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A Geometric Characterization of Fischer's Baby Monster

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Abstract. The sporadic simple group F_2 known as Fischer's Baby Monster acts flag-transitively on a rank 5 P-geometry $\mathcal{G}(F_2)$. P-geometries are geometries with string diagrams, all of whose nonempty edges except one are projective planes of order 2 and one terminal edge is the geometry of the Petersen graph. Let $\mathcal{K}$ be a flag-transitive P-geometry of rank 5. Suppose that each proper residue of $\mathcal{K}$ is isomorphic to the corresponding residue in $\mathcal{G}(F_2)$. We show that in this case $\mathcal{K}$ is isomorphic to $\mathcal{G}(F_2)$. This result realizes a step in classification of the flag-transitive P-geometries and also plays an important role in the characterization of the Fischer–Griess Monster in terms of its 2-local parabolic geometry.

Keywords: sporadic group, diagram geometry, simple connectedness, amalgams of groups.

1. Introduction

The geometry $\mathcal{G}(F_2)$ was constructed in [7]. In [13] the following description of this geometry was proposed. Let $K \cong F_2$. Then K contains an elementary abelian subgroup E of order 2^5 such that $N_K(E)/C_K(E) \cong L_5(2)$. Let $1 < E_1 < \cdots < E_5 = E$ be a chain of subgroups in E where $|E_i| = 2^i, 1 \le i \le 5$. Then the elements of type i in $\mathcal{G}(F_2)$ are all subgroups of K that are conjugate to E_i; two elements are incident if one of the corresponding subgroups contains another one. Notice that the truncation of $\mathcal{G}(F_2)$ by the elements of type 5 is exactly the minimal 2-local parabolic geometry of F_2 constructed in [21].

It follows directly from the definition that $\mathcal{G}(F_2)$ belongs to a string diagram and that the residue of an element of type 5 is the projective space $PG(4,2)$. It turns out that a rank-2 residue of type $\{4,5\}$ is the geometry of edges and vertices of the Petersen graph with the natural incidence relation. Thus $\mathcal{G}(F_2)$ is a flag-transitive P-geometry.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_5\}$ be a maximal flag in $\mathcal{G}(F_2)$ and K_i be the stabilizer of α_i in $K \cong F_2$. Then the K_i's are called the *maximal parabolic subgroups* associated with the action of K on $\mathcal{G}(F_2)$. Without loss of generality we can assume that $\alpha_i = E_i$ (clearly $K_i = N_K(E_i)$ in this case), $1 \le i \le 5$. Below we present a diagram of stabilizers where, under the node of type i, the structure of K_i is indicated. Here, $[2^n]$ stands for an arbitrary group of order 2^n.

1	2	3	4	P 5
Co_2	$S_3 \times \mathrm{Aut}(M_{22})$	$L_3(2) \times S_5$	$L_4(2) \times S_2$	$L_5(2)$
2^{1+22}	$2^2 \cdot [2^{30}]$	$2^3 \cdot [2^{32}]$	$2^4 \cdot [2^{30}]$	$2^5 \cdot [2^{25}]$

Other known examples of P-geometries relate to the sporadic simple groups $M_{22}, M_{23}, Co_2, J_4$ and to nonsplit extensions $3 \cdot M_{22}$ and $3^{23} \cdot Co_2$. Originally, interest in P-geometries was motivated by a relationship of these geometries with a class of 2-arc transitive graphs of girth 5 [8]. Recently, S.V. Shepectorov and the author reduced the classification problem of flag-transitive P-geometries to a treatment of the rank 5 case. Notice that $\mathcal{G}(F_2)$ is the only known rank 5 example.

Our main result is the following.

THEOREM A. *Let K be a group satisfying the following properties:*

(a) It is generated by subgroups K_1, K_2, and K_3 of shapes $2^{1+22}.Co_2$, $2^2.[2^{30}].(S_3 \times Aut(M_{22}))$ and $2^3.[2^{32}].(L_3(2) \times S_5)$, respectively. In K_2 and K_3, on the elementary abelian normal subgroups of order 2^2 and 2^3 their full automorphism groups are induced.
(b) $K_1 \cap K_2$ has index 3 in K_2.
(c) $K_1 \cap K_3$ and $K_2 \cap K_3$ both have index 7 in K_3 and they correspond to an incident point–line pair of a projective plane of order 2 acted on by the composition factor $L_3(2)$ of K_3.

Then $K \cong F_2$.

The concrete form of Theorem A is inspired by its application in the geometric characterization of the Monster (cf. [10]).

Let $\mathcal{K}$ be a rank 5 P-geometry and let K act flag-transitively on $\mathcal{K}$. Suppose that the residue of an element of type 1 in $\mathcal{K}$ is isomorphic to the P-geometry $\mathcal{G}(Co_2)$ (i.e., to the corresponding residue in $\mathcal{G}(F_2)$). Let K_1, K_2, and K_3 be the stabilizers in K of pairwise incident elements of types 1, 2, and 3, respectively. Then it can be deduced from results in [23] that for the group K and its subgroups K_i, $i = 1, 2, 3$, the hypothesis of Theorem A holds. So we have the following.

THEOREM B. *Let $\mathcal{K}$ be a flag-transitive P-geometry of rank 5 and suppose that the residue of an element of type 1 is isomorphic to the P-geometry $\mathcal{G}(Co_2)$. Then $\mathcal{K}$ is isomorphic to $\mathcal{G}(F_2)$.*

Now as a direct consequence of Theorem B we obtain the result announced in [12].

THEOREM C. *The geometry $\mathcal{G}(F_2)$ is simply connected.*

It is conjectured that $\mathcal{G}(F_2)$ is not 2-simply connected and that the automorphism group of its universal 2-cover is a nonsplit extension $3^{4371} \cdot F_2$.

Since the subgroups K_1, K_2, and K_3 are also parabolics in the maximal parabolic geometry of F_2 described in [20], Theorem A implies the simple connectedness of that geometry as well.

The group $K \cong F_2$ contains involutions a and b such that $C_K(a) \cong 2 \cdot {}^2E_6(2){:}2$ and $C_K(b) \cong 2^{1+22}.Co_2$. The involutions conjugate to a, form a class of $\{3,4\}$-transpositions in F_2 and the centralizer of b is conjugate to the subgroup K_1 from Theorem A. In [25] F_2 was characterized by the structure of the centralizer of a. The final step in his characterization relies on some unpublished results of B. Fischer. In [1] F_2 was characterized by the structure of the centralizer of b, i.e., it was shown that a finite simple group with an involution having such a centralizer contains another involution with centralizer isomorphic to $C_K(a)$. Thus a reduction to the characterization on G. Stroth was made. Finally, in [22] a self-contained characterization of F_2 as a finite group containing involutions a and b with the above structure of centralizers, is given.

