Journal of Algebraic Combinatorics, 19, 143–172, 2004

An Inequality Involving the Local Eigenvalues of a Distance-Regular Graph

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Received October 10, 2000; Revised March 24, 2003; Accepted April 15, 2003

Abstract. Let Γ denote a distance-regular graph with diameter $D \geq 3$, valency k, and intersection numbers a_i, b_i, c_i. Let X denote the vertex set of Γ and fix $x \in X$. Let Δ denote the vertex-subgraph of Γ induced on the set of vertices in X adjacent x. Observe Δ has k vertices and is regular with valency a_1. Let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_k$ denote the eigenvalues of Δ and observe $\eta_1 = a_1$. Let Φ denote the set of distinct scalars among $\eta_2, \eta_3, \ldots, \eta_k$. For $\eta \in \Phi$ let mult_η denote the number of times η appears among $\eta_2, \eta_3, \ldots, \eta_k$. Let λ denote an indeterminate, and let $p_0, p_1, \ldots, p_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying $p_0 = 1$ and

$$\lambda p_i = c_{i+1} p_{i+1} + (a_i - c_{i+1} + c_i) p_i + b_i p_{i-1} \quad (0 \leq i \leq D-1),$$

where $p_{-1} = 0$. We show

$$1 + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{p_{i-1}(\tilde{\eta})}{p_i(\tilde{\eta})(1+\tilde{\eta})} \text{mult}_\eta \leq \frac{k}{b_i} \quad (1 \leq i \leq D-1),$$

where we abbreviate $\tilde{\eta} = -1 - b_1(1+\eta)^{-1}$. Concerning the case of equality we obtain the following result. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where A denotes the adjacency matrix of Γ and E_i^* denotes the projection onto the ith subconstituent of Γ with respect to x. T is called the subconstituent algebra or the Terwilliger algebra. An irreducible T-module W is said to be *thin* whenever $\dim E_i^* W \leq 1$ for $0 \leq i \leq D$. By the *endpoint* of W we mean $\min\{i \mid E_i^* W \neq 0\}$. We show the following are equivalent: (i) Equality holds in the above inequality for $1 \leq i \leq D-1$; (ii) Equality holds in the above inequality for $i = D-1$; (iii) Every irreducible T-module with endpoint 1 is thin.

Keywords: distance-regular graph, association scheme, Terwilliger algebra, subconstituent algebra

2000 Mathematics Subject Classification: Primary 05E30; Secondary 05E35, 05C50

1. Introduction

In this paper Γ will denote a distance-regular graph with diameter $D \geq 3$, valency k, and intersection numbers a_i, b_i, c_i (see Section 2 for formal definitions). We recall the subconstituent algebra of Γ. Let X denote the vertex set of Γ and fix a "base vertex" $x \in X$. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where A denotes the adjacency matrix of Γ and E_i^* denotes the projection onto the ith subconstituent of Γ with respect to x. The algebra T is called the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [28]. Observe T has finite dimension. Moreover

T is semi-simple; the reason is each of A, E_0^*, $E_1^*, \ldots, E_D^*$ is symmetric with real entries, so T is closed under the conjugate-transpose map [16, p. 157]. Since T is semi-simple, each T-module is a direct sum of irreducible T-modules.

In this paper, we are concerned with the irreducible T-modules that possess a certain property. In order to define this property we make a few observations. Let W denote an irreducible T-module. Then W is the direct sum of the nonzero spaces among $E_0^*W, E_1^*W, \ldots, E_D^*W$. There is a second decomposition of interest. To obtain it we make a definition. Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of A, and for $0 \le i \le D$ let E_i denote the primitive idempotent of A associated with θ_i. Then W is the direct sum of the nonzero spaces among $E_0W, E_1W, \ldots, E_DW$. If the dimension of E_i^*W is at most 1 for $0 \le i \le D$ then the dimension of E_iW is at most 1 for $0 \le i \le D$ [28, Lemma 3.9]; in this case we say W is *thin*. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^*W \neq 0\}$. There exists a unique irreducible T-module with endpoint 0 [19, Proposition 8.4]. We call this module V_0. The module V_0 is thin; in fact $E_i^*V_0$ and E_iV_0 have dimension 1 for $0 \le i \le D$ [28, Lemma 3.6]. For a detailed description of V_0 see [9, 19]. In this paper, we are concerned with the thin irreducible T-modules with endpoint 1.

In order to describe the thin irreducible T-modules with endpoint 1 we define some parameters. Let $\Delta = \Delta(x)$ denote the vertex-subgraph of Γ induced on the set of vertices in X adjacent x. The graph Δ has k vertices and is regular with valency a_1. Let $\eta_1 \ge \eta_2 \ge \cdots \ge \eta_k$ denote the eigenvalues of the adjacency matrix of Δ. We call $\eta_1, \eta_2, \ldots, \eta_k$ the *local eigenvalues* of Γ with respect to x. We mentioned Δ is regular with valency a_1 so $\eta_1 = a_1$ and $\eta_k \ge -a_1$ [3, Proposition 3.1]. The eigenvalues $\eta_2, \eta_3, \ldots, \eta_k$ satisfy another bound. To give the bound we use the following notation. For any real number η other than -1 we define

$$\tilde{\eta} = -1 - b_1(1+\eta)^{-1}.$$

By [27, Theorem 1] we have $\tilde{\theta}_1 \le \eta_i \le \tilde{\theta}_D$ for $2 \le i \le k$. We remark $\tilde{\theta}_1 < -1$ and $\tilde{\theta}_D \ge 0$, since $\theta_1 > -1$ and $a_1 - k \le \theta_D < -1$ [25, Lemma 2.6]. Let W denote a thin irreducible T-module with endpoint 1. Observe E_1^*W is a 1-dimensional eigenspace for $E_1^*AE_1^*$; let η denote the corresponding eigenvalue. As we will see, η is one of $\eta_2, \eta_3, \ldots, \eta_k$. We call η the *local eigenvalue* of W. Let W' denote an irreducible T-module. Then W' and W are isomorphic as T-modules if and only if W' is thin with endpoint 1 and local eigenvalue η [31, Theorem 12.1].

Let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue η. The structure of W is described as follows [22, 31]. First assume $\eta = \tilde{\theta}_j$, where $j = 1$ or $j = D$. Then the dimension of W is $D - 1$. For $0 \le i \le D$, E_i^*W is zero if $i \in \{0, D\}$, and has dimension 1 if $i \notin \{0, D\}$. Moreover E_iW is zero if $i \in \{0, j\}$, and has dimension 1 if $i \notin \{0, j\}$. Next assume η is not one of $\tilde{\theta}_1, \tilde{\theta}_D$. Then the dimension of W is D. For $0 \le i \le D$, E_i^*W is zero if $i = 0$, and has dimension 1 if $1 \le i \le D$. Moreover E_iW is zero if $i = 0$, and has dimension 1 if $1 \le i \le D$. For a more complete description of the thin irreducible T-modules with endpoint 1 we refer the reader to [22, 31]. More general information on T and its modules can be found in [6–10, 12–14, 17–21, 24, 26, 28, 32].

In the present paper we obtain a finite sequence of inequalities involving the intersection numbers and local eigenvalues of Γ. We show equality is attained in each inequality if and only if every irreducible T-module with endpoint 1 is thin. We now state our inequalities. To do this we define some polynomials. Let λ denote an indeterminate, and let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra consisting of all polynomials in λ that have real coefficients. Let $p_0, p_1, \ldots, p_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying $p_0 = 1$ and

$$\lambda p_i = c_{i+1} p_{i+1} + (a_i - c_{i+1} + c_i) p_i + b_i p_{i-1} \quad (0 \leq i \leq D-1),$$

where $p_{-1} = 0$. One significance of these polynomials is that

$$p_i(A) = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D),$$

where $A_0, A_1, \ldots, A_D$ are the distance matrices of Γ. Let Φ denote the set of distinct scalars among $\eta_2, \eta_3, \ldots, \eta_k$. For $\eta \in \Phi$, let mult_η denote the number of times η appears among $\eta_2, \eta_3, \ldots, \eta_k$. We show

$$1 + \sum_{\eta \in \Phi} \mathrm{mult}_\eta = k, \quad 1 + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{\mathrm{mult}_\eta}{1 + \tilde{\eta}} = 0,$$

$$1 + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{\mathrm{mult}_\eta}{(1 + \tilde{\eta})^2} = \frac{k}{b_1}.$$

Our first main result is the inequality

$$1 + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{p_{i-1}(\tilde{\eta})}{p_i(\tilde{\eta})(1 + \tilde{\eta})} \mathrm{mult}_\eta \leq \frac{k}{b_i} \tag{1}$$

for $1 \leq i \leq D-1$. We remark on the terms in (1). Let η denote an element in Φ other than -1. We mentioned above that $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D$. If $\tilde{\theta}_1 \leq \eta < -1$ then $\tilde{\eta} \geq \theta_1$, and in this case $p_i(\tilde{\eta}) > 0$ for $0 \leq i \leq D-1$ [31, Lemma 4.5]. If $-1 < \eta \leq \tilde{\theta}_D$ then $\tilde{\eta} \leq \theta_D$, and in this case $(-1)^i p_i(\tilde{\eta}) > 0$ for $0 \leq i \leq D-1$ [31, Lemma 4.5]. In either case, the coefficient of mult_η in (1) is positive for $1 \leq i \leq D-1$.

Our second main result concerns the case of equality in (1). We prove the following are equivalent: (i) Equality holds in (1) for $1 \leq i \leq D-1$; (ii) Equality holds in (1) for $i = D-1$; (iii) Every irreducible T-module with endpoint 1 is thin.

Our two main results are found in Theorems 13.5 and 13.6.

2. Preliminaries concerning distance-regular graphs

In this section we review some definitions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1, 4, 23] or [28].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitean inner product $\langle\, ,\, \rangle$ defined by

$$\langle u, v\rangle = u^t \bar{v} \quad (u, v \in V), \tag{2}$$

where t denotes transpose and $-$ denotes complex conjugation. As usual, we abbreviate $\|u\|^2 = \langle u, u\rangle$ for all $u \in V$. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following formula will be useful. For all $B \in \mathrm{Mat}_X(\mathbb{C})$ and for all $u, v \in V$,

$$\langle Bu, v\rangle = \langle u, \bar{B}^t v\rangle. \tag{3}$$

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D = \max\{\partial(x, y) \mid x, y \in X\}$. We refer to D as the *diameter* of Γ. Let x, y denote vertices of Γ. We say x, y are *adjacent* whenever xy is an edge. Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever each vertex of Γ is adjacent to exactly k distinct vertices of Γ. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}| \tag{4}$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* of Γ. We abbreviate $c_i = p^i_{1i-1}$ $(1 \le i \le D)$, $a_i = p^i_{1i}$ $(0 \le i \le D)$, and $b_i = p^i_{1i+1}$ $(0 \le i \le D-1)$. For notational convenience, we define $c_0 = 0$ and $b_D = 0$. We note $a_0 = 0$ and $c_1 = 1$.

