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On the Twisted Cubic of $PG(3, q)$

G. LUNARDON lunardon@unina.it
Dipartimento di Matematica e Applicazioni, via Cintia, Complesso di Monte S. Angelo, 80126 Napoli, Italy

O. POLVERINO opolveri@unina.it; olga.polverino@unina2.it
Dipartimento di Matematica, Seconda Università degli Studi di Napoli, via Vivaldi n 43, 81100 Caserta, Italy

Abstract. In this paper we classify the lines of $PG(3, q)$ whose points belong to imaginary chords of the twisted cubic of $PG(3, q)$. Relying on this classification result, we obtain a complete classification of semiclassical spreads of the generalized hexagon $H(q)$.

Keywords: twisted cubics, generalized hexagons, coset geometries, spreads

1. Introduction

The twisted cubic of $PG(3, q)$, $q = p^h$, p prime, can be described as follows

$$\Sigma = \{(f_0(t), f_1(t), f_2(t), f_3(t)) : t \in GF(q) \cup \{\infty\}\},$$

where $f_0(t), \ldots, f_3(t)$ are linearly independent cubic polynomials over $GF(q)$.

Let $\bar{\Sigma}$ be the twisted cubic of $PG(3, \mathbb{F})$ defined by Σ, where $\mathbb{F}$ is the algebraic closure of $GF(q)$. A line of $PG(3, q)$ is a *chord* of Σ if its extension to $PG(3, \mathbb{F})$ contains two points of $\bar{\Sigma}$ (in the algebraic sense). There are three possibilities: the two points belong to Σ, or they are coincident, or they are conjugate over $GF(q^2)$. This is called a *real chord*, a *tangent* or an *imaginary chord*, respectively. By [4] Lemma 1, every point off Σ lies on exactly one chord. If $p \neq 3$, the tangents to Σ are self-polar lines of a non-singular symplectic polarity ω of $PG(3, q)$. An *axis* of Σ is a line l of $PG(3, q)$ whose polar line with respect to ω is a chord. We say that l is a real axis or an imaginary axis if l^ω is a real chord or an imaginary chord, respectively (for more details, see [7]). If $q \equiv 1 \pmod 3$ and l is an imaginary axis, then all points on l belong to some imaginary chord (see [6]). In Section 2 we prove the following result

Theorem 1 *If l is a line of $PG(3, q)$ whose points belong to imaginary chords of Σ, then either l is an imaginary chord or $q \equiv 1 \pmod 3$ and l is an imaginary axis.*

In Section 3, Theorem 1 is used to study semiclassical spreads of the generalized hexagon $H(q)$. Tits [11] constructs the generalized hexagon $H(q)$ as follows. Let $Q(6, q)$ be the parabolic quadric of $PG(6, q)$ with equation $X_3^2 = X_0X_4 + X_1X_5 + X_2X_6$. The points of $H(q)$ are all the points of $Q(6, q)$. The lines are those lines of $Q(6, q)$ whose Grassmann coordinates satisfy the equations $p_{34} = p_{12}$, $p_{35} = p_{20}$, $p_{36} = p_{01}$, $p_{03} = p_{56}$, $p_{13} = p_{64}$

and $p_{23} = p_{45}$. Two elements of $H(q)$ are *opposite* if they are at distance 6 in the incidence graph of $H(q)$. A *spread* of $H(q)$ is a set of $q^3 + 1$ mutually opposite lines of $H(q)$.

Let $Q^-(5, q)$ be an elliptic quadric intersection of $Q(6, q)$ with a 5-dimensional space. Let $\mathcal{S}$ be the set of lines of $H(q)$ contained in $Q^-(5, q)$. $\mathcal{S}$ is a spread of $H(q)$ called *Hermitian* [5].