Starting with the fact that K contains an involution b such that $C_K(b)$ is a 2-constrained group of shape $2^{1+22}.Co_2$ together with certain information concerning fusion in K of involutions from $O_2(C_K(b))$, it can be shown that K is a flag-transitive automorphism group of a rank 5 P-geometry (see Section 9 in [13]). Thus Theorem A has a close relation to the above-mentioned characterization of F_2 by centralizers of involutions. On the other hand, it is not assumed in Theorem A that K_1 is the full centralizer of an involution in K.

The paper is organized as follows. In Section 2 we collect some known facts concerning the Leech lattice and related groups, mainly the group Co_2. In Section 3 we establish some properties of subgroups K_1, K_2, and K_3 from Theorem A. These properties enable us to show that K acts flag-transitively on a rank 5 P-geometry $\mathcal{G}(K)$, one of whose residual geometries is isomorphic to the P-geometry $\mathcal{G}(Co_2)$ of the Conway group. The action of K on $\mathcal{G}(K)$ is defined in such a way that K_1, K_2, and K_3 are stabilizers of pairwise incident elements of type 1, 2, and 3, respectively.

In Section 4 we apply to $\mathcal{G}(K)$ a standard construction from [14] to obtain a C_4-subgeometry whose stabilizer $S \cong [2^{25}].Sp_8(2)$ induces on the subgeometry the flag-transitive action of $Sp_8(2)$.

In Section 5 we consider in K an involution σ and show that the centralizers of σ in the subgroups K_1, K_2, K_3, and S generate a subgroup $E \cong 2 \cdot {}^2E_6(2) : 2$. The subgroup E, acting on $\mathcal{G}(K)$ preserves a subgeometry $\mathcal{E}$, which is isomorphic to a truncated F_4-building on which E induces the natural action. Moreover, subgroup S, acting on the subgeometries conjugate to $\mathcal{E}$, has a length 120 orbit on which it induces a doubly transitive action of $Sp_8(2)$ on the cosets of $O_8^-(2)$. For identification of E we apply Tits's local characterization of the geometries of Lie type groups [26], [27].

In Section 6 we consider the action of K_1 on the set of all subgeometries that are conjugate to $\mathcal{E}$ and contain the element α_1 of type 1 stabilized by K_1. In this action $O_2(K_1)$ has all orbits of length 2. This enables us to define on the set of subgeometries passing through a fixed element of type 1 an equivalence relation with classes of size 2. After that we define on the set of subgeometries that are conjugate to $\mathcal{E}$ a graph $\Gamma = \Gamma(K)$ where two subgeometries are adjacent if they have an element of type 1 in common and are equivalent with respect to this element. We show that the valency of Γ is 3,968,055. The subgroup E acting on Γ stabilizes a vertex v and induces on the set $\Gamma(v)$ of vertices adjacent to v a primitive action of $E/\langle\sigma\rangle \cong {}^2E_6(2):2$. We show that the set of triangles of Γ splits under the action of K into two orbits. An edge of Γ lies in the 1,782 triangle from one of the orbits and in 44,352 from another one. Finally, we show that E acting on the set of vertices at distance 2 from v has an orbit $\Gamma_2^4(v)$ such that the stabilizer in E of a vertex w from this orbit is of the shape $2^{1+20}.U_4(3):2$ and the set $\Gamma(v) \cap \Gamma(w)$ is of size 648.

In terms of the graph $\Gamma(K)$ one can define a geometry $\mathcal{H}(K)$, which is a c-extension of the natural geometry of the group $E/\langle\sigma\rangle$ on which K acts flag-transitively. Notice that the geometry $\mathcal{H}(F_2)$ is presented in [2].

In Section 7 we collect certain results from [6], [15], and mainly from [22] concerning the Baby Monster graph $\Phi = \Gamma(F_2)$. We use these results in the next section as an information on the structure of $\Gamma(K)$ for an arbitrary group K satisfying Theorem A.

In Section 8 we complete the proof of Theorem A. We show that the diameter of Γ is 3 and determine the orbits of E on the vertex set of Γ. This information enables us to conclude that (1) K is nonabelian simple, (2) E and K_1 are the full centralizers in K of the corresponding involutions, and (3) the order of K is equal to the order of F_2. Now the isomorphism $K \cong F_2$ follows either from a characterization of the amalgams arising in flag-transitive action on rank 5 P-geometries [24] or form the characterizations of F_2 by the centralizers of involution [1], [22], [25].

As a consequence of our proof of Theorem A we obtain a characterization of the aforementioned c-extension $\mathcal{H}(F_2)$ of the natural ${}^2E_6(2)$-geometry. In particular it follows that the geometry is simply connected.

We recall that a computer construction of F_2 was announced in [15] and an independent computer-free existence proof of F_2 follows from Griess's construction of the Monster in [5].

2. Preliminary results

First, we recall some known properties of the Leech lattice and related groups (cf. [3], [4], [28]). The Mathieu group M_{22} has exactly two irreducible 10-dimensional GF(2)-modules: a factor module of the truncated Golay code and its dual, which is a section in the Golay cocode. They are also modules for $\mathrm{Aut}(M_{22})$. In order to simplify the terminology, we call these irreducible modules Golay code and Golay cocode, respectively. The orbit lengths on nonzero vectors are 77, 330, 616 in the code and 22, 231, 770 in the cocode.