For the rest of this paper we assume Γ is distance-regular with diameter $D \ge 3$.

By (4) and the triangle inequality,

$$p^h_{1j} = 0 \quad \text{if} \quad |h - j| > 1 \quad (0 \le h, j \le D). \tag{5}$$

Observe Γ is regular with valency $k = b_0$, and that $c_i + a_i + b_i = k$ for $0 \le i \le D$. Moreover $b_i > 0$ $(0 \le i \le D-1)$ and $c_i > 0$ $(1 \le i \le D)$. For $0 \le i \le D$ we abbreviate $k_i = p^0_{ii}$, and observe

$$k_i = |\{z \in X \mid \partial(x, z) = i\}|, \tag{6}$$

where x is any vertex in X. Apparently $k_0 = 1$ and $k_1 = k$. By [1, p. 195] we have

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \le i \le D). \tag{7}$$

The following formula will be useful [4, Lemma 4.1.7]:

$$p^1_{i,i+1} = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i} \quad (0 \le i \le D-1). \tag{8}$$

We recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \ne i \end{cases} \quad (x, y \in X).$$

We call A_i the ith *distance matrix* of Γ. For notational convenience we define $A_i = 0$ for $i < 0$ and $i > D$. We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ. We observe

$$A_0 = I, \tag{9}$$

$$\sum_{i=0}^{D} A_i = J, \tag{10}$$

$$\bar{A}_i = A_i \quad (0 \le i \le D), \tag{11}$$

$$A_i^t = A_i \quad (0 \le i \le D),$$

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \quad (0 \le i, j \le D), \tag{12}$$

where I denotes the identity matrix and J denotes the all 1's matrix. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. We refer to M as the *Bose-Mesner algebra* of Γ. Using (9) and (12) one can readily show $A_0, A_1, \ldots, A_D$ form a basis for M. By [4, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that

$$E_0 = |X|^{-1} J, \tag{13}$$

$$\sum_{i=0}^{D} E_i = I, \tag{14}$$

$$\bar{E}_i = E_i \quad (0 \le i \le D), \tag{15}$$

$$E_i^t = E_i \quad (0 \le i \le D), \tag{16}$$

$$E_i E_j = \delta_{ij} E_i \quad (0 \le i, j \le D). \tag{17}$$

We refer to $E_0, E_1, \ldots, E_D$ as the *primitive idempotents* of Γ. We call E_0 the *trivial idempotent* of Γ.

We recall the eigenvalues of Γ. Since $E_0, E_1, \ldots, E_D$ form a basis for M, there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Combining this with (17) we find $AE_i = E_i A = \theta_i E_i$ for $0 \le i \le D$. Using (11), (15) we find $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe $\theta_0, \theta_1, \ldots, \theta_D$ are distinct since A generates M. By [3, Proposition 3.1], $\theta_0 = k$

and $-k \leq \theta_i \leq k$ for $0 \leq i \leq D$. Throughout this paper we assume $E_0, E_1, \ldots, E_D$ are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_D$. We refer to θ_i as the *eigenvalue* of Γ associated with E_i. We call θ_0 the *trivial eigenvalue* of Γ. For $0 \leq i \leq D$ let m_i denote the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i). By (13) we find $m_0 = 1$. Using (14)–(17) we readily find

$$V = E_0V + E_1V + \cdots + E_DV \quad \text{(orthogonal direct sum).} \tag{18}$$

For $0 \leq i \leq D$, the space E_iV is the eigenspace of A associated with θ_i. We observe the dimension of E_iV is equal to m_i.

We record a fact about the eigenvalues θ_1, θ_D.

Lemma 2.1 ([25, *Lemma* 2.6]) *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Then*
(i) $-1 < \theta_1 < k$,
(ii) $a_1 - k \leq \theta_D < -1$.

Later in this paper we will discuss polynomials in one variable. We will use the following notation. We let λ denote an indeterminate, and we let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{R}$.

3. Two families of polynomials

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. In this section we recall two families of polynomials associated with Γ. To motivate things, we recall by (5) and (12) that

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \quad (0 \leq i \leq D), \tag{19}$$

where $b_{-1} = 0$ and $c_{D+1} = 0$. Let $f_0, f_1, \ldots, f_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying $f_0 = 1$ and

$$\lambda f_i = b_{i-1}f_{i-1} + a_if_i + c_{i+1}f_{i+1} \quad (0 \leq i \leq D-1), \tag{20}$$

where $f_{-1} = 0$. Let i denote an integer ($0 \leq i \leq D$). The polynomial f_i has degree i, and the coefficient of λ^i is $(c_1c_2\cdots c_i)^{-1}$. Comparing (19) and (20) we find $f_i(A) = A_i$.

We now recall some polynomials related to the f_i. Let $p_0, p_1, \ldots, p_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying

$$p_i = f_0 + f_1 + \cdots + f_i \quad (0 \leq i \leq D). \tag{21}$$

Let i denote an integer ($0 \leq i \leq D$). The polynomial p_i has degree i, and the coefficient of λ^i is $(c_1c_2\cdots c_i)^{-1}$. Moreover $p_i(A) = A_0 + A_1 + \cdots + A_i$. Setting $i = D$ in this and using (10) we find $p_D(A) = J$.

We record several facts for later use.

Lemma 3.1 ([22, *Theorem* 3.2]) *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with diameter* $D \geq 3$. *Let the polynomials* $p_0, p_1, \ldots, p_D$ *be from* (21). *Then* $p_0 = 1$ *and*

$$\lambda p_i = c_{i+1} p_{i+1} + (a_i - c_{i+1} + c_i) p_i + b_i p_{i-1} \quad (0 \leq i \leq D - 1),$$

where $p_{-1} = 0$.

Lemma 3.2 ([31, *Lemma* 4.5]) *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with diameter* $D \geq 3$ *and eigenvalues* $\theta_0 > \theta_1 > \cdots > \theta_D$. *Let the polynomials* p_i *be from* (21). *Then the following* (i)–(iv) *hold for all* $\theta \in \mathbb{R}$.

(i) *If* $\theta > \theta_1$ *then* $p_i(\theta) > 0$ *for* $0 \leq i \leq D$.
(ii) *If* $\theta = \theta_1$ *then* $p_i(\theta) > 0$ *for* $0 \leq i \leq D - 1$ *and* $p_D(\theta) = 0$.
(iii) *If* $\theta < \theta_D$ *then* $(-1)^i p_i(\theta) > 0$ *for* $0 \leq i \leq D$.
(iv) *If* $\theta = \theta_D$ *then* $(-1)^i p_i(\theta) > 0$ *for* $0 \leq i \leq D - 1$ *and* $p_D(\theta) = 0$.

4. The subconstituent algebra and its modules

In this section we recall some definitions and basic concepts concerning the subconstituent algebra and its modules. For more information we refer the reader to [6, 9, 10, 21, 22, 24, 28].

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. We recall the dual Bose-Mesner algebra of Γ. For the rest of this section, fix a vertex $x \in X$. For $0 \leq i \leq D$ we let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X). \tag{22}$$

We call E_i^* the ith *dual idempotent of* Γ *with respect to x.* We observe

$$\sum_{i=0}^{D} E_i^* = I, \tag{23}$$

$$\bar{E}_i^* = E_i^* \qquad (0 \leq i \leq D), \tag{24}$$

$$E_i^{*t} = E_i^* \qquad (0 \leq i \leq D), \tag{25}$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq D). \tag{26}$$

Using (23), (26), we find $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra of* Γ *with respect to x.* We recall the subconstituents of Γ. From (22) we find

$$E_i^* V = \mathrm{span}\,\{\hat{y} \mid y \in X, \partial(x, y) = i\} \quad (0 \leq i \leq D). \tag{27}$$

By (27) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum).} \tag{28}$$

Combining (27) and (6) we find

$$\dim E_i^* V = k_i \quad (0 \leq i \leq D). \tag{29}$$

We call $E_i^* V$ the ith *subconstituent of* Γ *with respect to* x.

We recall how M and M^* are related. By [28, Lemma 3.2] we find

$$E_h^* A_i E_j^* = 0 \quad \text{if and only if } p_{ij}^h = 0 \quad (0 \leq h, i, j \leq D). \tag{30}$$

Combining (30) and (5) we find

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1 \quad (0 \leq i, j \leq D), \tag{31}$$

$$E_i^* A_j E_1^* = 0 \quad \text{if } |i - j| > 1 \quad (0 \leq i, j \leq D). \tag{32}$$

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra of* Γ *with respect to* x [28]. We observe T has finite dimension. Moreover T is semi-simple; the reason is T is closed under the conjugate-transpose map [16, p. 157].

We now consider the modules for T. By a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. We refer to V itself as the *standard module* for T. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that

$$(\sigma B - B\sigma)W = 0 \quad \text{for all } B \in T. \tag{33}$$

The modules W, W' are said to be *isomorphic as T-modules* whenever there exists an isomorphism of T-modules from W to W'.

Let W denote a T-module and let W' denote a T-module contained in W. Using (3) we find the orthogonal complement of W' in W is a T-module. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. We mention any two nonisomorphic irreducible T-modules are orthogonal [9, Lemma 3.3].

Let W denote an irreducible T-module. Using (23)–(26) we find W is the direct sum of the nonzero spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$. Similarly using (14)–(17) we find W is the direct sum of the nonzero spaces among $E_0 W, E_1 W, \ldots, E_D W$. If the dimension of $E_i^* W$ is at most 1 for $0 \leq i \leq D$ then the dimension of $E_i W$ is at most 1 for $0 \leq i \leq D$ [28, Lemma 3.9]; in this case we say W is *thin*. Let W denote an irreducible T-module.

By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^* W \neq 0\}$. We adopt the following notational convention.

Definition 4.1 Throughout the rest of this paper we let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \ge 3$, valency k, intersection numbers a_i, b_i, c_i, Bose-Mesner algebra M, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. For $0 \le i \le D$ we let E_i denote the primitive idempotent of Γ associated with θ_i. We let V denote the standard module for Γ. We fix $x \in X$ and abbreviate $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $M^* = M^*(x)$, $T = T(x)$. We define

$$s_i = \sum_{\substack{y \in X \\ \partial(x,y)=i}} \hat{y} \quad (0 \le i \le D). \tag{34}$$

5. The T-module V_0

With reference to Definition 4.1, there exists a unique irreducible T-module with endpoint 0 [19, Proposition 8.4]. We call this module V_0. The module V_0 is described in [9, 19]. We summarize some details below in order to motivate the results that follow.

