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On the Finiteness of Near Polygons with 3 Points on Every Line

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Abstract. Let $\mathcal{S}$ be a near polygon of order (s, t) with quads through every two points at distance 2. The near polygon $\mathcal{S}$ is called semifinite if exactly one of s and t is finite. We show that $\mathcal{S}$ cannot be semifinite if $s = 2$ and derive upper bounds for t.

Keywords: near polygon, generalized quadrangle

1. Introduction

A near polygon is a partial linear space with the property that every line L contains a unique point $\pi_L(p)$ nearest to a given point p. Here distances are measured in the collinearity graph Γ. If d is the diameter of Γ, then the near polygon is called a near $2d$-gon. We always suppose that d is finite. A near 0-gon consists of one point, a near 2-gon is a line, and the class of the near quadrangles coincides with the class of the generalized quadrangles (GQ's), which were introduced by Tits in [8]. Near polygons themselves were introduced by Shult and Yanushka in [7] because of their relationship with certain systems of lines in Euclidean spaces. A near polygon is said to have order (s, t) if every line is incident with exactly $s + 1$ points, and if every point is incident with exactly $t + 1$ lines. Clearly, a near polygon is finite if both s and t are finite. If exactly one of s and t is finite, then the near polygon is called *semifinite*.

In this paper we only consider near polygons satisfying the following properties: (i) every line is incident with $s + 1 \geq 3$ points, (ii) every two points at distance 2 have at least two common neighbours. By Yanushka's Theorem (Proposition 2.5 of [7]), every two points at distance 2 are contained in a so-called *quad* which is a geodetically closed subGQ. More generally, we know that every two points x and y at mutual distance $i \in \{0, \ldots, d\}$ are contained in a unique geodetically closed sub near $2i$-gon $H(x, y)$ (Theorem 4 of [3]). It also can be proved that the near polygon has an order (s, t) (Lemma 19 of [3]).

An interesting problem is whether there are semifinite near polygons satisfying (i) and (ii). It is proved that the answer is negative for $(d, s) \in \{(2, 2), (2, 3), (3, 2)\}$. For GQ's of order $(2, t)$, the problem was solved by Cameron [5]. The case of GQ's of order $(3, t)$ was considered by Brouwer [1] and the case of near hexagons of order $(2, t)$ was treated by

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Brouwer et al. [2]. In Section 3 we show that similar techniques as in [2] and [5] provide upper bounds for the number of lines through a point for any near $2d$-gon of order $(2, t)$ satisfying (ii). As a corollary we have the following result which is the main theorem of this paper.

Main theorem. *Every near polygon of order* $(2, t)$ *satisfying* (ii) *is finite*.

In order to determine an upper bound for $t + 1$, we need to find an upper bound for the diameters of the graphs $\Gamma_d(x)$. This problem is solved in Section 2. In Section 4 we show how the use of certain geodetically closed sub near polygons can lead to better upper bounds for $t + 1$. These improved upper bounds might be useful to reduce the case load in future classifications.

2. Finiteness of the diameter of $\Gamma_d(x)$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a near $2d$-gon satisfying

(I) every line is incident with at least three points,
(II) every two points at distance 2 have at least two common neighbours, and let x be a fixed point of $\mathcal{S}$.

Lemma 1 *Let* u, v *and* w *be three points such that* $d(u, v) = d(v, w) = 1$ *and* $d(x, v) = d(x, w) = d(x, u) - 1$, *then every common neighbour* $\tilde{v} \neq v$ *of* u *and* w *has distance* $d(x, u)$ *from* x.

Proof: Let z be the point of vw at distance $d(x, u) - 2$ from x. The point on $u\tilde{v}$ collinear with z has distance $d(x, u) - 1$ from x. Hence $d(x, \tilde{v}) = d(x, u)$. □

Definitions A path $\gamma = (x_0, x_1, \ldots, x_k)$ in $\mathcal{S}$ is called *saw-edged* if the following three conditions are satisfied:

(1) $d(x, x_0) = d(x, x_k)$;
(2) $d(x, x_i) \in \{d(x, x_0), d(x, x_0) + 1\}$ for all $i \in \{0, \ldots, k\}$;
(3) if $d(x, x_i) = d(x, x_0) + 1$, then $d(x, x_{i-1}) = d(x, x_{i+1}) = d(x, x_0)$.

Let $l(\gamma) = k$ denote the length of γ. We call $\bar{l}(\gamma) = l(\gamma) + t(\gamma)$ the *modified length* of γ; here $t(\gamma)$ denotes the number of *teeth* of γ, i.e. the number of vertices x_i, $i \in \{0, \ldots, k\}$, at distance $d(x, x_0) + 1$ from x.