Let Λ be the Leech lattice. Let $\langle\ ,\ \rangle = (1/16)(\ ,\ )$ where $(\ ,\ )$ is the ordinary inner product. Let $\Lambda_n = \{\lambda | \lambda \in \Lambda, \langle\lambda,\lambda\rangle = n\}$ and $\overline{\Lambda} = \Lambda/2\Lambda$. Then $\overline{\Lambda}$ carries the structure of a 24-dimensional vector space over GF(2) and $\overline{\Lambda} = \overline{\Lambda}_0 \cup \overline{\Lambda}_2 \cup \overline{\Lambda}_3 \cup \overline{\Lambda}_4$ (recall that $\Lambda_1 = \emptyset$). Moreover, $\overline{\Lambda}$ is an irreducible self-dual module for the Conway group Co_1. The group Co_1 acting on $\overline{\Lambda}$ preserves a unique nontrivial quadratic form f where $f(\overline{\lambda}) = 0$ if and only if $\overline{\lambda} \in \overline{\Lambda}_i$ and i is even. Let $\overline{\lambda} \in \overline{\Lambda}_2$. Then the stablizer of $\overline{\lambda}$ in Co_1 is the group Co_2. We assume below that $\overline{\lambda}$ is the image of the vector $(4, 0^{22})$. Let $\Xi = \langle\overline{\lambda}\rangle\perp/\langle\overline{\lambda}\rangle$ where the orthogonal complement is defined with respect to f. Then Ξ is an irreducible 22-dimensional GF(2)-module for Co_2.

LEMMA 2.1. *Co_2 acting on the nonzero vectors of Ξ has exactly 5 orbits: Ξ_{22}, Ξ_{42}, Ξ_{44}, Ξ_{33}, and Ξ'_{33} with respective stabilizers isomorphic to $U_6(2).2$, $2^{10}.Aut(M_{22})$, $2_+^{1+8}.S_8$, $HS.2$ and $U_4(3).D_8$. Moreover Ξ'_{ij} contains images of vectors μ such that $\langle\mu,\mu\rangle = i$ and $\langle\lambda+\mu,\lambda+\mu\rangle = \pm j$.*

LEMMA 2.2. *Let M be a subgroup in Co_2 with shape $2^{10}\cdot Aut(M_{22})$. Then M stablizes an element from Ξ_{42} and $O_2(M)$ is the Golay code.*

Let μ be the element form Ξ_{42}, which is the image of the vector $(8, 0^{23})$ from Λ and let $M \cong 2^{10}\cdot \mathrm{Aut}(M_{22})$ be the stabilizer of μ in Co_2.

LEMMA 2.3. *M has exactly three orbits on Ξ_{22}, denoted by Ξ_{22}^4, Ξ_{22}^3, and Ξ_{22}^2. These orbits contain images of vectors of shapes $(0, 4^2, 0^{21})$, $(-3, 1^{23})$, and $(2^8, 0^{16})$, respectively. The corresponding stabilizers are isomorphic to $2^9.L_3(4).2$, $Aut(M_{22})$ and $[2^{10}].S_6$, respectively. These orbits contain images of vectors of shapes $(0^2, 4^2, 0^{20})$, $(2^8, 0^{16})$, $(0^2, 2^8, 0^{14})$, and $(1^{23}, -3)$, respectively.*

LEMMA 2.4. *M has exactly four orbits on $\Xi_{42} - \{\mu\}$. The corresponding stabilizers are isomorphic to $2^{4+10}.S_5$, $[2^9].S_6$, $[2^8].L_3(2)$, and $L_3(4)$, respectively.*

Let $\Sigma = \Sigma(\mu)$ be the orbit of M on $\Xi_{42} - \{\mu\}$ with stabilizer isomorphic to $2^{4+10}\cdot S_5$. Then Σ contains the images of all vectors from Λ_2 whose supports of size 2 are disjoint from $\{1, 2\}$ and whose nonzero components are equal to ± 4. Each such support corresponds to exactly two elements from Σ. Thus we have an equivalence relation on Σ with classes of size 2. These classes are indexed by the 2-element subsets of the set $\{3, 4, \cdots, 24\}$. It is clear that $O_2(M)$ preserves each class as a whole, whereas $M/O_2(M) \cong \mathrm{Aut}(M_{22})$ acts in the obvious way on the set of classes. The following lemma is a consequence of Lemma 2.4 and the above arguments.

LEMMA 2.5. *Let* $\{\mu, \nu_1, \nu_2\}$ *be a triple of elements from* Ξ_{42} *and suppose that its setwise stabilizer in* Co_2 *is of shape* $[2^{14}] \cdot (S_5 \times S_3)$. *Then* ν_1 *and* ν_2 *are equivalent elements from* $\Sigma(\mu)$.

The images under Co_2 of the triple form Lemma 2.5 will be called *special triples.* The special triples are closely related to both minimal and maximal 2-local parabolic geometries of Co_2 (cf. [20], [21]). In those geometries the elements of Ξ_{42} are points, whereas the special triples are lines. The following proposition is a consequence of the description of the natural representations of the 2-local geometries of Co_2 obtained in [14].

PROPOSITION 2.6. *Let* W *be a GF(2)-module for* Co_2, *which is generated by a set of one-dimensional subspaces indexed by the elements of* Ξ_{42}, *and suppose that the subspaces corresponding to a special triple generate a 2-dimensional subspace. Then* W *is isomorphic either to* $\langle\overline{\lambda}\rangle^{\perp}$ *or to* $\langle\overline{\lambda}\rangle^{\perp}/\langle\overline{\lambda}\rangle$, *where* $\overline{\lambda}$ *is the nonzero vector from* $\overline{\Lambda}$ *stabilized by* Co_2.

The group Co_2 acting on $\Xi = \langle\overline{\lambda}\rangle^{\perp}/\langle\overline{\lambda}\rangle$ preserves a unique nontrivial quadratic form, i.e., the one induced by f.

Let $\sigma \in \Xi_{22}$ and $U \cong U_6(2) \cdot 2$ be the stabilizer of σ in Co_2. Since Co_2 is primitive on Ξ_{22}, and in view of Lemma 2.1, we see that U does not stabilize a 2-dimensional subspace in Ξ. Thus we have the following.

LEMMA 2.7. *Let* $\sigma \in \Xi_{22}$, $U \cong U_6(2) \cdot 2$ *be the stabilizer of* σ *in* Co_2 *and* $\langle\sigma\rangle^{\perp}$ *be the subspace of* Ξ *dual to* $\langle\sigma\rangle$. *Then* $\langle\sigma\rangle^{\perp}$ *is an indecomposible module for* U.

We conclude the section by a description of the involutions in Co_2.

LEMMA 2.8. Co_2 *has exactly three conjugacy classes of involutions. The respective centralizers are isomorphic to* $2_+^{1+8} \cdot Sp_6(2)$, $(2_+^{1+6} \times 2^4) \cdot A_8$ *and* $2^{10} \cdot Aut(A_6)$.