The module V_0 is thin. In fact each of $E_i V_0$, $E_i^* V_0$ has dimension 1 for $0 \le i \le D$. We give two bases for V_0. The vectors $E_0\hat{x}, E_1\hat{x}, \ldots, E_D\hat{x}$ form a basis for V_0. These vectors are mutually orthogonal and $\|E_i\hat{x}\|^2 = m_i|X|^{-1}$ for $0 \le i \le D$. To motivate the second basis we make some comments. For $0 \le i \le D$ we have $s_i = A_i\hat{x}$, where s_i is from (34). Moreover $s_i = E_i^*\delta$, where $\delta = \sum_{y \in X} \hat{y}$. The vectors $s_0, s_1, \ldots, s_D$ form a basis for V_0. These vectors are mutually orthogonal and $\|s_i\|^2 = k_i$ for $0 \le i \le D$. With respect to the basis $s_0, s_1, \ldots, s_D$ the matrix representing A is

$$\begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{D-1} \\ \mathbf{0} & & & & c_D & a_D \end{pmatrix}. \tag{35}$$

The basis $E_0\hat{x}, E_1\hat{x}, \ldots, E_D\hat{x}$ and the basis $s_0, s_1, \ldots, s_D$ are related as follows. For $0 \le i \le D$ we have $s_i = \sum_{h=0}^{D} f_i(\theta_h)E_h\hat{x}$, where the f_i are from (20).

We define the matrix φ_0. Let $V_0^\perp$ denote the orthogonal complement of V_0 in V. Observe

$$V = V_0 + V_0^\perp \quad \text{(orthogonal direct sum)}.$$

Let φ_0 denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ such that $\varphi_0 - I$ vanishes on V_0 and such that φ_0 vanishes on $V_0^\perp$. In other words φ_0 is the orthogonal projection from V onto V_0. For all $y \in X$ we have

$$\varphi_0\hat{y} = k_i^{-1}s_i, \tag{36}$$

where $i = \partial(x, y)$. To see (36) observe $k_i^{-1}s_i$ is contained in V_0. Moreover $\hat{y} - k_i^{-1}s_i$ is orthogonal to each of $s_0, s_1, \ldots, s_D$ and hence is contained in $V_0^{\perp}$.

6. The local eigenvalues

A bit later in this paper we will consider the thin irreducible T-modules with endpoint 1. In order to discuss these we recall some parameters known as the local eigenvalues.

Definition 6.1 With reference to Definition 4.1, we let $\Delta = \Delta(x)$ denote the graph $(\breve{X}, \breve{R})$, where

$$\breve{X} = \{y \in X \mid \partial(x, y) = 1\},$$
$$\breve{R} = \{yz \mid y, z \in \breve{X},\ yz \in R\}.$$

We observe Δ is the vertex-subgraph of Γ induced on the set of vertices in X adjacent x. The graph Δ has exactly k vertices, where k is the valency of Γ. Also, Δ is regular with valency a_1. We let $\breve{A}$ denote the adjacency matrix of Δ. The matrix $\breve{A}$ is symmetric with real entries; therefore $\breve{A}$ is diagonalizable with all eigenvalues real. We let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_k$ denote the eigenvalues of $\breve{A}$. We mentioned Δ is regular with valency a_1 so $\eta_1 = a_1$ and $\eta_k \geq -a_1$ [3, Proposition 3.1]. We call $\eta_1, \eta_2, \ldots, \eta_k$ the *local eigenvalues* of Γ with respect to x.

For notational convenience we make the following definition.

Definition 6.2 With reference to Definition 4.1, we let Φ denote the set of distinct scalars among $\eta_2, \eta_3, \ldots, \eta_k$, where the η_i are from Definition 6.1. For $\eta \in \mathbb{R}$ we let mult_η denote the number of times η appears among $\eta_2, \eta_3, \ldots, \eta_k$. We observe $\text{mult}_\eta \neq 0$ if and only if $\eta \in \Phi$.

With reference to Definition 4.1, we consider the first subconstituent E_1^*V. By (29) the dimension of E_1^*V is k. Observe E_1^*V is invariant under the action of $E_1^*AE_1^*$. To illuminate this action we observe that for an appropriate ordering of the vertices of Γ,

$$E_1^*AE_1^* = \begin{pmatrix} \breve{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\breve{A}$ is from Definition 6.1. Apparently the action of $E_1^*AE_1^*$ on E_1^*V is essentially the adjacency map for Δ. In particular the action of $E_1^*AE_1^*$ on E_1^*V is diagonalizable with eigenvalues $\eta_1, \eta_2, \ldots, \eta_k$. We observe the vector s_1 from (34) is contained in E_1^*V. Using (35) we find s_1 is an eigenvector for $E_1^*AE_1^*$ with eigenvalue a_1. Let v denote a vector in E_1^*V. We observe the following are equivalent: (i) v is orthogonal to s_1; (ii) $Jv = 0$; (iii) $E_0v = 0$; (iv) $E_0^*Av = 0$; (v) $\varphi_0 v = 0$. Let U denote the orthogonal complement of s_1 in E_1^*V. We observe U has dimension $k - 1$. Using (3) we find U is invariant under $E_1^*AE_1^*$. Apparently the restriction of $E_1^*AE_1^*$ to U is diagonalizable with eigenvalues $\eta_2, \eta_3, \ldots, \eta_k$.

For $\eta \in \mathbb{R}$ let U_η denote the set consisting of those vectors in U that are eigenvectors for $E_1^* A E_1^*$ with eigenvalue η. We observe U_η is a subspace of U with dimension mult_η. We emphasize the following are equivalent: (i) $\mathrm{mult}_\eta \neq 0$; (ii) $U_\eta \neq 0$; (iii) $\eta \in \Phi$. By (3) and since $E_1^* A E_1^*$ is symmetric with real entries we find

$$U = \sum_{\eta \in \Phi} U_\eta \quad \text{(orthogonal direct sum).} \tag{37}$$

In Definition 6.1 we mentioned $\eta_1 = a_1$ and $\eta_k \geq -a_1$. We now recall some additional bounds satisfied by the local eigenvalues. To state the result we use the following notation.

Definition 6.3 With reference to Definition 4.1, for all $z \in \mathbb{R} \cup \infty$ we define

$$\tilde{z} = \begin{cases} -1 - \dfrac{b_1}{1+z}, & \text{if } z \neq -1, z \neq \infty \\ \infty, & \text{if } z = -1 \\ -1, & \text{if } z = \infty. \end{cases} \tag{38}$$

We observe $\tilde{\tilde{z}} = z$ for all $z \in \mathbb{R} \cup \infty$. By Lemma 2.1 neither of θ_1, θ_D is equal to -1, so $\tilde{\theta}_1 = -1 - b_1(1+\theta_1)^{-1}$ and $\tilde{\theta}_D = -1 - b_1(1+\theta_D)^{-1}$. By the data in Lemma 2.1 we have $\tilde{\theta}_1 < -1$ and $\tilde{\theta}_D \geq 0$.

Lemma 6.4 ([27, *Theorem* 1]) *With reference to Definitions* 4.1 *and* 6.1, *we have* $\tilde{\theta}_1 \leq \eta_i \leq \tilde{\theta}_D$ *for* $2 \leq i \leq k$.

We remark on the case of equality in the above lemma.

Lemma 6.5 ([5, *Theorem* 5.4, 22, *Theorem* 8.5]) *With reference to Definition* 4.1, *let* v *denote a nonzero vector in* U. *Then* (i)–(iii) *hold below.*

(i) *The vector* $E_0 v$ *is zero and each of* $E_2 v, E_3 v, \ldots, E_{D-1} v$ *is nonzero.*
(ii) $E_1 v = 0$ *if and only if* $v \in U_{\tilde{\theta}_1}$.
(iii) $E_D v = 0$ *if and only if* $v \in U_{\tilde{\theta}_D}$.

Corollary 6.6 *With reference to Definition* 4.1, *let* v *denote a nonzero vector in* U. *Then* (i), (ii) *hold below.*

(i) *If* $v \in U_{\tilde{\theta}_1}$ *or* $v \in U_{\tilde{\theta}_D}$ *then* Mv *has dimension* $D-1$.
(ii) *If* $v \notin U_{\tilde{\theta}_1}$ *and* $v \notin U_{\tilde{\theta}_D}$ *then* Mv *has dimension* D.

Proof: By (18) and since $E_0, E_1, \ldots, E_D$ form a basis for M, we find Mv has an orthogonal basis consisting of the nonvanishing vectors among $E_0 v, E_1 v, \ldots, E_D v$. Applying Lemma 6.5 we find that in case (i) exactly two of these vectors are zero. Similarly in case (ii) exactly one of these vectors is zero. The result follows. □

The following equations will be useful.

Lemma 6.7 *With reference to Definition* 4.1, *the following* (i)–(iii) *hold.*
(i) $1 + \sum_{\eta\in\Phi} \mathrm{mult}_\eta = k$.
(ii) $1 + \sum_{\eta\in\Phi,\eta\neq -1} \frac{\mathrm{mult}_\eta}{1+\tilde{\eta}} = 0$.
(iii) $1 + \sum_{\eta\in\Phi,\eta\neq -1} \frac{\mathrm{mult}_\eta}{(1+\tilde{\eta})^2} = \frac{k}{b_1}$.

Proof:

(i) There are $k-1$ elements in the sequence $\eta_2, \eta_3, \ldots, \eta_k$.
(ii) Each diagonal entry of $\breve{A}$ is zero so the trace of $\breve{A}$ is zero. Recall $\eta_1, \eta_2, \ldots, \eta_k$ are the eigenvalues of $\breve{A}$ so $\sum_{i=1}^{k} \eta_i = 0$. By this and since $\eta_1 = a_1$ we have $a_1 + \sum_{\eta\in\Phi} \eta\, \mathrm{mult}_\eta = 0$. In this equation, write each η in terms of $\tilde{\eta}$ using Definition 6.3 to obtain the result.
(iii) Recall Δ is regular with valency a_1, so each diagonal entry of $\breve{A}^2$ is a_1. Apparently the trace of $\breve{A}^2$ is ka_1, so $\sum_{i=1}^{k} \eta_i^2 = ka_1$. By this and since $\eta_1 = a_1$ we have $a_1^2 + \sum_{\eta\in\Phi} \eta^2\, \mathrm{mult}_\eta = ka_1$. Proceeding as in (ii) above we obtain the result. □

7. The local eigenvalue of a thin irreducible T-module with endpoint 1

In this section we make some comments concerning the thin irreducible T-modules with endpoint 1 and the local eigenvalues.

Definition 7.1 With reference to Definition 4.1, let W denote a thin irreducible T-module with endpoint 1. Observe E_1^*W is a 1-dimensional eigenspace for $E_1^*AE_1^*$; let η denote the corresponding eigenvalue. We observe E_1^*W is contained in E_1^*V and orthogonal to s_1, so $E_1^*W \subseteq U_\eta$. Apparently $U_\eta \neq 0$ so $\eta \in \Phi$. We refer to η as the *local eigenvalue* of W.

Lemma 7.2 ([31, *Theorem* 12.1]) *With reference to Definition* 4.1, *let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue η. Let W' denote an irreducible T-module. Then the following* (i), (ii) *are equivalent.*
(i) *W and W' are isomorphic as T-modules.*
(ii) *W' is thin with endpoint* 1 *and local eigenvalue η.*

Lemma 7.3 *With reference to Definition* 4.1, *for all $\eta \in \mathbb{R}$ we have*

$$U_\eta \supseteq E_1^* H_\eta, \tag{39}$$

where H_η denotes the subspace of V spanned by all the thin irreducible T-modules with endpoint 1 *and local eigenvalue η.*

Proof: Observe $E_1^*H_\eta$ is spanned by the E_1^*W, where W ranges over all thin irreducible T-modules with endpoint 1 and local eigenvalue η. For all such W the space E_1^*W is contained in U_η by Definition 7.1. The result follows. □

We remark on the dimension of the right-hand side in (39). To do this we make a definition.