Theorem 1 *Let* $y, z \in \mathcal{P}$ *such that* $d(x, y) = d(x, z)$. *If* y *and* z *are connected by a path of length* δ *only consisting of points at distance at most* $d(x, y)$ *from* x, *then* y *and* z *are connected by a saw-edged path* γ *with* $\bar{l}(\gamma) \leq \frac{3}{2}\delta$.

Proof: We use induction on δ. Clearly, the theorem holds if $\delta = 0$ or $\delta = 1$. Let $\delta = 2$ and let u and u' be two common neighbours of y and z. We may suppose that $d(x, u) = d(x, u') = d(x, y) - 1$. Choose now collinear points $v \in uz \setminus \{u, z\}$ and $v' \in yu' \setminus \{y, u'\}$, then the path (y, v', v, z) is saw-edged and has modified length 3. Suppose therefore that $\delta \geq 3$ and consider a path $y = x_0, x_1, \ldots, x_\delta = z$ for which $d(x, x_i) \leq d(x, y)$ for all $i \in \{0, \ldots, \delta\}$. If $d(x, x_i) = d(x, y)$ for some $i \in \{1, \ldots, \delta - 1\}$ then there exists a saw-edged path of modified length at most $\frac{3}{2}i + \frac{3}{2}(\delta - i) = \frac{3}{2}\delta$ connecting y and z. Suppose therefore that $d(x, x_i) < d(x, y)$ for all $i \in \{1, \ldots, \delta - 1\}$. By induction, x_1 and $x_{\delta-1}$ are connected by a saw-edged path γ_1 with $\bar{l}(\gamma_1) \leq \frac{3}{2}(\delta - 2) = \frac{3}{2}\delta - 3$. The path γ_1 can be extended to a path γ_2 of length $k = l(\gamma_1) + 2$ connecting y and z. By Lemma 1, the path γ_2 can be replaced by a path $\gamma_3 = (a_0, \ldots, a_k)$ which satisfies the following properties.

(a) $a_0 = y, a_k = z$.
(b) There are exactly $t(\gamma_1) + 1$ points $a_i, i \in \{0, \ldots, k\}$, satisfying $d(x, a_i) = d(x, y) - 1$; all the other points of the path γ_3 lie at distance $d(x, y)$ from x.
(c) If $d(x, a_i) = d(x, y) - 1$ for some $i \in \{0, \ldots, k\}$, then $d(x, a_{i-1}) = d(x, a_{i+1}) = d(x, y)$.

If $d(x, a_i) = d(x, y) - 1$ for some $i \in \{0, \ldots, k\}$, then the path (a_{i-1}, a_i, a_{i+1}) can be replaced by a saw-edged path of modified length at most 3. Hence y and z are connected by a saw-edged path of length at most

$$l(\gamma_3) - 2(t(\gamma_1) + 1) + 3(t(\gamma_1) + 1) = l(\gamma_1) + t(\gamma_1) + 3 = \bar{l}(\gamma_1) + 3 \leq \frac{3}{2}\delta. \qquad \square$$

Corollary 1
(a) *If y and z are points of $\mathcal{S}$ such that $d(x, y) = d(x, z)$, then they are connected by a saw-edged path γ with $\bar{l}(\gamma) \leq 3\, d(x, y)$.*
(b) *The subgraph $\Gamma_d(x)$ of Γ whose vertices are the points of $\mathcal{S}$ at distance d From x has diameter at most $\lfloor \frac{3}{2}d \rfloor$.*

3. Finiteness of $t + 1$

Let $\mathcal{S}$ be a near $2d$-gon satisfying the following properties:

(I) every line is incident with exactly 3 points,
(II) every two points at distance 2 have at least two common neighbours.

Let M be a positive integer for which the following holds:

(*) every geodetically closed sub near $2(d-1)$-gon H of $\mathcal{S}$ has order $(2, t_H)$ with $t_H \leq M$.

In this section, we will prove by induction that such a positive integer M always exists (notice that for $d = 2$, we can take $M = 0$). At this moment however, we need to assume the existence of such a number. Let $(2, t)$ be the order of $\mathcal{S}$. We now derive upper bounds for $t + 1$ using similar techniques as in [2] and [5].

Lemma 2 *If there is a cycle of length $2n + 1, n > 1$, in $\Gamma_d(x)$, then* $t + 1 \leq (2n + 1)(M + 1)$.