Let $\mathcal{S}$ be a spread of $H(q)$ and let L be a fixed line of $\mathcal{S}$. For each line M of $\mathcal{S} \setminus \{L\}$, the subspace $\langle L, M\rangle$ has dimension 3 and intersects $Q(6, q)$ in a nonsingular hyperbolic quadric. Let $\mathcal{R}_{L,M}$ be the regulus of $\langle L, M\rangle \cap Q(6, q)$ containing the lines L and M. The spread $\mathcal{S}$ of $H(q)$ is *locally Hermitian* with respect to L if $\mathcal{R}_{L,M}$ is contained in $\mathcal{S}$ for all lines M of $\mathcal{S}$ different from L. $\mathcal{S}$ is locally Hermitian with respect to all the lines of $\mathcal{S}$ if and only if it is a Hermitian spread (see [3]).

Let x be a fixed point of $Q(6, q)$ and denote by Γ_x the polar space whose points are the lines of $Q(6, q)$ incident with x and whose lines are the planes of $Q(6, q)$ incident with x. By construction, $\Gamma_x \simeq Q(4, q)$. If $\mathcal{S}$ is a locally Hermitian spread of $H(q)$ with respect to L and x is a point of L, then the set of lines $\mathcal{O}_x$, whose elements are either L or the transversals through x to the reguli of $\mathcal{S}$ containing L, is an ovoid of $\Gamma_x \simeq Q(4, q)$ (see [3]). We call $\mathcal{O}_x$ a *projection along reguli* of $\mathcal{S}$. If $\mathcal{O}_x$ is an elliptic quadric for all x in L, the spread $\mathcal{S}$ is called *semiclassical*.

In [3], Bloemen, Thas and Van Maldeghem have proved that, for q odd, a semiclassical spread of $H(q)$ is either Hermitian or isomorphic to the spread $\mathcal{S}_{[9]}$ constructed in [3]. Recently, there have been two (independent) constructions of non-Hermitian semiclassical spreads $\mathcal{S}_l$ in $H(q)$ for q an even power of 2; one in [6] by Cardinali, Lunardon, Polverino and Trombetti and one in [9] by Offer. It has first been shown by D. Luyckx (unpublished) that these two families are equivalent. Remark that the construction in [6] is valid for all $q \equiv 1 \pmod 3$, but for such odd q, the corresponding spreads are equivalent to $\mathcal{S}_{[9]}$.

In this paper, using the representation of $H(q)$ as a coset geometry and as an application of Theorem 1, we extend the classification result of Bloemen, Thas and Van Maldeghem of semiclassical spreads of $H(q)$ to the even characteristic case. We prove that a semiclassical spread of $H(q)$ is either Hermitian or isomorphic to $\mathcal{S}_l$, i.e. there is exactly one non-Hermitian semiclassical spread of $H(q)$ for all prime powers q congruent to 1 modulo 3.

2. Proof of Theorem 1

Proof: Take the twisted cubic Σ in the canonical form $\Sigma = \{(t^3, t^2, t, 1) : t \in GF(q)\} \cup \{(1, 0, 0, 0)\}$ and let l be a line of $PG(3, q)$ whose points belong to imaginary chords of Σ. Let P be the point of $PG(3, q)$ with coordinates $(\alpha - 1, 1, -1, 0)$ and suppose that $t^2 + t + \alpha$ is an irreducible polynomial over $GF(q)$. This implies that P belongs to the imaginary chord $\bar{l}$ of Σ with equations $x_0 + x_1 + \alpha x_2 = x_1 + x_2 + \alpha x_3 = 0$. Under the action of the collineation group K of $PGL(4, q)$ fixing Σ, there are two orbits of points on an imaginary chord if $q \equiv -1 \pmod 3$ and there is exactly one orbit if $q \not\equiv -1 \pmod 3$. In the former case the line l contains points of both orbits ([7], Corollary 5 to Lemma 21.1.3 and Corollary to Lemma 21.1.11). Hence, without loss of generality, we may assume that the point P belongs to l. Also, since l is not contained in any osculating plane, we can suppose that l contains the point $Q = (a, 0, a', 1)$ for some $a, a' \in GF(q)$. Thus, l has equations

$$\begin{cases} x_1 + x_2 - a'x_3 = 0 \\ x_0 + (1-\alpha)x_1 - ax_3 = 0 \end{cases}$$

In this case, l is an imaginary chord if $l = \bar{l}$, i.e. if $a = \alpha^2$ and $a' = -\alpha$, and it is easy to verify that l is an imaginary axis if $q \equiv 1 \pmod 3$, $a = (1-\alpha)^2/9$ and $a' = (-2-\alpha)/9$.