Definition 7.4 With reference to Definition 4.1, and from our discussion below (33), the standard module V can be decomposed into an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity with which W appears in V*, we mean the number of irreducible T-modules in the above decomposition which are isomorphic to W. We remark that this number is independent of the decomposition.

Definition 7.5 With reference to Definition 4.1, for all $\eta \in \mathbb{R}$ we let μ_η denote the multiplicity with which W appears in the standard module V, where W denotes a thin irreducible T-module with endpoint 1 and local eigenvalue η. If no such W exists we set $\mu_\eta = 0$.

Theorem 7.6 ([31, *Theorem* 12.6]) *With reference to Definition* 4.1, *for all* $\eta \in \mathbb{R}$ *the following scalars are equal*:
(i) *The scalar* μ_η *from Definition* 7.5.
(ii) *The dimension of* $E_1^* H_\eta$, *where* H_η *is from Lemma* 7.3.
Moreover

$$mult_\eta \geq \mu_\eta. \tag{40}$$

We consider the case of equality in (39) and (40).

Theorem 7.7 ([31, *Theorem* 12.9]) *With reference to Definition* 4.1, *the following* (i)–(iii) *are equivalent.*
(i) *Equality holds in* (39) *for all* $\eta \in \mathbb{R}$.
(ii) *Equality holds in* (40) *for all* $\eta \in \mathbb{R}$.
(iii) *Every irreducible* T*-module with endpoint* 1 *is thin.*

In summary we have the following.

Corollary 7.8 *With reference to Definition* 4.1, *suppose every irreducible* T*-module with endpoint* 1 *is thin. Then for all* $\eta \in \Phi$ *there exists a thin irreducible* T*-module with endpoint* 1 *and local eigenvalue* η. *The multiplicity with which this module appears in* V *is equal to* $mult_\eta$. *Up to isomorphism there are no further irreducible* T*-modules with endpoint* 1.

8. The space Mv for $v \in E_1^* V$

With reference to Definition 4.1, let η denote a scalar in Φ and let v denote a nonzero vector in U_η. We seek a criterion which determines when v is contained in $E_1^* H_\eta$, where H_η is from Lemma 7.3. Our criterion is Corollary 12.6. In order to develop this criterion we consider the space Mv. We begin by constructing a useful orthogonal basis for Mv.

As we proceed in this section, we will encounter scalars of the form $p_i(\tilde{\eta})$ appearing in the denominator of some rational expressions. To make it clear these scalars are nonzero, we begin with the following result.

Lemma 8.1 *With reference to Definition* 4.1, *let* η *denote a real number.*

(i) *If* $\tilde{\theta}_1 < \eta < -1$ *then* $\tilde{\eta} > \theta_1$. *If* $-1 < \eta < \tilde{\theta}_D$ *then* $\tilde{\eta} < \theta_D$. *In either case* $p_i(\tilde{\eta}) \neq 0$ *for* $0 \leq i \leq D$.

(ii) *If* $\eta = \tilde{\theta}_1$ *then* $\tilde{\eta} = \theta_1$. *If* $\eta = \tilde{\theta}_D$ *then* $\tilde{\eta} = \theta_D$. *In either case* $p_i(\tilde{\eta}) \neq 0$ *for* $0 \leq i \leq D-1$ *and* $p_D(\tilde{\eta}) = 0$.

Proof: Combine Definition 6.3 and Lemma 3.2. □

Definition 8.2 With reference to Definition 4.1, let η denote a real number ($\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D$) and let v denote a vector in U_η. We define the vectors $v_0, v_1, \ldots, v_{D-1}$ as follows.

(i) Suppose $\eta \neq -1$. Then

$$v_i = \sum_{h=0}^{i} \frac{p_h(\tilde{\eta})}{p_i(\tilde{\eta})} \frac{k_i b_i}{k_h b_h} p_h(A)v \quad (0 \leq i \leq D-1). \tag{41}$$

(ii) Suppose $\eta = -1$. Then

$$v_i = p_i(A)v \quad (0 \leq i \leq D-1). \tag{42}$$

(The polynomials p_i are from (21).)

Theorem 8.3 *With reference to Definition* 4.1, *let* η *denote a scalar in* Φ *and let* v *denote a nonzero vector in* U_η. *First assume* $\eta \neq \tilde{\theta}_1$, $\eta \neq \tilde{\theta}_D$. *Then the vectors* $v_0, v_1, \ldots, v_{D-1}$ *from Definition* 8.2 *form a basis for* Mv. *Next assume* $\eta = \tilde{\theta}_1$ *or* $\eta = \tilde{\theta}_D$. *Then* $v_{D-1} = 0$ *and* $v_0, v_1, \ldots, v_{D-2}$ *form a basis for* Mv.

Proof: For $0 \leq h \leq D$ the polynomial p_h has degree exactly h. By this and Definition 8.2 we find that for $0 \leq i \leq D-1$ the vector $v_i = g_i(A)v$, where g_i is a polynomial of degree exactly i. First assume $\eta \neq \tilde{\theta}_1$, $\eta \neq \tilde{\theta}_D$. We show $v_0, v_1, \ldots, v_{D-1}$ form a basis for Mv. By Corollary 6.6(ii) we find Mv has dimension D. From this and since A generates M, we find $v, Av, A^2v, \ldots, A^{D-1}v$ form a basis for Mv. By this and our initial comment the vectors $v_0, v_1, \ldots, v_{D-1}$ form a basis for Mv. Next assume $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$. Then $v_{D-1} = 0$ by [22, Theorem 9.6]. We show $v_0, v_1, \ldots, v_{D-2}$ form a basis for Mv. By Corollary 6.6(i) the space Mv has dimension $D-1$, so $v, Av, A^2v, \ldots, A^{D-2}v$ form a basis for Mv. By this and our initial comment $v_0, v_1, \ldots, v_{D-2}$ form a basis for Mv. □

The vectors v_i from Definition 8.2 are investigated in [22, 31] and a number of results are obtained. One result we will use is the following.

Theorem 8.4 ([22, *Lemma* 10.5, 31, *Theorem* 10.7]) *With reference to Definition* 4.1, *let* η *denote a real number* $(\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D)$ *and let* v *denote a vector in* U_η. *Then the vectors* $v_0, v_1, \ldots, v_{D-1}$ *from Definition* 8.2 *are mutually orthogonal. Moreover the square-norms of these vectors are given as follows.*
(i) *Suppose* $\eta \neq -1$. *Then*

$$\|v_i\|^2 = \frac{p_{i+1}(\tilde{\eta})c_{i+1}}{p_i(\tilde{\eta})(\tilde{\eta}+1)} \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i} \|v\|^2 \quad (0 \leq i \leq D-1). \tag{43}$$

(ii) *Suppose* $\eta = -1$. *Then*

$$\|v_i\|^2 = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i} \|v\|^2 \quad (0 \leq i \leq D-1).$$

9. The space Mv, continued

With reference to Definition 4.1, let η denote an element in Φ and let v denote a vector in U_η. In the last section we used v to define some vectors v_i. In this section we consider the v_i from a different point of view.

Lemma 9.1 *With reference to Definition* 4.1, *let* v *denote a vector in* E_1^*V. *Then*

$$E_i^* A_j v = 0 \quad \textit{if } |i-j| > 1 \quad (0 \leq i, j \leq D).$$

Proof: Let i, j be given and assume $|i-j| > 1$. Observe $E_i^* A_j E_1^* = 0$ by (32) so $E_i^* A_j E_1^* v = 0$. Observe $E_1^* v = v$ so $E_i^* A_j v = 0$. □

Lemma 9.2 *With reference to Definition* 4.1, *let* v *denote a vector in* E_1^*V *which is orthogonal to* s_1. *Then* $\sum_{j=0}^{D} E_i^* A_j v = 0$ *for* $0 \leq i \leq D$.

Proof: Observe $Jv = 0$ so $E_i^* J v = 0$. Eliminate J in this expression using (10) to get the result. □

Lemma 9.3 *With reference to Definition* 4.1, *let* v *denote a vector in* E_1^*V *which is orthogonal to* s_1. *Then*

$$p_i(A)v = E_{i+1}^* A_i v - E_i^* A_{i+1} v \quad (0 \leq i \leq D-1). \tag{44}$$

Moreover $p_D(A)v = 0$.

Proof: For $0 \le i \le D-1$ we have

$$\begin{aligned} p_i(A)v &= (A_0 + A_1 + \cdots + A_i)v \\ &= (E_0^* + E_1^* + \cdots + E_D^*)(A_0 + A_1 + \cdots + A_i)v \\ &= \sum E_r^* A_s v, \end{aligned} \tag{45}$$

where the sum is over all integers r, s such that $0 \le r \le i+1$, $0 \le s \le i$, and $|r-s| \le 1$. Cancelling terms in (45) using Lemma 9.2 we obtain (44). Recall $p_D(A) = J$ and $Jv = 0$ so $p_D(A)v = 0$. □

Theorem 9.4 *With reference to Definition* 4.1, *let* η *denote a real number* $(\tilde\theta_1 \le \eta \le \tilde\theta_D)$ *and let* $v \in U_\eta$. *Then for* $0 \le i \le D-1$ *the vector* v_i *from Definition* 8.2 *is given as follows.*
(i) *Suppose* $\eta \ne -1$. *Then*

$$v_i = E_{i+1}^* A_i v - \sum_{h=1}^{i} \frac{p_h(\tilde\eta)}{p_i(\tilde\eta)} \frac{k_i b_i}{k_h b_h} \left(E_h^* A_{h+1} v - \frac{p_{h-1}(\tilde\eta) b_h}{p_h(\tilde\eta) c_h} E_h^* A_{h-1} v \right).$$

(ii) *Suppose* $\eta = -1$. *Then*

$$v_i = E_{i+1}^* A_i v - E_i^* A_{i+1} v.$$

Proof:
(i) On the right-hand side in (41), first eliminate $p_h(A)v$ $(0 \le h \le i)$ using Lemma 9.3, then rearrange terms, and simplify the result using $E_0^* A v = 0$.
(ii) Combine (42) and (44). □

Theorem 9.5 *With reference to Definition* 4.1, *let* η *denote a real number* $(\tilde\theta_1 \le \eta \le \tilde\theta_D)$ *and let* $v \in U_\eta$. *Let* $v_0, v_1, \ldots, v_{D-1}$ *denote the corresponding vectors from Definition* 8.2.
(i) *Suppose* $\eta \ne -1$. *Then for* $0 \le i \le D-1$,

$$\|v_i\|^2 = \|E_{i+1}^* A_i v\|^2 + \sum_{h=1}^{i} \frac{p_h^2(\tilde\eta)}{p_i^2(\tilde\eta)} \frac{k_i^2 b_i^2}{k_h^2 b_h^2} \left\| E_h^* A_{h+1} v - \frac{p_{h-1}(\tilde\eta) b_h}{p_h(\tilde\eta) c_h} E_h^* A_{h-1} v \right\|^2.$$

(ii) *Suppose* $\eta = -1$. *Then for* $0 \le i \le D-1$,

$$\|v_i\|^2 = \|E_{i+1}^* A_i v\|^2 + \|E_i^* A_{i+1} v\|^2.$$

Proof: Take the square-norm in Theorem 9.4 and use (28). □

In the next two sections we analyze the formulae in Theorem 9.5. In Section 10 we consider these formulae when $\eta = \tilde\theta_1$ or $\eta = \tilde\theta_D$. In Section 11 we consider these formulae when $\tilde\theta_1 < \eta < \tilde\theta_D$.