Proof: Let $y_0, y_1, \dots, y_{2n+1} = y_0$ be a cycle of length $2n+1$ in $\Gamma_d(x)$. Let z_i, $i \in \{0, \dots, 2n\}$, denote the unique third point on the line $y_i y_{i+1}$. Suppose that $t+1 > (2n+1)(M+1)$. Then there exists a line L through x which is not contained in one of the sub near polygons $H(x, z_i)$. Now, let u be the point of L at distance $d-1$ from y_0. If $d(u, z_i) \leq d-1$ for a certain $i \in \{0, \dots, 2n\}$, then $d(x, z_i) = d-1$ implies that $d(w, z_i) = d-2$ for a certain point w on xu or that L is contained in $H(x, z_i)$, a contradiction. Hence $d(u, z_i) = d$ for every $i \in \{0, \dots, 2n\}$. Since $d(u, y_0) = d-1$ and $d(u, z_0) = d$, we necessarily have that $d(u, y_1) = d$. Since $d(u, y_1) = d$ and $d(u, z_1) = d$, $d(u, y_2) = d-1$. Repeating this argument several times, one finds that $d(u, y_{2n+1}) = d$, contradicting $d-1 = d(u, y_0) = d(u, y_{2n+1})$. Hence $t+1 \leq (2n+1)(M+1)$. ☐

Corollary 2 *At least one of the following possibilities occurs*:
(1) $t+1 \leq (2\lfloor \frac{3d}{2} \rfloor + 1)(M+1)$
(2) $\Gamma_d(x)$ *is a bipartite graph.*

Theorem 2 *If* $\Gamma_d(x)$ *is bipartite*, *then* $t+1 \leq \lfloor \frac{3d}{2} \rfloor (M+1)^2$.

Proof: Let $d'(\cdot,\cdot)$ denote the distance in $\Gamma_d(x)$. Let y be a fixed vertex of $\Gamma_d(x)$. For every $z \in \Gamma_d(x)$, let C_z be the set of lines through x containing a point of $\Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)$, and let D_z denote the set of all the other lines through x. If $d'(y, z)$ is even, then we put $A_z := D_z$ and $B_z := C_z$; otherwise $A_z := C_z$ and $B_z := D_z$. Clearly $A_y = \emptyset$. For two collinear points z and z' in $\Gamma_d(x)$, let $E_{z,z'}$ denote the set of all lines through x contained in $H(x, z'')$ with z'' the unique third point of the line zz'. Since $\Gamma_1(x) \cap (\Gamma_{d-1}(z) \cap \Gamma_{d-1}(z')) = \Gamma_1(x) \cap \Gamma_{d-2}(z'')$, we have that $A_{z'} = A_z \Delta E_{z,z'}$ (symmetrical difference). Hence for every point z of $\Gamma_d(x)$, $|A_z| \leq \lfloor \frac{3d}{2} \rfloor (M+1)$. Now, let z be a point of $\Gamma_d(x)$ for which $|A_z|$ is maximal. Let $A_z = \{L_1, \dots, L_{|A_z|}\}$. For every $i \in \{1, \dots, |A_z|\}$, let H_i denote the unique geodetically closed sub near polygon through z and the unique point of L_i at distance $d-1$ from z. If there would be a line zz', $z' \in \Gamma_d(x)$, through z not contained in any of these sub near polygons, then $|A_{z'}| = |A_z| + |E_{z,z'}| > |A_z|$, a contradiction. Hence $t+1 \leq \lfloor \frac{3d}{2} \rfloor (M+1)^2$. ☐

If $d = 1$, then $t+1 = 1$. For $d > 1$, successive application of Corollary 2 and Theorem 2 gives an upper bound for $t+1$ which only depends on d. This proves our Main Theorem. It is now also clear that a number M exists for which condition (*) holds.

4. Improved upper bounds for $t+1$

Again, let $\mathcal{S}$ be a near $2d$-gon satisfying (I) and (II). Lemma 2 provides an upper bound for $t+1$ in the case that there exists a point x for which $\Gamma_d(x)$ is not bipartite. This upper bound can be improved if certain geodetically closed sub near polygons exist, see Lemma 5. These improved upper bounds might be useful to reduce the case load in future classifications. As in the previous section let M denote a positive integer satisfying condition (*).

Let H be a geodetically closed sub near 2δ-gon of $\mathcal{S}$. A point x of $\mathcal{S}$ is called *classical* with respect to H if there exists a (necessarily unique) point $\pi(x) \in H$ such that $d(x, y) = d(x, \pi(x)) + d(\pi(x), y)$ for all $y \in H$.

Lemma 3 (Lemma 3.1 of [3]) *A point x of $\mathcal{S}$ is classical with respect to H if and only if there exists a point $y \in H$ at distance $d(x, H) + \delta$ from x.*

Classical points always exist at any admissible distance from H.