Every plane through l meets Σ in exactly one point off l. This implies that the plane $x_1 + x_2 - a'x_3 = 0$ contains only one point of Σ, namely $(1, 0, 0, 0)$, and hence $t^2 + t - a'$ is an irreducible polynomial over $GF(q)$. Moreover, for each $\lambda \in GF(q)$ there exists exactly one element $t \in GF(q)$ such that $P_t = (t^3, t^2, t, 1)$ belongs to the plane

$$\pi_\lambda : x_0 + (1-\alpha)x_1 - ax_3 + \lambda(x_1 + x_2 - a'x_3) = 0.$$

Therefore, for each $\lambda \in GF(q)$ there exists $t \in GF(q)$ such that

$$\lambda = \lambda(t) = \frac{-t^3 - (1-\alpha)t^2 + a}{(t^2 + t - a')} = \frac{p(t)}{q(t)}.$$

Hence $\lambda(t) = \lambda(t')$ implies $t = t'$. By a direct calculation, we get $\lambda(t) = \lambda(t')$ if and only if

$$\begin{aligned} &G(t, t') = p(t')q(t) - p(t)q(t') = (t - t') \cdot \\ &[t^2t'^2 + t^2t' + tt'^2 - a'(t^2 + t'^2) + (1 - \alpha - a')tt' + (a - a'(1-\alpha))(t + t') + a] = 0. \end{aligned}$$

Then $F(t, t') = G(t, t')/(t - t') \neq 0$ for any distinct $t, t' \in GF(q)$. If $F(t_0, t_0) = 0$, then $(t - t_0)^2 \mid G(t, t_0)$, and hence $(t - t_0)^2 \mid (p(t_0)q(t) - p(t)q(t_0)) = q(t_0)(-p(t) + \lambda(t_0)q(t))$. This implies that the plane $\pi_{\lambda(t_0)}$ either contains two points of Σ or is an osculating plane. Hence, $F(t, t') \neq 0$ for any $t, t' \in GF(q)$. Let $X = t$ and $Y = t'$ and let Γ be the algebraic curve with affine equation

$$\begin{aligned} F(X, Y) = X^2Y^2 + X^2Y + XY^2 - a'(X^2 + Y^2) + (1 - \alpha - a')XY \\ + (a - a'(1-\alpha))(X + Y) + a = 0 \end{aligned} \tag{1}$$

Since $F(X, Y) \neq 0$ for each $X, Y \in GF(q)$, Γ has only two $GF(q)$-rational points: $X_\infty = (0, 1, 0)$ and $Y_\infty = (0, 0, 1)$, which are isolated double points with tangents, respectively, $Y = \varepsilon_i$ and $X = \varepsilon_i$ $(i = 1, 2)$, where $\varepsilon_i^2 + \varepsilon_i - a' = 0$. If Γ is absolutely irreducible, then it has genus $g \leq 1$, and, if N_q is the number of $GF(q)$-rational points of Γ, by the Hasse-Weil bound ([10]), we get

$$2 = N_q \geq q + 1 - 2g\sqrt{q} \geq q - 2\sqrt{q} + 1$$

which is not possible for $q > 5$. Then, if $q > 5$, Γ is absolutely reducible. In this case, we may assume Γ is the union of two conics C_1 and C_2, both passing through X_∞ and Y_∞, with affine equations, respectively, $G_1(X, Y) = 0$ and $G_2(X, Y) = 0$. Since $F(X, Y) = F(Y, X)$, we have that either $G_1(Y, X) = G_2(X, Y)$ or $G_1(Y, X) = G_1(X, Y)$ and $G_2(Y, X) = G_2(X, Y)$.