LEMMA 2.9. *Let* $M \cong 2^{10} \cdot Aut(M_{22})$ *be a subgroup of* Co_2, τ *be an involution form the orbit of length 77 of* M *on* $O_2(M)$, *and* S *be the centralizer of* τ *in* Co_2. *Then* $S \cong 2_+^{1+8} \cdot Sp_6(2)$; S *has a unique orbit of length 63 on* $\Xi_4 2$ *and a unique orbit of length 28 on* Ξ_{22}.

3. Reconstruction of the P-geometry

Let K be a group and K_i i=1,2,3 be its subgroups satisfying the hypothesis of Theorem A. In this section we deduce some information about the structure of these subgroups. Using this information we will show that K acts flag-transitively on a rank 5 P-geometry in such a way K_1, K_2, and K_3 are maximal parabolics. Some arguments in this section are rather similar to those in Section 3 of [10].

Let E_i be the normal subgroup of order 2^i in K_i, $1 \leq i \leq 3$. Let $Q = O_2(K_1)$ be the extraspecial group with center E_1 and set $\bar{Q} = Q/E_1$ be the elementary abelian 2-group of rank 22. The well-known properties of extraspecial groups imply the following.

LEMMA 3.1. *Let T be a subgroup of K_1 containing $O_2(K_1)$. Then E_1 is the center of T.*

Since K_2 contains a unique conjugacy class of subgroups of index 3, $K_1 \cap K_2$ centralizes an involution from E_2. Thus, by Lemma 3.1, we have the following.

LEMMA 3.2. *$E_1 \leq E_2$; in particular E_1 is not normal in K_2.*

The following lemma is a direct consequence of condition (c) in Theorem A.

LEMMA 3.3. *K_1 and K_2 generate K.*

Let Δ be a graph on the set of (right) cosets of K_1 in K where two cosets are adjacent if they both have nonempty intersections with a coset of K_2. Let $\alpha \in \Delta$ be the coset containing the identity element. Since α is actually the same as K_1, the latter is the stabilizer of α in K and K_2 stabilizers a triangle $t = \{\alpha, \beta, \gamma\}$ and induces S_3 on t. Let T be the set of all images of t under K and $T(\alpha)$ be the set of triangles from T passing through α.

LEMMA 3.4. *The action induced by K_1 on $T(\alpha)$ is similar to the primitive action of Co_2 on the cosets of $2^{10}.Aut(M_{22})$.*

By Lemma 2.2 the action of Co_2 on $T(\alpha)$ is similar to its action on Ξ_{42}.

LEMMA 3.5. *The subgroup $Q = O_2(K_1)$ induces a nontrivial action on the set $\Delta(\alpha)$ of vertices adjacent to α in Δ.*

Proof. Since K_2 induces S_3 on t, it contains an element k that stabilizes γ and permutes α and β. By Lemma 3.3 $K = \langle K_1, k \rangle$. Suppose that Q does not act on $\Delta(\alpha)$. Then Q is contained in the elementwise stabilizer $K(t)$ of the triangle t. By Lemma 3.1 the center of $K(t)$ is E_1. This means that E_1 is normal in K. Hence E_1 is normal in K_2, a contradiction to Lemma 3.2. □

Thus Q acts nontrivially on $\Delta(\alpha)$. It is clear that the orbits of this action are of length 2. Moreover, if $\{\varepsilon, \delta\}$ is such an orbit then $\{\alpha, \varepsilon, \delta\} \in T(\alpha)$ and each triangle from $T(\alpha)$ can be obtained in this manner.

Now let us turn to the action of K_3 on Δ. Let Ω be the orbit of K_3 containing α. By condition (c) $|\Omega| = 7$ and $K_2 \cap K_3$ has on Ω an orbit of length 3, which contains α. It is clear that the latter orbit coincides with the triangle t that is stabilized by K_2. Hence Ω contains exactly seven triangles from T and these triangles are the lines of a projective plane on Ω. Let $t_1 = t, t_2, t_3$ be the triangles from Ω that lie in $T(\alpha)$. Then $K_1 \cap K_3$ stablizes $\{t_1, t_2, t_3\}$ as a whole and by Lemma 2.5 the triple $\{t_1, t_2, t_3\}$ corresponds to a special triple of elements in Ξ_{42}. In particular we have the following.

LEMMA 3.6. *K_3 is the full stabilizer of Ω in K.*

The subgroup $K_1 \cap K_3$ contains Q and it induces S_4 on $\Omega - \{\alpha\}$.

LEMMA 3.7. *Q induces on $\Omega - \{\alpha\}$ the elementary abelian group of order 4.*

Now we come to one of the central results of the section. First recall some notations from Section 2: Λ is the Leech lattice; $\overline{\Lambda} = \Lambda/2\Lambda$; for $U \subseteq \overline{\Lambda}$ by $U^{\perp}$ we denote the orthogonal complement of U with respect to the unique nontrivial quadratic form f on $\overline{\Lambda}$ preserved by Co_1; $\overline{\lambda}$ is the nonzero vector stabilized by Co_2.

PROPOSITION 3.8. *As a GF(2)-module for Co_2, $\tilde{Q}$ is isomorphic to the module $\Xi = \langle\overline{\lambda}\rangle^{\perp}/\langle\overline{\lambda}\rangle$.*

Proof. From the description of the orbits of Q on $\Delta(\alpha)$ it follows that the module dual to $\tilde{Q}$ contains a collection of one-dimensional subspaces indexed by the elements of Ξ_{42}. By Lemma 3.7 the subspaces corresponding to a special triple generate a two-dimensional subspace. By Proposition 2.6 $\tilde{Q}$ is either $\langle\overline{\lambda}\rangle^{\perp}$ or $\langle\overline{\lambda}\rangle^{\perp}/\langle\overline{\lambda}\rangle$. Since the dimension of $\tilde{Q}$ is 22 and Ξ is self-dual the result follows. □

Let ϕ: $Q \longrightarrow \Xi$ and ψ: $\langle\overline{\lambda}\rangle^{\perp} \longrightarrow \Xi$ be the surjective mappings that commute with the action of $Co_2 \cong K_1/Q$.

Let us show that E_2 and E_3 are contained in Q. Really, $C_{K_1}(E_2) \cong 2^2.[2^{30}].\mathrm{Aut}(M_{22})$ and by Lemma 2.7, $E_2 \leq Q$. Since $K_1 \cap K_3$ acts transitively on $E_3 - E_1$ this implies $E_3 \leq Q$. Now, by Lemmas 2.2 and 2.5, we have the following.