10. The local eigenvalues $\tilde{\theta}_1, \tilde{\theta}_D$

In this section we consider the implications of Theorem 9.5 in the case $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$. We acknowledge all the results in the present section appear in [22]. Nevertheless, we will give short proofs to indicate how things follow from Theorem 9.5.

Theorem 10.1 ([22, *Lemma* 9.4, *Theorem* 9.6, *Theorem* 10.5]) *With reference to Definition* 4.1, *let $j = 1$ or $j = D$ and define $\eta = \tilde{\theta}_j$. Then for all $v \in U_\eta$,*

$$E_h^* A_{h+1} v = \frac{p_{h-1}(\theta_j) b_h}{p_h(\theta_j) c_h} E_h^* A_{h-1} v \quad (1 \leq h \leq D-1), \tag{46}$$

$$E_{i+1}^* A_i v = \sum_{h=0}^{i} \frac{p_h(\theta_j)}{p_i(\theta_j)} \frac{k_i b_i}{k_h b_h} p_h(A) v \quad (0 \leq i \leq D-1), \tag{47}$$

$$\| E_{i+1}^* A_i v \|^2 = \frac{p_{i+1}(\theta_j) c_{i+1}}{p_i(\theta_j)(\theta_j + 1)} \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i} \|v\|^2 \quad (0 \leq i \leq D-1). \tag{48}$$

Moreover both sides of (47) *and* (48) *are 0 for $i = D-1$.*

Proof: We assume $v \neq 0$; otherwise the result is trivial. Now $U_\eta \neq 0$ so $\eta \in \Phi$. We assume $\eta = \tilde{\theta}_j$ so $\eta \neq -1$ by Definition 6.3. Let the vectors $v_0, v_1, \ldots, v_{D-1}$ be as in (41), with $\eta = \tilde{\theta}_j$. Consider the equation in Theorem 9.5(i) for $i = D-1$. Recall $v_{D-1} = 0$ by Theorem 8.3, so both sides are 0. On the right-hand side each term is nonnegative so each term is 0. By Lemma 8.1 we find $p_h(\tilde{\eta}) \neq 0$ for $1 \leq h \leq D-1$. By (7) we have $k_{D-1} b_{D-1} \neq 0$. From these comments we obtain

$$\left\| E_h^* A_{h+1} v - \frac{p_{h-1}(\theta_j) b_h}{p_h(\theta_j) c_h} E_h^* A_{h-1} v \right\|^2 = 0 \quad (1 \leq h \leq D-1)$$

and (46) follows. Evaluating the equation in Theorem 9.4(i) using (41) and (46) we obtain (47). Evaluating the equation in Theorem 9.5(i) using (43) and (46) we obtain (48). We mentioned above that $v_{D-1} = 0$, and the last assertion of the theorem follows from this. □

Theorem 10.2 ([22, *Theorem* 9.8]) *With reference to Definition* 4.1, *let $j = 1$ or $j = D$ and define $\eta = \tilde{\theta}_j$. Then for all nonzero $v \in U_\eta$, the space Mv is a thin irreducible T-module with endpoint* 1 *and local eigenvalue η.*

Proof: We first show Mv is a T-module. It is clear Mv is closed under M. We show Mv is closed under M^*. We assume $\eta = \tilde{\theta}_j$ so $\eta \neq -1$ by Definition 6.3. Let the vectors v_i be as in (41), where $\eta = \tilde{\theta}_j$. By Theorem 8.3 the vectors $v_0, v_1, \ldots, v_{D-2}$ form a basis for Mv. From (47) we find

$$v_i = E_{i+1}^* A_i v \quad (0 \leq i \leq D-2). \tag{49}$$

Combining this with (26) we find Mv is closed under M^*. Recall M and M^* generate T so Mv is a T-module. We show Mv is irreducible. From (49) and since the v_i form a basis for Mv, we find v is a basis for E_1^*Mv. In particular E_1^*Mv has dimension 1. Since Mv is a T-module it is a direct sum of irreducible T-modules. It follows there exists an irreducible T-module W' such that $W' \subseteq Mv$ and such that $E_1^*W' \neq 0$. We show $W' = Mv$. Observe $E_1^*W' \subseteq E_1^*Mv$, and we mentioned E_1^*Mv has dimension 1, so $E_1^*W' = E_1^*Mv$. Now apparently $v \in E_1^*W'$. Observe W' is M-invariant, so $Mv \subseteq W'$, and it follows $W' = Mv$. In particular Mv is irreducible. We mentioned the vectors (49) form a basis for Mv. It follows E_i^*Mv is 0 for $i \in \{0, D\}$ and has dimension 1 for $1 \leq i \leq D-1$. Apparently Mv is thin with endpoint 1. We mentioned v is a basis for E_1^*Mv. From the construction v is an eigenvector for $E_1^*AE_1^*$ with eigenvalue η. It follows Mv has local eigenvalue η. □

Corollary 10.3 ([22, *Lemma* 11.2]) *With reference to Definition* 4.1, *let* $j = 1$ *or* $j = D$ *and define* $\eta = \tilde{\theta}_j$. *Then* $U_\eta = E_1^*H_\eta$, *where* H_η *is from Lemma* 7.3.

Proof: The inclusion $U_\eta \supseteq E_1^*H_\eta$ is from Lemma 7.3. We now show $U_\eta \subseteq E_1^*H_\eta$. We assume $U_\eta \neq 0$; otherwise the result is trivial. Let v denote a nonzero vector in U_η. We show $v \in E_1^*H_\eta$. By Theorem 10.2 we find Mv is a thin irreducible T-module with endpoint 1 and local eigenvalue η, so $Mv \subseteq H_\eta$. Of course $v \in Mv$ so $v \in H_\eta$. From the construction $v \in E_1^*V$ so $v = E_1^*v$. It follows $v \in E_1^*H_\eta$. We have now shown $U_\eta \subseteq E_1^*H_\eta$ and the result follows. □

11. The local eigenvalues η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_D$)

In this section we consider the implications of Theorem 9.5 for the case $\tilde{\theta}_1 < \eta < \tilde{\theta}_D$.

Lemma 11.1 *With reference to Definition* 4.1, *let* η *denote a real number* ($\tilde{\theta}_1 < \eta < \tilde{\theta}_D$) *and let* $v \in U_\eta$.
(i) *Suppose* $\eta \neq -1$. *Then for* $0 \leq i \leq D-1$,

$$\|E_{i+1}^*A_iv\|^2 \leq \frac{p_{i+1}(\tilde{\eta})c_{i+1}}{p_i(\tilde{\eta})(\tilde{\eta}+1)}\,\frac{b_1b_2\cdots b_i}{c_1c_2\cdots c_i}\,\|v\|^2. \tag{50}$$

(ii) *Suppose* $\eta = -1$. *Then for* $0 \leq i \leq D-1$,

$$\|E_{i+1}^*A_iv\|^2 \leq \frac{b_1b_2\cdots b_i}{c_1c_2\cdots c_i}\,\|v\|^2. \tag{51}$$

Proof: Combine Theorems 8.4 and 9.5. □

We now consider the case of equality in (50) or (51).

Lemma 11.2 *With reference to Definition* 4.1, *let* η *denote a real number* $(\tilde{\theta}_1 < \eta < \tilde{\theta}_D)$ *and let* $v \in U_\eta$. *First assume* $\eta \neq -1$. *Then for* $0 \leq i \leq D-1$ *the following* (i)–(iii) *are equivalent.*

(i) *Equality holds in* (50).

(ii) $E_h^* A_{h+1} v = \dfrac{p_{h-1}(\tilde{\eta}) b_h}{p_h(\tilde{\eta}) c_h} E_h^* A_{h-1} v \quad (1 \leq h \leq i).$

(iii) $E_{i+1}^* A_i v = \displaystyle\sum_{h=0}^{i} \frac{p_h(\tilde{\eta})}{p_i(\tilde{\eta})} \frac{k_i b_i}{k_h b_h} p_h(A) v.$

Next assume $\eta = -1$. *Then for* $0 \leq i \leq D-1$ *the following* (i′)–(iii′) *are equivalent.*

(i′) *Equality holds in* (51).

(ii′) $E_i^* A_{i+1} v = 0.$

(iii′) $E_{i+1}^* A_i v = p_i(A) v.$

Proof: First assume $\eta \neq -1$.

(i)⇒(ii) Let the vector v_i be as in (41). We assume equality in (50); combining this with (43) we find $\|v_i\|^2 = \|E_{i+1}^* A_i v\|^2$. Therefore in the equation of Theorem 9.5(i), in the sum on the right each term is 0. We examine these terms. By Lemma 8.1 we find $p_h(\tilde{\eta}) \neq 0$ for $1 \leq h \leq i$. By (7) we have $k_i b_i \neq 0$. From these comments we obtain

$$\left\| E_h^* A_{h+1} v - \frac{p_{h-1}(\tilde{\eta}) b_h}{p_h(\tilde{\eta}) c_h} E_h^* A_{h-1} v \right\|^2 = 0 \quad (1 \leq h \leq i)$$

and the result follows.

(ii)⇒(iii) Evaluate the equation of Theorem 9.4(i) using (41).

(iii)⇒(i). Let the vector v_i be as in (41). We assume $E_{i+1}^* A_i v = v_i$ so $\|E_{i+1}^* A_i v\|^2 = \|v_i\|^2$. Combining this with (43) we obtain equality in (50).

The proof for the case $\eta = -1$ is similar, and omitted. □

12. Stable vectors

In Lemma 11.1 we obtained a sequence of inequalities. In this section we consider the case when equality is attained in each of the inequalities. In order to treat this case we make a definition.

Definition 12.1 With reference to Definition 4.1, let η denote a real number $(\tilde{\theta}_1 < \eta < \tilde{\theta}_D)$ and let $v \in U_\eta$. We define what it means for v to be *stable*. First assume $\eta \neq -1$. Then we say v is *stable* whenever

$$E_i^* A_{i+1} v = \frac{p_{i-1}(\tilde{\eta}) b_i}{p_i(\tilde{\eta}) c_i} E_i^* A_{i-1} v \quad (1 \leq i \leq D-1). \tag{52}$$

Next assume $\eta = -1$. Then we say v is *stable* whenever

$$E_i^* A_{i+1} v = 0 \quad (1 \leq i \leq D-1). \tag{53}$$

Lemma 12.2 *With reference to Definition* 4.1, *let* η *denote a real number* $(\tilde{\theta}_1 < \eta < \tilde{\theta}_D)$. *Then the set of stable vectors in* U_η *is a subspace of* U_η.

