resulting scaled matrix is precisely our original matrix A. We obtain by Lemma 6.2 that

$$M(AD_{n-3}; \mathrm{wt}_B) = (1+a^2)^{k_4} M(AD_{n-3}; \mathrm{wt}_A), \tag{6.5}$$

where k_4 is an integer.

By (6.1)–(6.5) we obtain, after working out the explicit values of $k_1, \ldots, k_4$ that

$$\mathrm{M}(AD_n; \mathrm{wt}_A) = (1 + a^2)^{\alpha_n} \mathrm{M}(AD_{n-3}; \mathrm{wt}_A),$$

where $\alpha_{3k} = \alpha_{3k+1} = 4k$, and $\alpha_{3k+2} = 4k + 2$. Repeated application of this proves the Theorem.   □

7. The Aztec dragon

Consider the lattice of hexagons, squares and equilateral triangles illustrated in figure 22. Propp [8] considered a family of regions R_n called *Aztec dragons* on this lattice (R_6 is outlined in figure 22) and conjectured that the number of their tilings is $2^{n(n+1)}$. This was first proved, in work not yet published, by Ben Wieland (as announced in [8]). In this section we use the Reduction Theorem to obtain a weighted version of this result.

There are two types of tiles. Since each hexagon must be paired with a square, and each triangle also with a square, the number of tiles of each type is constant across all tilings of a given dragon. Thus, our weighted version (Theorem 7.1 below) will not concern the natural weight in the original set-up, but a certain other weight that is natural once we rephrase the problem.

Let R_n' be the region obtained from R_n by including n square-triangle tiles along its northwestern boundary, as indicated in figure 23. Clearly, these tiles are forced to be part of any tiling of the enlarged region. Enumerating the tilings of R_n is equivalent therefore to enumerating tilings of R_n'.

As in the previous sections, consider the dual graph of R_n' (for $n = 6$ this is pictured in figure 24). It is easy to see that this graph can be deformed isomorphically to the one illustrated in figure 25.

Furthermore, by employing the "vertex-splitting" trick (see e.g. Lemma 1.3 of [1]), the latter graph is readily seen to have its perfect matchings identified with those of the graph
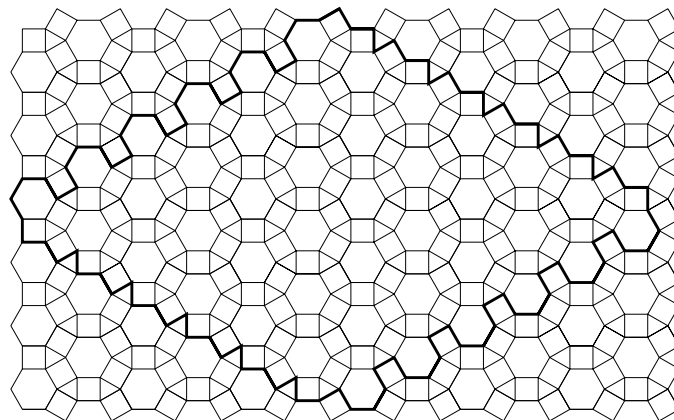

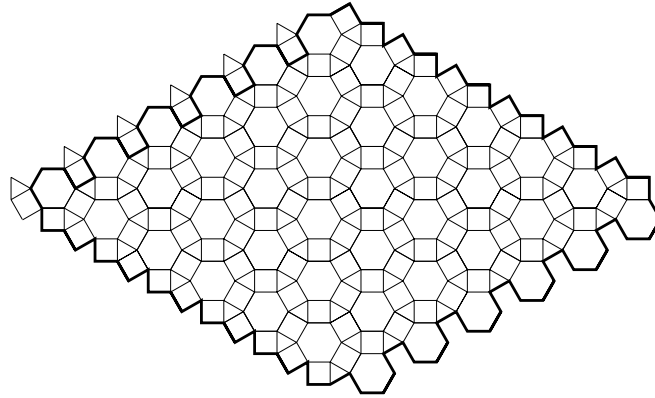*Figure 22.*   The Aztec dragon R_6.
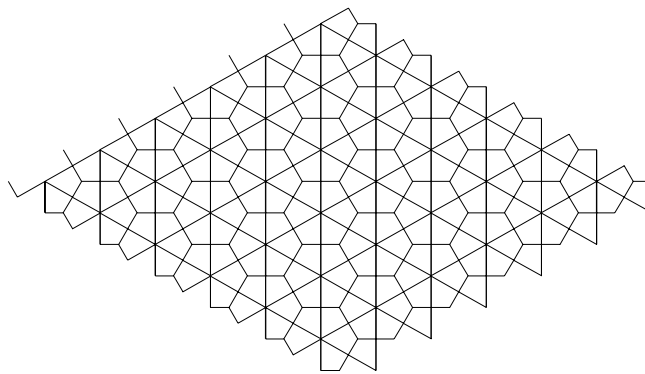
Figure 23. The region R_6'.