Suppose that $G_1(Y, X) = G_2(X, Y)$ and let $G_1(X, Y) = XY + AX + BY + C$. Since the tangents to C_1 and C_2 at the points X_∞ and Y_∞ are, respectively, the lines $Y = \varepsilon_i$ and $X = \varepsilon_i$ $(i = 1, 2)$, we may assume $A = -\varepsilon_1$ and $B = -\varepsilon_2$. Hence, Γ has equation

$$(XY - \varepsilon_1 X - \varepsilon_2 Y + C)(XY - \varepsilon_2 X - \varepsilon_1 Y + C) = 0 \tag{2}$$

Comparing the coefficients of Eqs. (1) and (2) and noting that $\varepsilon_1 + \varepsilon_2 = -1$ and $\varepsilon_1 \cdot \varepsilon_2 = -a'$, we obtain

$$\begin{cases} 2C = -3a' - \alpha \\ C = a - a'(1 - \alpha) \\ C^2 = a \end{cases} \tag{$*$}$$

Also, since the affine intersection points of C_1 and C_2 are not $GF(q)$-rational points, we have that $X^2 + X + C$ is irreducible over $GF(q)$. If q is even, from $(*)$ we get either $a = \alpha^2$ and $a' = -\alpha$ or $a = 1 + \alpha^2 = (1 - \alpha)^2/9$ and $a' = -\alpha = (-2 - \alpha)/9$. In the former case l is an imaginary chord. In the latter case, since $C = 1 + \alpha$ and $X^2 + X + C$ is irreducible over $GF(q)$, we have $q \equiv 1 (\text{mod } 3)$ and hence l is an imaginary axis. If q is odd, with some calculation from $(*)$ we get

$$\begin{cases} 9a'^2 + 2a'(1 + 5\alpha) + \alpha^2 + 2\alpha = 0 \\ a = \dfrac{-a'(1 + 2\alpha) - \alpha}{2} \end{cases} \tag{$**$}$$

If $q \equiv 0 (\text{mod } 3)$, from $(**)$ we get $a' = -\alpha$ and $a = \alpha^2$, so l is an imaginary chord. If $q \not\equiv 0 (\text{mod } 3)$, from $(**)$ we get either $a' = -\alpha$ and $a = \alpha^2$ or $a' = (-2 - \alpha)/9$ and $a = (1 - \alpha)^2/9$. In the latter case, since $X^2 + X + C$ is irreducible over $GF(q)$, we obtain $q \equiv 1 (\text{mod } 3)$, and hence l is an imaginary axis.

On the other hand, suppose $G_1(Y, X) = G_1(X, Y)$ and $G_2(Y, X) = G_2(X, Y)$. Let $G_1(X, Y) = XY + AX + AY + B$ and $G_2(X, Y) = XY + A'Y + A'X + B'$. We may assume $A = -\varepsilon_1$ and $A' = -\varepsilon_2$ and this implies that Γ has equation

$$(XY - \varepsilon_1(X + Y) + B)(XY - \varepsilon_2(X + Y) + B') = 0 \tag{3}$$

With an argument similar to the previous one, we obtain

$$\begin{cases} B + B' = 1 - \alpha + a' \\ \varepsilon_1 B' + \varepsilon_2 B = a'(1 - \alpha) - a \\ B \cdot B' = a \end{cases}$$

From the above system we get

$$a^2 + a(a' + 2a'\alpha + \alpha) - (a'^3 + a'^2\alpha(1 - \alpha)) = 0. \tag{4}$$