Lemma 4 *For every $i \in \{0, \ldots, d - \delta\}$, there exists a point x at distance i from H for which (x, H) is classical.*

Proof: The lemma clearly is true if $i = 0$. Suppose therefore that $i > 0$. Let y be a point at distance $i - 1$ from H for which (y, H) is classical. Let z be a point of H at distance $i + \delta - 1$ from x. Since $i + \delta - 1 \leq d - 1$, there exists a point $x \sim y$ at distance $i + \delta$ from z. Since z has distance at most δ to any point of H, $d(x, H) \geq i$. Hence $d(x, H) = i$ since $d(y, H) = i - 1$. The result now follows from Lemma 3. □

The following lemma is an improvement of Lemma 2.

Lemma 5 *If there exists a $(2n + 1)$-cycle in $\Gamma_\delta(x) \cap H$ for some point $x \in H$, then $t + 1 \leq (2n + 1)(M + 1) - 2n(d - \delta)$.*

Proof: By the previous lemma there exists a point y at distance $d(y, x) = d(y, H) = d - \delta$ from x such that (y, H) is classical. If $H(x, y)$ has order $(2, N)$, then a similar reasoning as in the proof of Lemma 2 yields $(t+1)-(N+1) \leq (2n+1)((M+1)-(N+1))$. Hence $t+1 \leq (2n + 1)(M + 1) - 2n(N + 1)$. The lemma now follows since $N + 1 \geq d - \delta$. □

We will now give two instances where Lemma 5 can be applied and leads to better upper bounds for $t + 1$. As any quad Q of $\mathcal{S}$ has an order $(2, t_Q)$, Q must be isomorphic to either the (3×3)-grid, $W(2)$ or $Q(5, 2)$, see e.g. [6]. If H is isomorphic to $Q(5, 2)$, then $\Gamma_2(x) \cap H$ contains a path of length 5, see [6]. Another example is when H is a hex isomorphic to the near hexagon $T_5^*(\mathcal{K})$ which we will describe now. Let Π_∞ be a PG(5, 3) which is embedded as a hyperplane in Π. Consider in Π_∞ the set $\mathcal{K}$ of 12 points determined by the columns of the following matrix:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}$$

Let $T_5^*(\mathcal{K})$ denote the incidence structure whose points are the points of $\Pi \setminus \Pi_\infty$, whose lines are those lines L of Π_∞ satisfying $|L \cap \Pi_\infty| = |L \cap \mathcal{K}| = 1$, and whose incidence relation is the one derived from Π. By [4] and [7], $T_5^*(\mathcal{K})$ is a near hexagon of order $(2, 11)$. Let x be a point of $\Pi \setminus \Pi_\infty$ and take a 3-dimensional subspace α of Π_∞ for which $|\alpha \cap \mathcal{K}| = 4$, e.g., let α be generated by the first four columns of the matrix M. Let β

be a plane generated by three points of $\alpha \cap \mathcal{K}$. Let $\gamma \notin \{\alpha, \langle x, \beta\rangle\}$ be a 3-dimensional subspace of $\langle x, \alpha\rangle$ through β. The points of $\gamma \setminus \Pi_\infty$ determine a sub near hexagon C of $T_5^*(\mathcal{K})$ which is isomorphic to a generalized cube. The points of C can be labeled with the triples (i, j, k), $i, j, k \in \{-1, 0, 1\}$, such that $(i, j, k) \sim (i', j', k')$ if and only if these triples agree in exactly two positions. If $(0, 0, 0)$ is the unique point of $\gamma \setminus \Pi_\infty$ collinear with x, then

- $d(x, (0, 0, 0)) = 1$,
- $d(x, (1, 0, 0)) = d(x, (-1, 0, 0)) = d(x, (0, 1, 0)) = d(x, (0, -1, 0)) = d(x, (0, 0, 1)) = d(x, (0, 0, -1)) = 2$,
- $d(x, (1, 1, 0)) = d(x, (1, -1, 0)) = d(x, (-1, 1, 0)) = d(x, (-1, -1, 0)) = d(x, (1, 0, 1)) = d(x, (1, 0, -1)) = d(x, (-1, 0, 1)) = d(x, (-1, 0, -1)) = d(x, (0, 1, 1)) = d(x, (0, 1, -1)) = d(x, (0, -1, 1)) = d(x, (0, -1, -1)) = 3$.

We may suppose that $d(x, (1, 1, 1)) = 3$. Then $d(x, (1, 1, -1)) = d(x, (-1, 1, 1)) = d(x, (1, -1, 1)) = d(x, (-1, -1, -1)) = 2$ and $d(x, (-1, -1, 1)) = d(x, (-1, 1, -1)) = d(x, (1, -1, -1)) = 3$. The closed path $(0, -1, -1), (1, -1, -1), (1, 0, -1), (1, 0, 1), (1, 1, 1), (0, 1, 1), (0, -1, 1), (0, -1, -1)$ has length 7 and is completely contained in $\Gamma_3(x)$.

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