Figure 24. The dual graph of R_6'.

in figure 26 (this figure should be regarded as a subgraph of the grid graph; in particular, it has vertices at the midpoints of the segments of length 2).

However, the graph we obtained this way is just the periodic weighting wt_A of the Aztec diamond AD_{2n}, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, we have

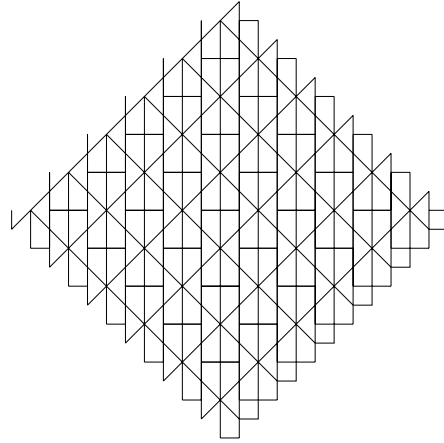$$\mathrm{T}(R_n) = \mathrm{M}(AD_{2n}; \mathrm{wt}_A). \tag{7.1}$$

Figure 25. An isomorphic deformation.
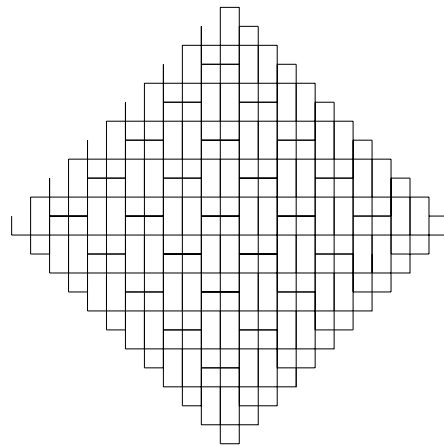

Figure 26. A subgraph of AD_{12}.

One can also regard the obtained graph (shown in figure 26 for $n = 6$) as a subgraph of AD_{2n}. Weight the vertical edges in this subgraph by 1, and the horizontals by a, i.e. consider the weight wt_B on AD_{2n}, where

$$
B = \begin{bmatrix} a & 1 & a & 1 \\ 1 & 0 & 1 & a \\ 0 & 1 & a & 1 \\ 1 & a & 1 & a \end{bmatrix}.
$$

Theorem 7.1 $M(AD_{2n}; \mathrm{wt}_B) = (1 + a^2)^{n(n+1)}$.

Corollary 7.2 $T(R_n) = 2^{n(n+1)}$.

Proof: This follows from (7.1) and the case $a = 1$ of Theorem 7.1. □

Proof of Theorem 7.1: By applying the Reduction Theorem twice and collecting cell-factors one obtains

$$M(AD_{2n}; \mathrm{wt}_B) = \Delta^{3n-1} M(AD_{2n-2}; \mathrm{wt}_C), \tag{7.2}$$

where $\Delta = 1 + a^2$ and $C = d^{(2)}(B)$ is the matrix

$$C = \begin{bmatrix} a & 1 & a/\Delta & 1/\Delta \\ \Delta & 0 & 1 & a \\ 0 & \Delta & a & 1 \\ 1 & a & 1/\Delta & a/\Delta \end{bmatrix}.$$

By Lemma 4.4, we can factor out $1/\Delta$ from the first and last rows of C to obtain

$$M(AD_{2n-2}; \mathrm{wt}_C) = \left((1/\Delta)^{2n-2}\right)^n M(AD_{2n-2}; \mathrm{wt}_D), \tag{7.3}$$

where

$$D = \begin{bmatrix} a\Delta & \Delta & a & 1 \\ \Delta & 0 & 1 & a \\ 0 & \Delta & a & 1 \\ \Delta & a\Delta & 1 & a \end{bmatrix}.$$

In turn, by Lemma 6.2 we can factor out Δ from the first two columns of D to get

$$M(AD_{2n-2}; \mathrm{wt}_D) = \left(\Delta^{2n-1}\right)^{n-1} M(AD_{2n-2}; \mathrm{wt}_B). \tag{7.4}$$