LEMMA 3.9. *$\phi(E_2)$ is an element of Ξ_{42} while $\phi(E_3)$ is a special triple.*

Without loss of generality we assume that $\phi(E_2)$ is the element μ, which is the image of the vector $(8, 0^{23})$. In this case $\phi(E_3) - \{\mu\}$ is a pair of equivalent elements from $\Sigma(\mu)$ (cf. notation after Lemma 2.5).

Let $\overline{\Lambda}_4^8$, $\overline{\Lambda}_4^4$, and $\overline{\Lambda}_2^4$ be the subsets of $\overline{\Lambda}$ containing the images of vectors of the shape $(8, 0^{23})$, $(4^4, 0^{20})$, and $(4^2, 0^{22})$, respectively. A direct calculation in the Leech lattice (or even just in the Golay code) proves the following.

LEMMA 3.10. *$\overline{\mathrm{M}} = \{0\} \cup \overline{\Lambda}_4^8 \cup \overline{\Lambda}_4^4 \cup \overline{\Lambda}_2^4$ is a subspace of $\langle\overline{\lambda}\rangle^{\perp}$.*

By definition the distinguished vector $\overline{\lambda}$ stabilized by Co_2 is contained in $\overline{\mathrm{M}}$. Put $R = \phi^{-1}(\psi(\overline{\mathrm{M}}))$. Since the quadratic form f vanishes on $\overline{\mathrm{M}}$, the subgroup R is elementary abelian. In addition both E_2 and E_3 are contained in R.

LEMMA 3.11. *R is a normal subgroup of K_2.*

Proof. Let $N = C_{K_2}(E_2)$. Then $N \cong 2^2.[2^{30}].\mathrm{Aut}(M_{22})$ and N is an index 2 subgroup of $K_1 \cap K_2$. Let Ξ_{22}^4 be the orbit of length 44 of $M \cong 2^{10}.\mathrm{Aut}(M_{22})$ on Ξ_{22} (cf. Lemma 2.3) and let $\sigma \in \phi^{-1}(\Xi_{22}^4)$. By definition $\sigma \in R$ and by Lemma 2.3 $C_N(\sigma) \cong [2^{30}].L_3(4).2$. Let ε be any other involution from K_2, which does not lie in $\phi^{-1}(\Xi_{22}^4)$. Then by Lemmas 2.3, 2.4, and 2.8 $C_N(\varepsilon) \ncong C_N(\sigma)$. Since N is normal in K_2, the latter preserves $\phi^{-1}(\Xi_{22}^4)$ as a whole. Now the claim follows from the fact that $E_2 \cup \phi^{-1}(\Xi_{22}^4)$ generates R. □

Let $\bar{R} = R/E_2$. Then it is easy to see that $\bar{R}$ is an irreducible GF(2)-module for $\mathrm{Aut}(M_{22}) \cong N/O_2(N)$, isomorphic to the Golay cocode. This implies the following.

LEMMA 3.12. *$O_{2,3}(K_2)$ commutes with $\bar{R}$.*

Now we are in a position to prove the following.

PROPOSITION 3.13. *There exists a rank 5 P-geometry $\mathcal{G}(K)$ on which K acts flag-transitively in such a way that K_1, K_2, and K_3 are stabilizers of pairwise incident elements of types 1, 2, and 3, respectively.*

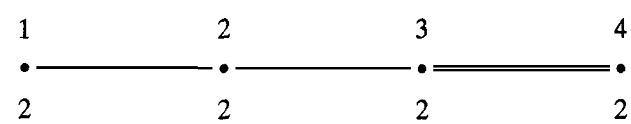
Proof. Let us consider the rank 4 P-geometry $\mathcal{G}(Co_2)$ of Conway's second group in its natural representation in Ξ (see [13]). Let $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be a maximal flag such that $\phi^{-1}(\alpha_i) = E_i$ for $i = 2,3$ (cf. Lemma 3.9). Put $E_j = \phi^{-1}(\alpha_j)$ for $j = 4,5$. Then it follows from [13] that $E_4, E_5 \leq R$. Since both E_4 and E_5 contain E_2, by Lemma 3.12, these subgroups are normalized by $O_{2,3}(K_2)$. Notice that the latter is not contained in K_1. Let K_j be the subgroup generated by the normalizers of E_j in K_i for $1 \leq i \leq j-1$, $j = 4,5$. Then $C_{K_j}(E_j) \leq K_1 \cap K_2 \cap K_3$ and K_j induces $L_j(2)$ on E_j. Let Δ be the graph on the cosets of K_1 in K defined above, and let α be the vertex stabilized by K_1. Let Ξ_i be the orbit of α under K_i, $1 \leq i \leq 5$. Then, clearly, $\Xi_1 = \{\alpha\}$, $\Xi_2 \in T(\alpha)$, $\Xi_3 = \Omega$. By the above arguments $|\Xi_i| = 2^i - 1$, the subgraph of Δ induced by Ξ_i is complete and $\Xi_i \subseteq \Xi_j$ for $1 \leq i < j \leq 5$. Also, it is easy to see that K_i is the full stabilizer of Ξ_i in K. Now the desired P-geometry $\mathcal{G}(K)$ has all images of the subsets Ξ_i, $1 \leq i \leq 5$ under K as the element set. Two elemets are incident if one of the subsets contains another one. By definition $\mathcal{G}(K)$ is a geometry belonging to a string diagram and it is easy to check that the residue of an element of type 1 is isomorphic to the P-geometry $\mathcal{G}(Co_2)$. K acts flag transitively on $\mathcal{G}(K)$ so that the K_i's are the maximal parabolic subgroups. □

4. $Sp_8(2)$-subgeometry

As a result of Section 3, we have a P-geometry $\mathcal{G}(K)$ of rank 5 on which K acts flag-transitively and the residue of an element of type 1 in this geometry is isomorphic to the P-geometry $\mathcal{G}(Co_2)$. It is shown in [14] that such a geometry contains a subgeometry $\mathcal{L}$ which belongs to the diagram C_4 and that the stabilizer of $\mathcal{L}$ in K induces on $\mathcal{L}$ a flag-transitive automorphism group $Sp_8(2)$. Since we need some information about $\mathcal{L}$ and its embedding into $\mathcal{G}(K)$, we recall below the procedure of its construction from [14].