If $a' = -\alpha$, from (4) we get $a = \alpha^2$. If $a' \neq -\alpha$, treating (4) as a quadratic equation in a, we find that for q odd its discriminant is

$$\Delta = (1 + 4a')(a' + \alpha)^2$$

and, since $t^2 + t - a'$ is irreducible over $GF(q)$, Δ is a non-square over $GF(q)$. For q even, the S-invariant of (4) is

$$S = \frac{a'^3 + a'^2\alpha + a'^2\alpha^2}{a'^2 + \alpha^2}$$

and since $t^2 + t - a'$ is irreducible over $GF(q)$, $Tr(a') = 1$. So, we get

$$Tr(S) = Tr(a') + Tr(\alpha + \alpha^2) + Tr\left(\frac{\alpha^2}{a' + \alpha} + \frac{\alpha^4}{a'^2 + \alpha^2}\right) = 1 + 0 + 0 = 1.$$

Hence in both cases Eq. (4) has no solution in $GF(q)$. This proves the theorem for $q > 5$.

Finally, for $q \leq 5$, if we require the function $\lambda(t)$ is one-to-one and none of the planes $\pi_{\lambda(t)}$ is an osculating plane, by a direct calculation we get that either the line l is an imaginary chord or $q = 4$ and l is an imaginary axis. □

3. Coset geometries and semiclassical spreads

Let L be a fixed line of $H(q)$ and denote by E^L the group of automorphisms of $H(q)$ generated by all the collineations fixing L pointwise and stabilizing all the lines through any point of L. The group E^L has order q^5 and acts regularly on the set of the lines of $H(q)$ at distance 6 from L (see, e.g. [2] or [12]). A spread $\mathcal{S}$ of $H(q)$ containing L is a *translation spread* with respect to L, if for each $x \in L$ there is a subgroup of E^L which preserves $\mathcal{S}$ and acts transitively on the lines of $\mathcal{S}$ at distance 4 from M, for all lines M of $H(q)$ incident with x and different from L. By [9] Theorem 5, a spread $\mathcal{S}$ of $H(q)$ is a translation spread with respect to L if and only if the stabilizer of $\mathcal{S}$ in E^L has order q^3.

Let $[\infty] = \langle(1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 1)\rangle$. The lines of $H(q)$ at distance 6 from $[\infty]$ are the lines

$$[a, b, c, d, e] = \langle(c + bd, a, 1, b, 0, d, b^2 - da), (d^2 + eb, -b, 0, -d, 1, e, -ae - c - 2bd)\rangle$$

of $Q(6, q)$ [3]. Denote by $\theta(a, b, c, d, e)$ the element of $E^{[\infty]}$ which maps the line $[0, 0, 0, 0, 0]$ to the line $[a, b, c, d, e]$. The group $Z = \{\theta(0, 0, c, 0, 0) \mid c \in GF(q)\}$ is contained in the center of $E^{[\infty]}$ when $q = 3^r$, and it is the center of $E^{[\infty]}$ when $q \neq 3^r$. Note that the lines $[\infty]$ and $M = [0, 0, 0, 0, 0]$ of $H(q)$ are at distance 6. Denote by $x_\infty = (1, 0, 0, 0, 0, 0, 0)$ and $x_t = (t, 0, 0, 0, 0, 0, 1)$, $t \in GF(q)$, the points of the line

$[\infty]$. For each $t \in \tilde{F} = GF(q) \cup \{\infty\}$, there is a unique chain, say $(x_t, N_t, y_t, R_t, z_t, M)$, of lenght 5 joining x_t and M. Put

$$A_4(t) = \{g \in E^{[\infty]} \mid N_t^g = N_t\}$$
$$A_3(t) = \{g \in E^{[\infty]} \mid y_t^g = y_t\}$$
$$A_2(t) = \{g \in E^{[\infty]} \mid R_t^g = R_t\}$$
$$A_1(t) = \{g \in E^{[\infty]} \mid z_t^g = z_t\}.$$

Then, $A_1(t) < A_2(t) < A_3(t) < A_4(t)$ and $A_3(t) = A_2(t)Z$. Moreover, $|A_1(t)| = q$, $|A_2(t)| = q^2$, $|A_3(t)| = q^3$ and $|A_4(t)| = q^4$ (see, e.g. [2] or [12]).