By (7.2)–(7.4) we obtain the recurrence

$$M(AD_{2n}; \mathrm{wt}_B) = \Delta^{2n} M(AD_{2n-2}; \mathrm{wt}_B).$$

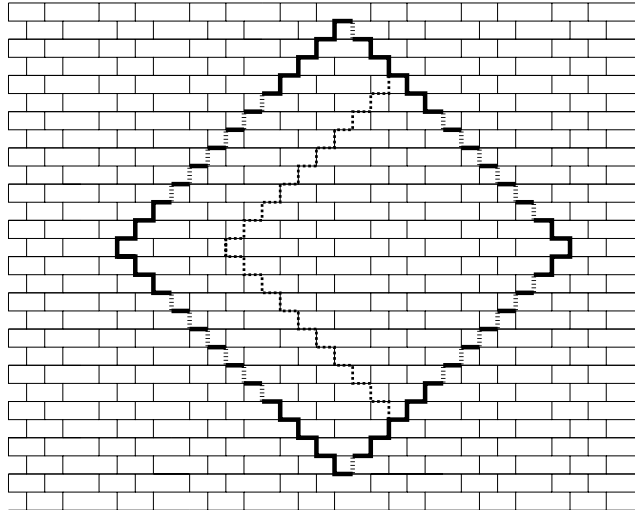Repeated application of this proves the statement of the Theorem. □

Figure 27.   The region B_{13}.

Remark 7.3   The number $2^{n(n+1)}$ of perfect matchings of the above subgraph of AD_{2n} is approximately equal to the square root of the total number $2^{n(2n+1)}$ of perfect matchings of the full AD_{2n}.

8.   Powers of 3

We have seen in the previous sections families of (unweighted) graphs whose number of perfect matchings are perfect or near-perfect powers of the base 2, 5 or 13. What about 3? Matt Blum has considered (as described in an e-mail message to James Propp dated
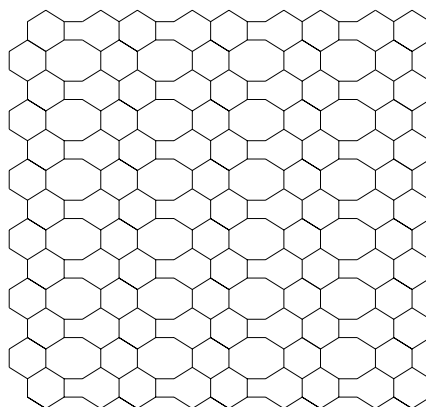

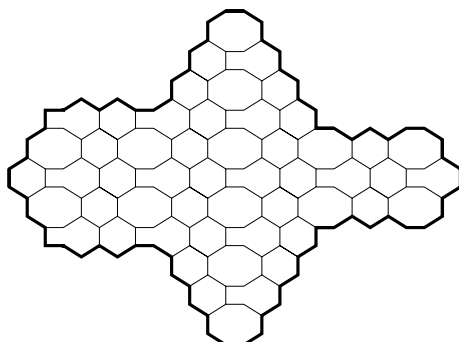*Figure 28.*   A deformation of the lattice.

Figure 29. B_{13} with forced edges removed.

December 1997) a family of subgraphs of the square grid for which he noticed (from data on concrete cases) that the number of perfect matchings seems to be always a power of 3 or twice a power of 3. We prove in this section that this is indeed the case.

Consider, following Blum, the sublattice of the square grid lattice showed in figure 27 and view it as an infinite graph S. Draw the boundary of an Aztec diamond of order n on this lattice in such a way that the easternmost edge has an embedded hexagon east of it. Let B_n be the induced subgraph of S spanned by the vertices lying inside or on the Aztec diamond boundary (the graph inside the larger contour in figure 27 is B_{13}). An alternative way to view B_n is as a subgraph of the lattice showed in figure 28, which is just a deformation of the lattice considered by Blum (the graph obtained from B_{13} after removing the forced edges is shown in this guise in figure 29).