Let $\Phi = \{\alpha_1, \cdots, \alpha_4\}$ be a flag in $\mathcal{G}(K)$, where α_i is of type i, $1 \le i \le 4$. We assume below that the subgroup K_i from Theorem A is the stabilizer of α_i in K, $1 \le i \le 3$. Let B be the stabilizer of Φ in K. Let X_j be the stabilizer of the flag $\Phi - \{\alpha_i\}$ in K, $1 \le i \le 3$ and X_4 be the subgroup of index 5 in the stabilizer of the flag $\Phi - \{\alpha_4\}$, which contains B. Then B has index 3 in X_i for $1 \le i \le 4$. Let X be the subgroup of K generated by the subgroups X_i, $1 \le i \le 4$. Then $\mathcal{L} = (X, B; (X_i)_{1 \le i \le 4})$ is a chamber system [27] and it is easy to check that it belongs to a string diagram where all nonempty edges except the edge $\{3,4\}$ are projective planes of order 2.

To determine the edge $\{3,4\}$ one should consider the subgroup $X_{34} = \langle X_3, X_4 \rangle$. With a suitable choice of Φ, this subgroup is contained in $K_1 \cap K_2$ and it is not so difficult to see in the latter group that $X_{34}/O_2(X_{34}) \cong Sp_4(2) \cong S_6$. Hence the edge $\{3,4\}$ is the generalized quadrangle of order (2,2) and $\mathcal{L}$ has the following diagram:



Now one can identify $X_{234} = \langle X_2, X_3, X_4 \rangle$ as a subgroup of K_1. It turns out that X_{234} is the preimage of an involution centralizer in $Co_2 \cong K_1/O_2(K_1)$ with the shape $2_+^{1+8}.Sp_6(2)$. Therefore the residue of an element of type 1 in $\mathcal{L}$ is the $Sp_6(2)$-building. By [27] this implies that $\mathcal{L}$ is the $Sp_8(2)$-building. Finally, by the construction, X acts flag-transitively on $\mathcal{L}$. Thus we have the following.

PROPOSITION 4.1. *$\mathcal{G}(K)$ contains a subgeometry $\mathcal{L}$ isomorphic to the $Sp_8(2)$-building. The stabilizer of $\mathcal{L}$ in K contains a Sylow 2-subgroup of K_i for i=1,2,3 and induces on $\mathcal{L}$ the group $Sp_8(2)$.*

The following two lemmas are consequences of the construction of $\mathcal{L}$.

LEMMA 4.2. *The element α_3 is contained in exactly five subgeometries that are conjugate to $\mathcal{L}$ and K_3 induces S_5 on these subgeometries.*

LEMMA 4.3. *Let S be the stabilizer of the subgeometry $\mathcal{L}$ and $S_1 = S \cap K_1$. Then S_1 contains $O_2(K_1)$ and its image in $K_1/O_2(K_1) \cong Co_2$ is the centralizer of a central involution.*

5. ${}^2E_6(2)$-subgeometry

In this section we show that K contains a subgroup E of the shape $2 \cdot {}^2E_6(2) : 2$, which preserves in $\mathcal{G}(K)$ a subgeometry isomorphic to a truncated F_4-building on which E induces the natural action. Subgroup E will be constructed as follows. We consider a suitable involution σ in Q (to be more precise, in $\phi^{-1}(\Xi_{22}^4)$) and show that the centralizers of σ in K_1, K_2, K_3, and S generate a subgroup E of the above shape. Here S is the stabilizer of a certain $Sp_8(2)$-subgeometry constructed in the previous section. After determination of the structure of the aforementioned centralizers, and of their mutual intersections, we will apply Tits's local characterization of the geometries of the Lie-type groups [26], [27].

We start with a lemma that follows from the fact that the orbit Ξ_{22} of Co_2 on Ξ has length divisible by 4 and from general properties of extraspecial groups.

LEMMA 5.1. *Let T be a Sylow 2-subgroup of K_1. Then each orbit of T on Ξ_{22} has length divisible by 4 and each orbit of T on $\phi^{-1}(\Xi_{22})$ has length divisible by 8.*

Put $C_1 = \phi^{-1}(\Xi_{22})$ and $C_2 = \phi^{-1}(\Xi_{22}^4)$.

LEMMA 5.2. *C_i is a conjugacy class of involutions in K_i, $i=1,2$.*

Proof. The quadratic form on Ξ preserved by Co_2 vanishes on Ξ_{22} so C_1 and C_2 consist of involutions. Now C_1 is a conjugacy class of K_1 by Lemmas 2.1 and 5.1. Ξ_{22}^4 is contained in $\psi(\overline{M})$ where $\overline{M}$ is the subspace defined in Lemma 3.13. So by Lemmas 2.3 and 3.11 K_2 stabilizes $\phi^{-1}(\Xi_{22}^4)$ as a whole. By Lemma 5.1 K_2 acts transitively on this set. □

Now we intend to construct a conjugacy class C_3 of involutions in K_3 such that $C_3 \subseteq C_2$.

In what follows for a group A we put $\overline{A} = A/O_2(A)$.

Let $L = K_1 \cap K_2$. Then L acts transitively on the set of 231 elements of type 3 incident to α_2 as well as on the set Ξ_{22}^4 of size 44. Notice that $\overline{L} \cong \mathrm{Aut}(M_{22})$ has a unique primitive permutation representation of degree less than or equal to 44, i.e., the natural representation of degree 22. Also, $\overline{L}$ has a unique permutation representation of degree 231, i.e., the action on the pairs of points from the natural representation. This means that $L \cap K_3$ stabilizes in Ξ_{22}^4 a subset Θ of size 4 and all other orbits of $L \cap K_3$ on Ξ_{22}^4 are of length divisible by 20. By Lemma 5.1 Θ is an orbit of $L \cap K_3$. By Lemma 2.3 the length of any orbit of $L \cap K_3$ on $\Xi_{22} - \Xi_{22}^4$ is divisible by 16. Now since $L \cap K_3$ has index 3 in $K_1 \cap K_3$, we conclude that Θ is an orbit of $K_1 \cap K_3$. By Lemma 3.13 $O_{2,3}(K_2)$ commutes with $\phi^{-1}(\Xi_{22}^4)/E_2$ and it is clear from the above construction that $E_2 \subseteq \phi^{-1}(\Theta)$. Now since $K_3 = \langle K_1, \cap K_3, O_{2,3}(K_2)\rangle$, and by Lemma 5.1, we have the following.