Proof: Let v denote a stable vector in U_η and let α denote a complex scalar. Then αv is a stable vector in U_η. Let v' denote a stable vector in U_η. Then $v + v'$ is a stable vector in U_η. □

Lemma 12.3 *With reference to Definition* 4.1, *let* η *denote a real number* $(\tilde{\theta}_1 < \eta < \tilde{\theta}_D)$ *and let* $v \in U_\eta$. *If* $\eta \neq -1$ *then the following* (i)–(iii) *are equivalent.*
(i) *Equality holds in* (50) *for* $0 \leq i \leq D-1$.
(ii) *Equality holds in* (50) *for* $i = D-1$.
(iii) v *is stable.*
If $\eta = -1$ *then the following* (i′)–(iii′) *are equivalent.*
(i′) *Equality holds in* (51) *for* $0 \leq i \leq D-1$.
(ii′) *Equality holds in* (51) *for* $i = D-1$.
(iii′) v *is stable.*

Proof: First assume $\eta \neq -1$.

(i) ⇒ (ii) Clear.
(ii) ⇒ (iii) Observe Lemma 11.2(i) holds at $i = D-1$ so Lemma 11.2(ii) holds at $i = D-1$. Therefore (52) holds so v is stable by Definition 2.1.
(iii) ⇒ (i) The vector v is stable so (52) holds. Apparently Lemma 11.2(ii) holds for $0 \leq i \leq D-1$ so equality holds in (50) for $0 \leq i \leq D-1$.

Now assume $\eta = -1$.

(i′) ⇒ (ii′) Clear.
(ii′) ⇒ (iii′) Applying Lemma 11.2(i′),(ii′) at $i = D-1$, we find $E_{D-1}^* A_D v = 0$. We claim

$$E_i^* A_{i+1} v = 0 \quad (0 \leq i \leq D-1). \tag{54}$$

To obtain (54) we use the equation

$$E_{i-1}^* A E_i^* A_{i+1} E_1^* = b_i E_{i-1}^* A_i E_1^* \quad (1 \leq i \leq D-1). \tag{55}$$

To verify (55), observe corresponding entries agree. Indeed for all $y, z \in X$, on either side of (55) the yz entry is equal to b_i if $\partial(x, y) = i-1$, $\partial(x, z) = 1$, $\partial(y, z) = i$, and 0 otherwise. We now have (55). Applying (55) to v we find

$$E_{i-1}^* A E_i^* A_{i+1} v = b_i E_{i-1}^* A_i v \quad (1 \leq i \leq D-1).$$

Apparently

$$E_i^* A_{i+1} v = 0 \to E_{i-1}^* A_i v = 0 \quad (1 \le i \le D-1)$$

and (54) follows. Now (53) holds so v is stable by Definition 12.1.
(iii′) ⇒(i′) We show equality holds in (51) for $0 \le i \le D-1$. Let i be given. We assume $i \ge 1$; otherwise the result is trivial. Observe $E_i^* A_{i+1} v = 0$ by (53) and since v is stable. Applying Lemma 11.2 (i′), (ii′) we find equality holds in (51). □

In the next two lemmas we present several more characterizations of the stable condition.

Lemma 12.4 *With reference to Definition* 4.1, *let η denote a real number* $(\tilde{\theta}_1 < \eta < \tilde{\theta}_D)$ *and let $v \in U_\eta$. If $\eta \ne -1$ then the following (i)–(iii) are equivalent.*

(i) $E_{i+1}^* A_i v = \sum_{h=0}^{i} \frac{p_h(\tilde{\eta})}{p_i(\tilde{\eta})} \frac{k_i b_i}{k_h b_h} p_h(A) v \quad (0 \le i \le D-1).$

(ii) $E_D^* A_{D-1} v = \sum_{h=0}^{D-1} \frac{p_h(\tilde{\eta})}{p_{D-1}(\tilde{\eta})} \frac{k_{D-1} b_{D-1}}{k_h b_h} p_h(A) v.$

(iii) *v is stable.*

If $\eta = -1$ then the following (i′)–(iii′) *are equivalent.*

(i′) $E_{i+1}^* A_i v = p_i(A) v \quad (0 \le i \le D-1).$

(ii′) $E_D^* A_{D-1} v = p_{D-1}(A) v.$

(iii′) *v is stable.*

Proof: First assume $\eta \ne -1$.

(i) ⇒ (ii) Clear.
(ii) ⇒ (iii) Observe Lemma 11.2(iii) holds at $i = D-1$ so Lemma 11.2(ii) holds at $i = D-1$. Now (52) holds so v is stable by Definition 12.1.
(iii) ⇒ (i) The vector v is stable so (52) holds. Apparently Lemma 11.2(ii) holds for $0 \le i \le D-1$ so Lemma 11.2(iii) holds for $0 \le i \le D-1$. The result follows.

Now assume $\eta = -1$.

(i′) ⇒ (ii′) Clear.
(ii′) ⇒ (iii′) Observe Lemma 11.2(iii′) holds at $i = D-1$ so equality holds in (51) at $i = D-1$. Applying Lemma 12.3(ii′), (iii′) we find v is stable.
(iii′) ⇒ (i′) We show $E_{i+1}^* A_i v = p_i(A) v$ for $0 \le i \le D-1$. Let the integer i be given. We assume $i \ge 1$; otherwise the result is trivial. Observe $E_i^* A_{i+1} v = 0$ by (53) and since v is stable. Applying Lemma 11.2(ii′), (iii′) we find $E_{i+1}^* A_i v = p_i(A) v$. □

Theorem 12.5 *With reference to Definition* 4.1, *let η denote a scalar in Φ $(\eta \ne \tilde{\theta}_1$, $\eta \ne \tilde{\theta}_D)$. Then for all nonzero $v \in U_\eta$ the following* (i), (ii) *are equivalent.*

(i) *v is stable.*

(ii) *v is contained in a thin irreducible T-module with endpoint* 1 *and local eigenvalue η.*

Proof: (i) $\Rightarrow$ (ii): We show Mv is a thin irreducible T-module with endpoint 1 and local eigenvalue η. We first show Mv is a T-module. It is clear Mv is closed under M. We show Mv is closed under M^*. Let the vectors $v_0, v_1, \ldots, v_{D-1}$ be as in Definition 8.2. By Theorem 8.3 we find $v_0, v_1, \ldots, v_{D-1}$ is a basis for Mv. We assume v is stable, so by Lemma 12.4 we find

$$v_i = E^*_{i+1} A_i v \quad (0 \le i \le D-1). \tag{56}$$

Combining (56) and (26) we see Mv is closed under M^*. Recall M and M^* generate T so Mv is a T-module. We show Mv is irreducible. By (56) and since the v_i form a basis for Mv, we see v is a basis for $E^*_1 Mv$. In particular $E^*_1 Mv$ has dimension 1. Since Mv is a T-module it is a direct sum of irreducible T-modules. It follows there exists an irreducible T-module W' such that $W' \subseteq Mv$ and such that $E^*_1 W' \neq 0$. We show $W' = Mv$. Observe $E^*_1 W' \subseteq E^*_1 Mv$, and we mentioned $E^*_1 Mv$ has dimension 1, so $E^*_1 W' = E^*_1 Mv$. Now apparently $v \in E^*_1 W'$. Observe W' is M-invariant, so $Mv \subseteq W'$, and it follows $W' = Mv$. In particular Mv is irreducible. We mentioned the vectors in (56) form a basis for Mv. It follows $E^*_i Mv$ is 0 for $i = 0$ and has dimension 1 for $1 \le i \le D$. Apparently Mv is thin with endpoint 1. We mentioned v is a basis for $E^*_1 Mv$. From the construction v is an eigenvector for $E^*_1 A E^*_1$ with eigenvalue η. It follows Mv has local eigenvalue η.

(ii) $\Rightarrow$ (i): To show v is stable we apply Lemma 12.4. We show v satisfies Lemma 12.4(i) if $\eta \neq -1$ and Lemma 12.4(i′) if $\eta = -1$. To do this, let the vectors $v_0, v_1, \ldots, v_{D-1}$ be as in Definition 8.2. We show $E^*_{i+1} A_i v = v_i$ for $0 \le i \le D-1$. By assumption v is contained in a thin irreducible T-module with endpoint 1 and local eigenvalue η. We denote this module by W. By (31) we find $A^i v$ is contained in $E^*_1 W + \cdots + E^*_{i+1} W$ for $0 \le i \le D-1$. Also for $0 \le i \le D-1$, v_i is a linear combination of $v, Av, \ldots, A^i v$, so v_i is contained in $E^*_1 W + \cdots + E^*_{i+1} W$. By this and since $v_0, v_1, \ldots, v_{D-1}$ are linearly independent, we find

$$v_0, v_1, \ldots, v_i \text{ is a basis for } E^*_1 W + E^*_2 W + \cdots + E^*_{i+1} W \quad (0 \le i \le D-1). \tag{57}$$

For the rest of this proof, fix an integer $i (0 \le i \le D-1)$. We show v_i is contained in $E^*_{i+1} W$. To see this, recall $E^*_1 W, \ldots, E^*_D W$ are mutually orthogonal. Therefore $E^*_{i+1} W$ is the orthogonal complement of $E^*_1 W + \cdots + E^*_i W$ in $E^*_1 W + \cdots + E^*_{i+1} W$. Recall v_i is orthogonal to each of $v_0, v_1, \ldots, v_{i-1}$. By (57) the vectors $v_0, v_1, \ldots, v_{i-1}$ form a basis for $E^*_1 W + \cdots + E^*_i W$, so v_i is orthogonal to $E^*_1 W + \cdots + E^*_i W$. Apparently v_i is contained in $E^*_{i+1} W$, as desired. We show $E^*_{i+1} A_i v = v_i$. We mentioned the vector v_i is a linear combination of $v, Av, \ldots, A^i v$. In this combination the coefficient of $A^i v$ is $(c_1 c_2 \cdots c_i)^{-1}$ in view of our comments below (21). Similarly $A_i v$ is a linear combination of $v, Av, \ldots, A^i v$, and in this combination the coefficient of $A^i v$ is $(c_1 c_2 \cdots c_i)^{-1}$. Apparently $A_i v - v_i$ is a linear combination of $v, Av, \ldots, A^{i-1} v$. From this and our above comments $A_i v - v_i$ is contained in $E^*_1 W + \cdots + E^*_i W$, so $E^*_{i+1}(A_i v - v_i) = 0$. We already showed $v_i \in E^*_{i+1} W$ so $E^*_{i+1} v_i = v_i$. Now apparently $E^*_{i+1} A_i v = v_i$. We have now shown v satisfies Lemma 12.4(i) if $\eta \neq -1$ and Lemma 12.4(i′) if $\eta = -1$. Applying that lemma we find v is stable. $\square$

Corollary 12.6 *With reference to Definition* 4.1, *let* η *denote a real number* ($\tilde{\theta}_1 < \eta < \tilde{\theta}_D$). *Then the set of stable vectors in* U_η *is equal to* $E_1^* H_\eta$, *where* H_η *is from Lemma* 7.3.