Table 1.   The first thirty values of $T(B_n)$.

n	$T(B_n)$	Factorization	n	$T(B_n)$	Factorization
1	1	1	16	531441	3^{12}
2	2	2	17	43046721	3^{16}
3	6	$2 \cdot 3$	18	10460353203	3^{21}
4	6	$2 \cdot 3$	19	10460353203	3^{21}
5	6	$2 \cdot 3$	20	10460353203	3^{21}
6	6	$2 \cdot 3$	21	10460353203	3^{21}
7	27	3^3	22	7625597484987	3^{27}
8	486	$2 \cdot 3^5$	23	5559060566555523	3^{33}
9	486	$2 \cdot 3^5$	24	5559060566555523	3^{33}
10	486	$2 \cdot 3^5$	25	5559060566555523	3^{33}
11	486	$2 \cdot 3^5$	26	5559060566555523	3^{33}
12	6561	3^8	27	243153330918113857602	$2 \cdot 3^{40}$
13	531441	3^{12}	28	79766443076872509863361	3^{48}
14	531441	3^{12}	29	79766443076872509863361	3^{48}
15	531441	3^{12}	30	79766443076872509863361	3^{48}

Theorem 8.1   *For n ≥ 31 we have*

$$M(B_n) = 3^{4x_n} M(B_{n-30}),\tag{8.1}$$

where $x_{5k+1} = 4k - 12$, $x_{5k+2} = 4k - 10$ and $x_{5k+3} = x_{5k+4} = x_{5k+5} = 4k - 8$, $k \geq 6$.

The values of $M(B_n)$ for $n = 1, \ldots, 30$ are given in Table 1. By (8.1) one obtains that $M(B_n)$ is either a perfect power of 3 or twice a perfect power of 3, for all n.

Proof: The graph B_n is the periodic weighting of AD_n with period indicated by the dashed contour in figure 17. Rotating the indicated subgraph by 180° (due to our choice of the "base" edge being easternmost), we read off the period to be the 20×20 matrix

$$
A =
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}.
$$

By Lemma 3.5, the periodic weightings arising by successive applications of the Reduction Theorem have periods given by the orbit of A under the operator d defined in Section 3. Using a computer algebra program (for instance Maple) one can easily compute $d^{(i)}(A)$ for successive values of i. One finds that all entries of $d^{(30)}(A)$ are either 0 or integer powers of 9, with the position of the zeroes matching perfectly the positions of the zeros of A. The

exponents of 9 form the pattern

$$
\begin{bmatrix}
4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 \\
1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty \\
-2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 \\
-1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty \\
-\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 \\
-1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty \\
-\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 \\
1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 \\
-\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 \\
0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 \\
-\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 \\
-1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 \\
-\infty & 1 & -\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 \\
1 & -\infty & 1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 \\
-\infty & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 \\
1 & -\infty & -1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty \\
2 & -1 & -4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 \\
-1 & -\infty & 0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty \\
-4 & 0 & 4 & 1 & -2 & -1 & -\infty & -1 & -\infty & 1 & -\infty & 0 & -\infty & -1 & -\infty & 1 & -\infty & 1 & 2 & -1 \\
0 & -\infty & 1 & -\infty & -1 & -\infty & -1 & -2 & 1 & 4 & 0 & -4 & -1 & 2 & 1 & -\infty & 1 & -\infty & -1 & -\infty
\end{bmatrix}
$$

$$(8.2)$$

Denote the matrix (8.2) by $B = (b_{ij})_{1 \le i,j \le 20}$ (the position corresponding to a zero of A is recorded in B by $-\infty$).

The form of B suggests the following generalization of our original weight period A. Consider the matrix $A(q) = (a_{ij}(q))_{1 \le i,j \le 20}$ given by

$$a_{ij}(q) = q^{b_{ij}}, \quad 1 \le i,j \le 20,$$

with the convention that $q^{-\infty} = 0$. By construction, $A(1)$ is our original weight period A.

Applying the operator d successively thirty times to $A(q)$ (a simple task with the aid of Maple), we obtain

$$d^{(30)}(A(q)) = A(9q). \tag{8.3}$$

Therefore, by the Reduction Theorem and Lemma 3.5 we obtain that $M(AD_n; wt_{A(q)})$ is equal to $M(AD_{n-30}; wt_{A(9q)})$ multiplied by a product of cell-factors. The latter product is obtained multiplying together all cell-factors from the (thirtyfold!) application of the Reduction Theorem. With the assistance of Maple it is not hard to find this product explicitly. One obtains

$$M(AD_n; wt_{A(q)}) = 3^{4y_n} M(AD_{n-30}; wt_{A(9q)}), \quad n \ge 31, \tag{8.4}$$

where $y_{10k+1} = 8k - 13$, $y_{10k+2} = y_{10k+6} = 8k - 9$, $y_{10k+3} = y_{10k+7} = 8k - 7$, $y_{10k+4} = y_{10k+5} = 8k - 8$, $y_{10k+8} = 8k - 5$ and $y_{10k+9} = y_{10k+10} = 8k - 4$, for $k \ge 3$.