LEMMA 5.3. *$C_3 = \phi^{-1}(\Theta)$ is a conjugacy class of involutions in K_3 of size 8 and K_3 induces a 2-group on C_3.*

Let σ be an involution from C_3. Let P_i be the centralizer of σ in K_i, $1 \le i \le 3$. Then by the above construction we have the following:

$$P_1 \cong 2^{2+20}.U_6(2).2 \quad P_2 \cong [2^{30}].(S_3 \times L_3(4).2) \quad P_3 \cong [2^{32}].(L_3(2) \times S_5)$$

We will need the following.

LEMMA 5.4. *Let T be a Sylow 2-subgroup of $K_1 \cap K_3$. Then T has a unique orbit of length 4 on Ξ_{22} and this orbit coincides with Θ.*

Proof. Let Δ be such an orbit of length 4. Without loss of generality we can assume that T is a Sylow 2-subgroup in $K_1 \cap K_2$. Then, by Lemma 2.3, $\Delta \subseteq \Xi_{22}^4$. Now $K_1 \cap K_2$ acting on Ξ_{22}^4 preserves an imprimitivity system with classes of size 2 and $O_2(K_1 \cap K_2)$ permutes elements in the classes. The action induced on the set of equivalence classes is the natural degree 22 permutation action of $\mathrm{Aut}(M_{22})$. A direct calculation in the latter group shows that its Sylow 2-subgroup has orbits of length 2, 4, and 16. Thus, the result follows. □

The next lemma follows from the structure of P_1 and P_2.

LEMMA 5.5. *Group P_2, acting on the set of elements of type 3 incident to α_2, has exactly two orbits whose lengths are 21 and 210. The element α_3 lies in the former of the orbits.*

Let us fix an $Sp_8(2)$-subgeometry containing the flag $\{\alpha_1, \alpha_2, \alpha_3\}$. Let S be the stabilizer in K of this subgeometry and let $P_4 = C_S(\sigma)$.

Put $E = \langle P_i | 1 \le i \le 4\rangle$. We will show that $E \cong 2 \cdot {}^2E_6(2):2$ and that P_i are the maximal parabolics associated with the natural action of E on the F_4-building. We start with description of the minimal parabolics Q_i. By definition Q_i is the intersection of P_j for $1 \le j \le 4$, $j \ne i$. Put $B = Q_i \cap Q_j$ and $Q_{ij} = \langle Q_i, Q_j\rangle$ for $i \ne j$.

Subgroup P_3 induces $L_3(2)$ on the set of elements of type 1 and 2 incident to α_3 and S_5 on the set of $Sp_8(2)$-subgeometries containing α_3 (cf. Lemma 4.2 and 5.3). This implies the following.

LEMMA 5.6. *Subgroup B is of the shape $[2^{37}] \cdot S_3$, moreover, $\overline{Q}_1 \cong \overline{Q}_2 \cong S_3 \times S_3$, $\overline{Q}_4 \cong S_5$, $\overline{Q}_{14} \cong \overline{Q}_{24} \cong S_3 \times S_5$.*

Lemma 5.5 and the shape of P_2 imply the following.

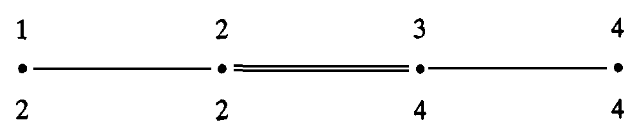
LEMMA 5.7. $\overline{Q}_3 \cong S_5$, $\overline{Q}_{34} \cong L_3(4).2$, $\overline{Q}_{13} \cong S_3 \times S_5$.

Now let us determine the structure of $\overline{Q}_{23}$.

LEMMA 5.8. $\overline{Q}_{23} \cong U_4(2).2$.

Proof. It follows from Lemmas 2.9 and 4.3 that $S_1 = S \cap K_1$ acting on Ξ_{22} has an orbit Φ of length 28. Then $\overline{S}_1 \cong Sp_6(2)$ acts in this orbit as it acts on the set of minus forms in the symplectic vector space W with stabilizer isomorphic to $O_6^-(2) \cong U_4(2).2$. Let us show that $\Theta \subseteq \Phi$. In fact, the image of $S_1 \cap K_3$ in $\overline{S}_1$ is the stabilizer of a two-dimensional isotropic subspace in W and it has an orbit Δ of length 4 on Φ. This orbit consists of the forms that vanish on the isotropic subspace. Since $S_1 \cap K_3$ contains a Sylow 2-subgroup of $K_1 \cap K_3$ we can apply Lemma 5.4 and conclude that $\Delta = \Theta$. Hence the image of $P_1 \cap P_4$ in $\overline{S}_1$ contains at least $U_4(2)$. On the other hand, $Q_{23} \leq P_1 \cap P_4$ and, by Lemma 5.6 and 5.7, we can see that $Q_{23} = P_1 \cap P_4 \cong [2^{31}] \cdot U_4(2) \cdot 2$. □

By Lemmas 5.6, 5.7, and 5.8 we have a chamber system $\mathcal{E} = (E, B, \{Q_i\}_{1 \leq i \leq 4})$, which corresponds to the following diagram:



Now it is straightforward to see that Q_2, Q_3, and Q_4 generate P_1, whereas Q_1, Q_2, and Q_3 generate in S a subgroup of index 120 and of the shape $[2^{25}].O_8^-(2)$. Notice that in the latter case the whole group S cannot be generated since $S \cap P_1$ is a proper subgroup of $S \cap K_1$. Now analogously to the case of $Sp_8(2)$-subgeometries we see that in $\mathcal{E}$ all C_3-residues are buildings, so by [27] $\mathcal{E}$ is a building itself. Since the shapes of the maximal parabolics are known, and application of the classification of buildings in [26] enables us to identify the action of E on $\mathcal{E}$ with the group ${}^2E_6(2):2$. It is clear that the order 2 subgroup $\langle\sigma\rangle$ is in the kernel of the action. We claim the extension $E/\langle\sigma\rangle$ by $\langle\sigma\rangle$ is nonsplit. Indeed, P_1 contains a Sylow 2-subgroup of E and by Lemma 2.7 $O_2(P_1/E_1)$ is an indecomposible GF(2)-module. So the claim follows and in view of [4] we have the following.