Proof: Assume $\eta \in \Phi$; otherwise U_η and H_η are both 0. Let U'_η denote the set of stable vectors in U_η. We show $U'_\eta = E_1^* H_\eta$. We first show $U'_\eta \subseteq E_1^* H_\eta$. Assume $U'_\eta \neq 0$; otherwise the result is trivial. Let v denote a nonzero vector in U'_η. We show $v \in E_1^* H_\eta$. By Theorem 12.5, v is contained in a thin irreducible T-module with endpoint 1 and local eigenvalue η. This module is contained in H_η so $v \in H_\eta$. By construction $v \in E_1^* V$ so $v = E_1^* v$. It follows $v \in E_1^* H_\eta$. We have now shown $U'_\eta \subseteq E_1^* H_\eta$. We now show $U'_\eta \supseteq E_1^* H_\eta$. Observe $E_1^* H_\eta$ is spanned by the $E_1^* W$, where W ranges over all thin irreducible T-modules with endpoint 1 and local eigenvalue η. For all such W the space $E_1^* W$ is contained in U'_η by Theorem 12.5. It follows $U'_\eta \supseteq E_1^* H_\eta$. We have now shown $U'_\eta = E_1^* H_\eta$. $\square$

13. The main results

In this section we prove the main results of the paper, which are Theorems 13.5 and 13.6. To prepare for these results we make a definition.

With reference to Definition 4.1, let η denote an element of Φ and observe

$$V = U_\eta + U_\eta^\perp \quad \text{(orthogonal direct sum)},$$

where $U_\eta^\perp$ denotes the orthogonal complement of U_η in V. Let F_η denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ such that $F_\eta - I$ vanishes on U_η and such that F_η vanishes on $U_\eta^\perp$. In other words F_η is the orthogonal projection from V onto U_η. We make a few observations about F_η. We claim $F_\eta^t = \bar{F}_\eta$. To see this observe $\langle F_\eta u, v\rangle = \langle F_\eta u, F_\eta v\rangle$ for all $u, v \in V$. From this and (2) we find $F_\eta^t = F_\eta^t \bar{F}_\eta$. Taking the conjugate-transpose of this we find $\bar{F}_\eta = F_\eta^t \bar{F}_\eta$ so $F_\eta^t = \bar{F}_\eta$. Let y denote a vertex in X such that $\partial(x, y) \neq 1$. We claim $F_\eta \hat{y} = 0$. To see this, observe $\hat{y}$ is orthogonal to $E_1^* V$ and $U_\eta \subseteq E_1^* V$ so $\hat{y} \in U_\eta^\perp$. Now $F_\eta \hat{y} = 0$ by the definition of F_η.

Lemma 13.1 *With reference to Definition* 4.1, *for* $0 \leq i \leq D - 1$ *we have*

$$0 = \sum_{\substack{y \in X \\ \partial(x,y)=1}} \left(\|E_{i+1}^* A_i \hat{y}\|^2 - \|E_{i+1}^* A_i \varphi_0 \hat{y}\|^2 - \sum_{\eta \in \Phi} \|E_{i+1}^* A_i F_\eta \hat{y}\|^2 \right), \tag{58}$$

where φ_0 *is from* (36).

Proof: In view of (37),

$$E_1^* = \varphi_0 E_1^* + \sum_{\eta \in \Phi} F_\eta. \tag{59}$$

Applying $E^*_{i+1}A_i$ to each term in (59) we obtain

$$E^*_{i+1}A_iE^*_1 = E^*_{i+1}A_i\varphi_0E^*_1 + \sum_{\eta\in\Phi} E^*_{i+1}A_iF_\eta. \tag{60}$$

For all $Y, Z \in \mathrm{Mat}_X(\mathbb{C})$ let (Y, Z) denote the trace of $Y\bar{Z}^t$. We observe $(\,,)$ is a positive definite Hermitean form on $\mathrm{Mat}_X(\mathbb{C})$. We claim that in (60) the terms to the right of the equals sign are mutually orthogonal with respect to $(\,,)$. To see this, observe that for $\eta \in \Phi$ we have $\varphi_0E^*_1F_\eta = 0$. It follows $E^*_{i+1}A_i\varphi_0E^*_1$ and $E^*_{i+1}A_iF_\eta$ are orthogonal with respect to $(\,,)$. Further observe that for distinct $\eta, \eta' \in \Phi$ we have $F_\eta F_{\eta'} = 0$ in view of (37). It follows $E^*_{i+1}A_iF_\eta$ and $E^*_{i+1}A_iF_{\eta'}$ are orthogonal with respect to $(\,,)$. We have now shown that in (60), the terms to the right of the equals sign are mutually orthogonal with respect to $(\,,)$. It follows

$$\|E^*_{i+1}A_iE^*_1\|^2 = \|E^*_{i+1}A_i\varphi_0E^*_1\|^2 + \sum_{\eta\in\Phi} \|E^*_{i+1}A_iF_\eta\|^2, \tag{61}$$

where we abbreviate $\|Z\|^2 = (Z, Z)$ for all $Z \in \mathrm{Mat}_X(\mathbb{C})$. To obtain (58) from (61) we observe $\|Z\|^2 = \sum_{y\in X} \|Z\hat{y}\|^2$ for all $Z \in \mathrm{Mat}_X(\mathbb{C})$. Evaluating (61) using this we routinely obtain (58). □

In the next few lemmas we evaluate the terms in (58).

Lemma 13.2 *With reference to Definition* 4.1, *let y denote a vertex in X such that* $\partial(x, y) = 1$. *Then*

$$\|E^*_{i+1}A_i\hat{y}\|^2 = \frac{b_1b_2\cdots b_i}{c_1c_2\cdots c_i} \quad (0 \le i \le D-1). \tag{62}$$

Proof: Observe $E^*_{i+1}A_i\hat{y} = \sum \hat{z}$, where the sum is over all vertices $z \in X$ such that $\partial(x, z) = i + 1$ and $\partial(y, z) = i$. There are $p^1_{i,i+1}$ such vertices, so (62) follows in view of (8). □

Lemma 13.3 *With reference to Definition* 4.1, *let y denote a vertex in X such that* $\partial(x, y) = 1$. *Then for* $0 \le i \le D-1$ *we have*

$$E^*_{i+1}A_i\varphi_0\hat{y} = c_{i+1}k^{-1}s_{i+1}, \tag{63}$$

where s_{i+1} is from (34) *and φ_0 is from* (36). *Moreover*

$$\|E^*_{i+1}A_i\varphi_0\hat{y}\|^2 = \frac{b_1b_2\cdots b_i}{c_1c_2\cdots c_i}\,\frac{c_{i+1}}{k}. \tag{64}$$

Proof: To get (63) observe $\varphi_0 \hat{y} = k^{-1} s_1$ by (36). Also $s_1 = A\hat{x}$, so $\varphi_0 \hat{y} = k^{-1} A\hat{x}$. Evaluating the left-hand side of (63) using this, and simplifying the result using (19), we routinely obtain the right-hand side of (63). We now have (63). To get (64) take the square norm in (63) and evaluate the result using $\|s_j\|^2 = k_j$ and (7). □

Referring to (58), so far we have evaluated the terms on the left and in the middle. To evaluate the terms on the right we will use (48), Lemma 11.1, and the following result.

Lemma 13.4 *With reference to Definition* 4.1, *for all* $\eta \in \Phi$ *we have*

$$\sum_{\substack{y \in X \\ \partial(x,y)=1}} \|F_\eta \hat{y}\|^2 = \text{mult}_\eta. \tag{65}$$

Proof: We show both sides of (65) equal the trace of F_η. We start with the left-hand side. Let y denote any vertex in X. Observe the yy entry of $F_\eta^t \bar{F}_\eta$ is equal to $\|F_\eta \hat{y}\|^2$. We mentioned earlier $F_\eta^t = F_\eta^t \bar{F}_\eta$. Combining these facts we find the yy entry of F_η is equal to $\|F_\eta \hat{y}\|^2$. Recall $F_\eta \hat{y} = 0$ if $\partial(x, y) \neq 1$. From these comments we find the left-hand side of (65) is equal to the trace of F_η. We show the right-hand side of (65) is equal to the trace of F_η. Recall F_η is the orthogonal projection from V onto U_η. Therefore U_η is an eigenspace for F_η with eigenvalue 1 and $U_\eta^\perp$ is an eigenspace for F_η with eigenvalue 0. Since the trace of F_η is equal to the sum of its eigenvalues, we see the trace of F_η is equal to the dimension of U_η. This dimension is just mult_η. We have now shown the right-hand side of (65) is equal to the trace of F_η. The result follows. □

Evaluating the terms in (58) using our above comments we obtain the following theorem.

Theorem 13.5 *With reference to Definition* 4.1, *for* $1 \leq i \leq D-1$ *we have*

$$1 + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{p_{i-1}(\tilde{\eta})}{p_i(\tilde{\eta})(1+\tilde{\eta})} \text{mult}_\eta \leq \frac{k}{b_i}. \tag{66}$$

(*The polynomials* p_i *are from Lemma* 3.1 *and the scalars* $\tilde{\eta}$ *are from Definition* 6.3. *The set* Φ *and the scalars* mult_η *are from Definition* 6.2.)

Proof: Let the integer i be given. We evaluate the terms in (58). The terms on the left are found in Lemma 13.2. The terms in the middle are found in (64). We now consider the terms on the right. Let y denote a vertex in X such that $\partial(x, y) = 1$, and pick any $\eta \in \Phi$. We evaluate $\|E_{i+1}^* A_i F_\eta \hat{y}\|^2$. First suppose $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$. Then $\|E_{i+1}^* A_i F_\eta \hat{y}\|^2$ is obtained by setting $v = F_\eta \hat{y}$ in (48). Next suppose $\eta \neq \tilde{\theta}_1$ and $\eta \neq \tilde{\theta}_D$. In this case we obtain an upper bound for $\|E_{i+1}^* A_i F_\eta \hat{y}\|^2$ by setting $v = F_\eta \hat{y}$ in Lemma 11.1. Evaluating (58) using these comments, and simplifying the result using Lemma 13.4, we find

$$k \leq c_{i+1} + \text{mult}_{-1} + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{p_{i+1}(\tilde{\eta}) c_{i+1}}{p_i(\tilde{\eta})(1+\tilde{\eta})} \text{mult}_\eta. \tag{67}$$

We simplify (67) a bit. Let $\eta \in \Phi$ and assume $\eta \neq -1$. Setting $\lambda = \tilde{\eta}$ in Lemma 3.1 we find

$$\tilde{\eta} p_i(\tilde{\eta}) = c_{i+1} p_{i+1}(\tilde{\eta}) + (a_i - c_{i+1} + c_i) p_i(\tilde{\eta}) + b_i p_{i-1}(\tilde{\eta}). \tag{68}$$

Dividing (68) by $p_i(\tilde{\eta})$ we find

$$\frac{p_{i+1}(\tilde{\eta}) c_{i+1}}{p_i(\tilde{\eta})} = \tilde{\eta} - a_i + c_{i+1} - c_i - \frac{p_{i-1}(\tilde{\eta}) b_i}{p_i(\tilde{\eta})}. \tag{69}$$

Evaluating the right-hand side of (67) using (69), and simplifying the result using Lemma 6.7 we routinely obtain (66). □

We now consider the case of equality in (66).