Define a point-line geometry $H = (\mathbf{P}, \mathbf{L}, I)$ as follows:

$$\mathbf{P} = \{(t), A_3(t)g, A_1(t)g : g \in E^{[\infty]}, t \in \tilde{F}\}$$
$$\mathbf{L} = \{[\infty], A_4(t)g, A_2(t)g, g : g \in E^{[\infty]}, t \in \tilde{F}\}$$

where $[\infty]$ and (t) are symbols, and the incidences are: $[\infty]I(t)$, $A_4(t)I(t)$, $gIA_1(t)g$ for all $t \in \tilde{F}$ and $g \in E^{[\infty]}$, whereas $A_i(t)gIA_{i+1}(v)h$ if and only if $t = v$ and $g \in A_{i+1}(v)h$ with $i = 1, 2, 3$ and for all $g, h \in E^{[\infty]}$ and $t, v \in \tilde{F}$. Then, H is isomorphic to $H(q)$ (see, e.g., [2]).

The group

$$\bar{E} = E^{[\infty]}/Z = \{(x, y, z, t) \mid x, y, z, t \in GF(q)\}$$

is elementary abelian, and can be regarded as a four-dimensional vector space over $GF(q)$. For each element g of $E^{[\infty]}$, let g^* be the preimage in $E^{[\infty]}$ of the 1-space of $\bar{E}$ spanned by $\bar{g} = gZ$. If $\bar{g}, \bar{h}$ are elements of $\bar{E}$, then $(\bar{g}, \bar{h}) = [g, h]$ defines an alternating $GF(q)$-bilinear form on $\bar{E}$; if q is even, $\bar{g} \mapsto g^2$ defines a quadratic form associated with (,). Thus, $\bar{E}$ is endowed with a symplectic or an orthogonal geometry. If $[g, h] = 1$, then $[g^*, h^*] = 1$. Thus, maximal elementary abelian subgroups of $E^{[\infty]}$ are preimages of maximal totally isotropic (or singular) 2-spaces of $\bar{E}$.

Let $PG(3, q)$ be the 3-dimensional projective space associated with the $GF(q)$-vector space $\bar{E}$. As $A_3(t) = A_2(t)Z$, $A_3(t)$ is a maximal elementary abelian subgroup of $E^{[\infty]}$ for each $t \in \tilde{F}$. Thus, $L_t = A_3(t)/Z$ is a totally isotropic (or singular) line of the projective space $PG(3, q)$. Denote by P_t, L_t, α_t, respectively, the point $A_1(t)Z/Z$, the line $A_3(t)/Z$ and the plane $A_4(t)/Z$ of $PG(3, q)$. The set $\Sigma = \{P_t : t \in \tilde{F}\}$ is a twisted cubic of $PG(3, q)$ and L_t is the tangent line to Σ at P_t and α_t is the osculating plane to Σ at P_t. Moreover, we may suppose $P_\infty = (0, 0, 0, 1)$ and $P_t = (1, -t, t^2, t^3)$, $t \in GF(q)$ (see [1]).

If τ is an automorphism of $H(q)$ fixing $[\infty]$ then $\tau E^{[\infty]}\tau^{-1} = E^{[\infty]}$. Since the automorphism of $E^{[\infty]}$ defined by $g \mapsto \tau g \tau^{-1}$ preserves the families $\{A_i(t) \mid t \in \tilde{F}\}$ $(i = 1, 2, 3, 4)$, the automorphisms τ induce, by conjugation, the collineation group K of $PGL(4, q)$ fixing the twisted cubic Σ.