Table 2.   The first thirty values of M(AD_n; wt$_{A(q)}$).

n	M(AD_n; wt$_{A(q)}$)	Factorization	n	M(AD_n; wt$_{A(q)}$)	Factorization
1	q^2	q^2	16	$531441q^2$	$3^{12}q^2$
2	$2q^{-2}$	$2q^{-2}$	17	$43046721q^2$	$3^{16}q^2$
3	$6q^{-2}$	$2 \cdot 3q^{-2}$	18	$10460353203q^2$	$3^{21}q^2$
4	6	$2 \cdot 3$	19	10460353203	3^{21}
5	6	$2 \cdot 3$	20	10460353203	3^{21}
6	$6q^2$	$2 \cdot 3q^2$	21	$10460353203q^2$	$3^{21}q^2$
7	$27q^2$	3^3q^2	22	$7625597484987q^{-2}$	$3^{27}q^{-2}$
8	$486q^2$	$2 \cdot 3^5q^2$	23	$5559060566555523q^{-2}$	$3^{33}q^{-2}$
9	486	$2 \cdot 3^5$	24	5559060566555523	3^{33}
10	486	$2 \cdot 3^5$	25	5559060566555523	3^{33}
11	$486q^2$	$2 \cdot 3^5q^2$	26	$5559060566555523q^2$	$3^{33}q^2$
12	$6561q^{-2}$	3^8q^{-2}	27	$243153309181138857602q^2$	$2 \cdot 3^{40}q^2$
13	$531441q^{-2}$	$3^{12}q^{-2}$	28	$79766443076872509863361q^2$	$3^{48}q^2$
14	531441	3^{12}	29	79766443076872509863361	3^{48}
15	531441	3^{12}	30	79766443076872509863361	3^{48}

Recurrence (8.4) determines M(AD_n; wt$_{A(q)}$) for all n provided we know it for $n \leq 30$. This can be easily found using Maple to carry out successive applications of the Reduction Theorem. One obtains the values in Table 2. These clearly specialize to the values in Table 1 when $q = 1$. It is straightforward to check that for $q = 1$ recurrence (8.4) and these initial values imply (8.1).   □

9.   Concluding remarks

As mentioned in the proof of Theorem 6.1, a result analogous to Lemma 4.4 holds for the columns of the edge-weight array. It is easy to see that together with Lemma 6.2 this analog of Lemma 4.4 implies that *single* columns of the edge-weight array can be scaled with a simple multiplicative effect on the matching generating function of a periodically weighted Aztec diamond. The same is true for the rows of the edge-weight array.

A natural equivalence relation arises this way on the set of period matrices. In order to obtain a recurrence for the values M(AD_n; wt$_A$), it is enough to find an iterate d$^{(k)}(A)$ in the equivalence class of A.

Sometimes it is useful to have an alternative approach, as illustrated in the proof of Theorem 8.1, where it was crucial for the proof that we first generalized the original weight— by introduction of the parameter q. This larger class of matrices had the desired property that a convenient iterate (the thirtieth) was in the same class, yielding thus a recurrence.

All the problems treated in this paper led, in their unweighted form, to periodic weightings of Aztec diamonds with periods given by certain 0–1 even by even matrices (except for the case of Aztec dungeons and fortresses, when entries equal to 1/2 also arise). A natural

problem to consider would be to classify the 0–1 matrices that give rise to nice enumeration results when regarded from the point of view of the approach described in this paper. Two ways for this to come about are mentioned in the previous two paragraphs. Given the similarity of the examples considered in Sections 6 and 8, one natural starting point would concern sublattices of the square grid obtained by deleting parallel edges in some more general simple pattern.

Acknowledgment

Notes

1. Indeed, the statement of Theorem 2.3 clearly applies when H is an arbitrary subgraph of the square grid. By performing "vertex splittings" (see e.g. [1]) if necessary, any planar bipartite graph can be viewed as a subgraph of the square grid, with unaffected matching generating function.
2. In figure 2(d) the graph H' is indeed an induced *weighted* subgraph, with the missing edges weighted by 0 (see the definition of the derived weight wt$'$).

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