Theorem 13.6 *With reference to Definition* 4.1, *the following* (i)–(iii) *are equivalent.*
(i) *Equality holds in* (66) *for* $1 \leq i \leq D - 1$.
(ii) *Equality holds in* (66) *for* $i = D - 1$.
(iii) *Every irreducible* T*-module with endpoint* 1 *is thin.*

Proof: (i)⇒(ii): Clear.
(ii) ⇒ (iii): By Theorem 7.7 it suffices to show equality holds in (39) for all $\eta \in \mathbb{R}$. Let η be given. We show $U_\eta = E_1^* H_\eta$. Assume $\eta \in \Phi$; otherwise U_η and H_η are both 0. We assume $\eta \neq \tilde{\theta}_1$, $\eta \neq \tilde{\theta}_D$; otherwise we are done by Corollary 10.3. By the construction $U_\eta \supseteq E_1^* H_\eta$; we show $U_\eta \subseteq E_1^* H_\eta$. Since $U_\eta = F_\eta V$, and since the vectors $\{\hat{y} \mid y \in X\}$ span V, it suffices to show $F_\eta \hat{y} \in E_1^* H_\eta$ for all $y \in X$. Let the vertex y be given. We assume $F_\eta \hat{y} \neq 0$; otherwise the result is trivial. Observe $\partial(x, y) = 1$; otherwise $F_\eta \hat{y} = 0$. By Corollary 12.6 the set $E_1^* H_\eta$ consists of all the stable vectors in U_η. To show $F_\eta \hat{y} \in E_1^* H_\eta$ we show $F_\eta \hat{y}$ is stable. To do this we examine the proof of Theorem 13.5. We must have equality in (67) for $i = D - 1$. From the discussion above (67), we find that if $\eta \neq -1$ then equality holds in (50) for $i = D - 1$ and $v = F_\eta \hat{y}$. Similarly if $\eta = -1$ then equality holds in (51) for $i = D - 1$ and $v = F_\eta \hat{y}$. Applying Lemma 12.3 we find $F_\eta \hat{y}$ is stable as desired.
(iii) ⇒ (i): We show equality holds in (66) for $1 \leq i \leq D - 1$. To do this we examine the proof of Theorem 13.5. Apparently it suffices to show (67) holds with equality for $1 \leq i \leq D - 1$. Pick any vertex $y \in X$ such that $\partial(x, y) = 1$. Pick any $\eta \in \Phi$ and assume $\eta \neq \tilde{\theta}_1$, $\eta \neq \tilde{\theta}_D$. Recall $F_\eta V = U_\eta$ so $F_\eta \hat{y} \in U_\eta$. By assumption and Theorem 7.7 we have $U_\eta = E_1^* H_\eta$. Now apparently $F_\eta \hat{y} \in E_1^* H_\eta$. By this and Corollary 12.6 we find $F_\eta \hat{y}$ is stable. Suppose for the moment $\eta \neq -1$. Applying Lemma 12.3(i),(iii) we find equality holds in (50) for $v = F_\eta \hat{y}$ and $1 \leq i \leq D - 1$. Next suppose $\eta = -1$. Applying Lemma 12.3(i′),(iii′) we find equality holds in (51) for $v = F_\eta \hat{y}$ and $1 \leq i \leq D - 1$. From these comments and from the discussion above (67), we find equality holds in (67) for $1 \leq i \leq D - 1$. It follows equality holds in (66) for $1 \leq i \leq D - 1$. □

14. An example

We illustrate the results of this paper for the Johnson graph $J(D, N)$ [4, p. 255]. This graph is defined as follows. Let D and N denote integers with $D \geq 3$ and $N > 2D$. Let Ω denote a set with cardinality N. The Johnson graph $J(D, N)$ has vertex set consisting of all subsets of Ω that have cardinality D. Let y, z denote vertices of $J(D, N)$. Then y, z are adjacent in $J(D, N)$ whenever the cardinality of $y \cap z$ is $D - 1$. It is routine to check $J(D, N)$ is distance-regular with diameter D and intersection numbers

$$c_i = i^2, \quad a_i = i(N - 2i), \quad b_i = (D - i)(N - D - i)$$

for $0 \leq i \leq D$. Since $k = b_0$ we find $k = D(N - D)$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of $J(D, N)$. It is known $\theta_i = (D - i)(N - D - i) - i$ for $0 \leq i \leq D$ [4, p. 256]. In particular $\theta_1 = D(N - D) - N$ and $\theta_D = -D$. Using this and Definition 6.3 we find $\tilde{\theta}_1 = -2$ and $\tilde{\theta}_D = N - D - 2$.

Let X denote the vertex set of $J(D, N)$ and fix $x \in X$. We compute the corresponding set Φ from Definition 6.2. Let Δ denote the vertex-subgraph of $J(D, N)$ induced on the set of vertices in X adjacent x. One finds Δ is the Cartesian product of complete graphs $K_D \times K_{N-D}$. Apparently the eigenvalues of Δ are $N - 2$ (with multiplicity 1), $N - D - 2$ (with multiplicity $D - 1$), $D - 2$ (with multiplicity $N - D - 1$), and -2 (with multiplicity $(D - 1)(N - D - 1)$). Since the multiplicity of $N - 2$ as an eigenvalue for Δ is 1, we find Φ consists of the three scalars $N - D - 2$, $D - 2$, -2.

In the table below we display the elements η of Φ. For each $\eta \in \Phi$ we display the scalars mult_η and $\tilde{\eta}$, along with the coefficient of mult_η from (66).

η	mult_η	$\tilde{\eta}$	$\frac{p_{i-1}(\tilde{\eta})}{p_i(\tilde{\eta})(1+\tilde{\eta})}$
$N - D - 2$	$D - 1$	$-D$	$\frac{i}{(D-1)(D-i)}$
$D - 2$	$N - D - 1$	$D - N$	$\frac{i}{(N-D-1)(N-D-i)}$
-2	$(D - 1)(N - D - 1)$	$D(N - D) - N$	$\frac{i^2}{(D-1)(N-D-1)(D-i)(N-D-i)}$

Using the data in the above table we find equality holds in (66) for $1 \leq i \leq D - 1$. Let $T = T(x)$ denote the subconstituent algebra of $J(D, N)$ with respect to x. Applying Theorem 13.6 we find all the irreducible T-modules with endpoint 1 are thin. By Corollary 7.8 we find that for all $\eta \in \Phi$ there exists an irreducible T-module with endpoint 1 and local eigenvalue η. If $\eta = N - D - 2$ or $\eta = -2$ this module has dimension $D - 1$. If $\eta = D - 2$ this module has dimension D. In all three cases the multiplicity with which this module appears in the standard module V is equal to mult_η. Up to isomorphism there are no further irreducible T-modules with endpoint 1.

For more detail on the irreducible T-modules for $J(D, N)$ see [30, Example 6.1].

15. Directions for further research

In this section we give some suggestions for further research.

Problem 15.1 Investigate the implications of Theorem 13.5 for the case in which Φ has at most two elements. For this case the subgraph Δ from Definition 6.1 is strongly-regular [4, p. 3].

Problem 15.2 Referring to Definition 4.1, assume Γ is bipartite. Find results reminiscent of Theorem 13.5 and Theorem 13.6 that relate the intersection numbers of Γ, the irreducible T-modules with endpoint 2, and the eigenvalues of the subgraph $\Delta_2 = (\check{X}, \check{R})$, where

$$\check{X} = \{y \in X \mid \partial(x, y) = 2\},$$
$$\check{R} = \{yz \mid y, z \in \check{X},\ \partial(y, z) = 2\},$$

and where ∂ denotes the path-length distance function for Γ. See the work of Curtin [9, 10, 15] for some results in this direction.

Problem 15.3 Referring to the inequality (66), for $2 \le i \le D-1$ the expression on the left-hand side does not involve the intersection numbers a_j, b_j for $j \ge i$ or the intersection number c_j for $j \ge i+1$. Apparently by that inequality b_i is bounded above by an expression involving the following: (i) $k, b_1, \ldots, b_{i-1}$; (ii) $c_1, c_2, \ldots, c_i$; (iii) the set Φ; (iv) the scalars mult_η $(\eta \in \Phi)$. Use this information to bound the rate of decrease of the sequence $k, b_1, b_2, \ldots, b_{D-1}$.

Problem 15.4 Referring to Definition 4.1, suppose there exists at least one irreducible T-module with endpoint 1 that is not thin. However, let us assume these modules are "scarce" in the following sense. Let the set Φ be from Definition 6.2. Fix $\alpha \in \Phi$ and assume equality holds in (39) for all $\eta \in \Phi\backslash\alpha$. (Compare this with Theorem 7.7.) For $1 \le i \le D-1$ find a lower bound for the right-hand side of (66) minus the left-hand side of (66).

Problem 15.5 For this problem let $\text{mult} : \mathbb{R} \to \mathbb{R}$ denote any function. For $\eta \in \mathbb{R}$ let mult_η denote the image of η under mult. Let Φ denote the set of real numbers η such that $\text{mult}_\eta \neq 0$. Let Γ denote a distance-regular graph with diameter $D \ge 3$, valency k, intersection numbers a_i, b_i, c_i and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. We say mult is *feasible* for Γ whenever the following six conditions hold: (i) Φ is finite; (ii) mult_η is positive for $\eta \in \Phi$; (iii) $-a_1 \le \eta \le a_1$ for $\eta \in \Phi$; (iv) $\tilde{\theta}_1 \le \eta \le \tilde{\theta}_D$ for $\eta \in \Phi$; (v) the three equations of Lemma 6.7 hold; (vi) the inequality (66) holds for $1 \le i \le D-1$. Let F denote the set of functions from $\mathbb{R}$ to $\mathbb{R}$ that are feasible for Γ. For each vertex of Γ we get an element of F by combining Definition 6.1, Definition 6.2, Lemma 6.4, Lemma 6.7, and Theorem 13.5. In particular F is nonempty. The set F is determined by the intersection numbers of Γ. Therefore the condition that F is nonempty gives a feasibility condition on the intersection numbers of Γ. The set F is convex. In other words for all pairs of functions $\text{mult} \in F$,

$\mathrm{mult}' \in F$, and for all $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq 1$, the function $\alpha\mathrm{mult} + (1-\alpha)\mathrm{mult}'$ is contained in F. We suggest using linear programming to investigate F [2].

Acknowledgments

The author would like to thank Eric Egge and Mark MacLean for giving this manuscript a careful reading and offering many valuable suggestions.

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