Let $\mathcal{S}$ be a locally Hermitian spread of $H \simeq H(q)$ with respect to the line $[\infty]$. Then we can write $\mathcal{S} = G \cup \{[\infty]\}$ where $G = \{\theta(a, b, c, f(a, b), g(a, b)) \mid a, b, c \in GF(q)\}$

and f and g are functions from $GF(q) \times GF(q)$ to $GF(q)$ (see [6] and [9]). If the line $\theta[0, 0, 0, 0, 0]$ belongs to $\mathcal{S}$, then $\mathcal{S}$ is a translation spread with respect to $[\infty]$ if and only if G is a subgroup of $E^{[\infty]}$ of order q^3 ([9] Theorem 5). By [9] Theorem 7, $\mathcal{S}$ is semiclassical if and only if f and g are $GF(q)$-linear functions. From [9] Theorem 5 and Theorem 7, we have the following

Lemma 1 *All semiclassical spreads of $H(q)$ are translation spreads.*

If G is a subgroup of order q^3 of $E^{[\infty]}$ containing Z and $\mathcal{S} = G \cup \{[\infty]\}$, then the following results are known:

(a) ([6], Theorem 4) $\mathcal{S}$ is a translation spread of $H \simeq H(q)$ with respect to $[\infty]$ if and only if all the points of $S = \{\langle\bar{g}\rangle \mid \bar{g} \in G/Z\} \subset PG(3, q)$ lie on imaginary chords of Σ.
(b) ([6], Corollary 3) $\mathcal{S}$ is a semiclassical spread of $H \simeq H(q)$ if and only if G/Z defines a line of $PG(3, q)$ whose points belong to imaginary chords of Σ.
(c) ([6], Theorem 5) $\mathcal{S}$ is an Hermitian spread of $H \simeq H(q)$ if and only if G/Z defines an imaginary chord of Σ.

If $q \equiv 1 \pmod 3$ and l is an imaginary axis of Σ, then all the points on l belong to some imaginary chord, and hence, if $l = \{\langle\bar{g}\rangle \mid \bar{g} \in G/Z\}$, the subgroup G defines a semiclassical spread $\mathcal{S}_l = G \cup \{[\infty]\}$ of $H \simeq H(q)$ which is not Hermitian (see [6]). Since there is exactly one orbit of imaginary axes under the action of K, the semiclassical spreads $\mathcal{S}_l$ are all equivalent. As noted in [6], for q odd, $\mathcal{S}_l$ is isomorphic to the spread $\mathcal{S}_{[9]}$ constructed in [3]. Also, as noted in the introduction, the new examples of semiclassical spreads of $H(2^{2e})$ constructed in [9] are equivalent to $\mathcal{S}_l$.

Theorem 2 *A semiclassical spread $\mathcal{S}$ of $H(q)$ is either Hermitian or $q \equiv 1 \pmod 3$ and $\mathcal{S}$ is isomorphic to $\mathcal{S}_l$ for l an imaginary axis.*

Proof: Let $\mathcal{S} = G \cup \{[\infty]\}$ be a semiclassical spread of $H \simeq H(q)$. Then G/Z defines a line l of $PG(3, q)$ whose points belong to imaginary chords of Σ. By Theorem 1, either l is an imaginary chord or $q \equiv 1 \pmod 3$ and l is an imaginary axis. In the former case $\mathcal{S}$ is an Hemitian spread; in the latter case $\mathcal{S} = \mathcal{S}_l$. □

Corollary 1 *If $\mathcal{S}$ is a translation spread of $H(2^r)$, then $\mathcal{S}$ is either Hermitian or $q \equiv 1 \pmod 3$ and $\mathcal{S}$ is isomorphic to $\mathcal{S}_l$ for l an imaginary axis.*

Proof: By [6] Corollary 1, all translation spreads of $H(2^r)$ are semiclassical. Then the proof follows from Theorem 2. □

Luyckx and Thas have recently classified the semiclassical 1-systems of $Q(6, q), q = 2^r$, not contained in an elliptic quadric $Q^-(5, q)$ (see [8] for more details), proving there are exactly $\frac{q-2}{2}$ inequivalent examples under the action of the subgroup of $PGL(7, q)$ stabilizing $Q(6, q)$. Applying Theorem 2, exactly one example of such 1-systems of $Q(6, q)$ is a spread of $H(q)$, and this forces $q = 2^{2e}$.

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