PROPOSITION 5.9. *The geometry $\mathcal{G}(K)$ contains a subgeometry $\mathcal{E}$ whose full stabilizer in K is a subgroup $E \cong 2\cdot{}^2E_6(2){:}2$. The subgeometry is isomorphic to the truncated F_4-building on which E induces the natural action. The stabilizer S of an $Sp_8(2)$-subgeometry acting on the set of subgeometries conjugate to $\mathcal{E}$ has an orbit of length 120 on which it induces a doubly transitive action of $Sp_8(2)$ on the cosets of $O_8^-(2)$.*

6. A graph

Let us consider the set Π of the subgeometries conjugate to $\mathcal{E}$, i.e, the set of images of $\mathcal{E}$ under the action of K. Let $\Pi(\alpha_1)$ be the subset of Π consisting of the subgeometries containing α_1. Since E is flag-transitive on $\mathcal{E}$, it is easy to see that K_1 acts transitively on $\Pi(\alpha_1)$ and $P_1 \cong 2^{2+20}.U_6(2):2$ is the point stabilizer in this action. Since $O_2(K_1)$ intersects P_1 by a subgroup of index 2, we see that the orbits of $O_2(K_1)$ on $\Pi(\alpha_1)$ are all of length 2. Thus we have an equivalence relation on $\Pi(\alpha_1)$ with classes of size 2.

Now define a graph $\Gamma = \Gamma(K)$ having Π as the set of vertices in which two subgeometries are adjacent if they have an element α of type 1 in common and are equivalent with respect to the equivalence relation on $\Pi(\alpha)$ defined above. So each element of type 1 in the subgeometry $\mathcal{E}$ gives rise a subgeometry adjacent to $\mathcal{E}$ in Γ. Since E acts primitively on the set of elements of type 1 in $\mathcal{E}$ and, by the construction, distinct subgeometries have distinct sets of elements of type 1, we see that there is a bijection between the set of elements of type 1 in $\mathcal{E}$ and the set of subgeometries adjacent to $\mathcal{E}$ in Γ.

Let v be a vertex of Γ corresponding to $\mathcal{E}$. Let $\Gamma(v)$ be the set of vertices adjacent to v in Γ. By the above paragraph and Proposition 5.9 we have the following.

LEMMA 6.1. *K acts vertex- and edge-transitively on Γ and E is the stabilizer of a vertex v in this action. The action of E on the set $\Gamma(v)$ of vertices adjacent to v is similar to its action on the set of elements of type 1 in $\mathcal{E}$. In particular, the valency of Γ is equal to 3,968,055.*

Let us consider the action of E on the set of elements of type 1 in $\mathcal{E}$ (notice that $\langle\sigma\rangle$ is in the kernel of the action). The elements of type 1, 2, 3, and 4 in $\mathcal{E}$ will be called points, lines, planes, and simplecta, respectively. We use the term *containment* for the incidence between the elements. Thus we are interested in the action of E on the set of points. A detailed description of this representation can be found in [22]. We start with the following.

LEMMA 6.2. *Subgroup E acting on the point set of $\mathcal{E}$ has rank 5 with the subdegrees 1, 1,782, 44,352, 2,097,152, and 1,824,768. The respective 2-point stabilizers are isomorphic to $2^{2+20}.U_6(2){:}2$, $[2^{30}].L_3(4){:}2$, $[2^{25}].U_4(2){:}2$, $2.U_6(2){:}2$, and $[2^{20}].L_3(4){:}2$.*

Subdegree 1782 corresponds to the collinearity graph, i.e., to the graph where two points are adjacent if they are on a common line. The structure of the collinearity graph with respect to a point u is given in Figure 1. Here the number of vertices in a box B adjacent to a fixed vertex in a box A is indicated around A on the edge (or loop) joining A and B.

The image in $E/\langle\sigma\rangle$ of the stabilizer of a point u is of the shape $2^{1+20}.U_6(2){:}2$. Let $\gamma(u)$ denote the unique nontrivial element in the center of that group. Then the following proposition holds (cf. [22]).

LEMMA 6.3. *Let $w \in \Sigma_i^{(')}(u)$. Then the product $\gamma(u)\cdot\gamma(w)$ is of order i, $i = 2, 3, 4$.*

Subgroup P_4, which is the stabilizer of a simplecton in its action on the collinearity graph, has a unique orbit Δ of length 119. Orbit Δ consists of all points in the simplecton. Suppose that u is contained in Δ and let w be another point form Δ. Then w is either collinear to u or is contained in the suborbit of length 44,352 and both possibilities take place.

Now, by Proposition 5.9, subgroup S has an orbit Σ of length 120 on the vertex set of Γ. If S is the stabilizer of an $Sp_8(2)$-subgeometry that contains the flag $\{\alpha_1, \alpha_2, \alpha_3\}$ (by the construction $\mathcal{E}$ contains this flag as well), then Σ contains the vertex v. Now S contains a Sylow 2-subgroup of K_1 and hence it contains $O_2(K_1)$. This means that S contains an element that moves v to a vertex adjacent to v, i.e., to the vertex that is equivalent to v in $\Pi(\alpha_1)$. Since the action of S on Σ is doubly transitive, we see that $\Sigma \subseteq \{v\} \cup \Gamma(v)$. Hence $\Sigma - \{v\}$ is an orbit of length 119 of P_4 on $\Gamma(v)$ and by the above paragraph we can assume that $\Sigma = \{v\} \cup \Delta$. Thus we have the following.

LEMMA 6.4. *Subgroup S acting on Γ stabilizes a complete subgraph Σ on 120 vertices containing v. Moreover, $\Sigma - \{v\}$ is a simplecton in $\mathcal{E}$.*

As a direct consequence of the above lemma we have the following.

LEMMA 6.5. *If two vertices from $\Gamma(v)$ correspond to points lying in a common simplecton then they are adjacent.*

Let us consider the action of K on Γ. The elementwise stabilizer of vertices $x, y, z, \ldots$ in this action is denoted by $K(x, y, z, \ldots)$. By Proposition 5.9 and Lemma 6.1, the action of $K(v)$ on $\Gamma(v)$ is determined up to similarity. The center of $K(v)$ is of order 2. The unique nontrivial element of this center is denoted by the same symbol v. Thus each vertex of Γ corresponds to an involution of K. At this point we cannot claim that with distinct vertices distinct involutions are associated, but later we show that this is the case. On the other hand, the involution corresponding to adjacent vertices are distinct and they commute. This observation together with Lemmas 6.3 and 6.5 imply the following.

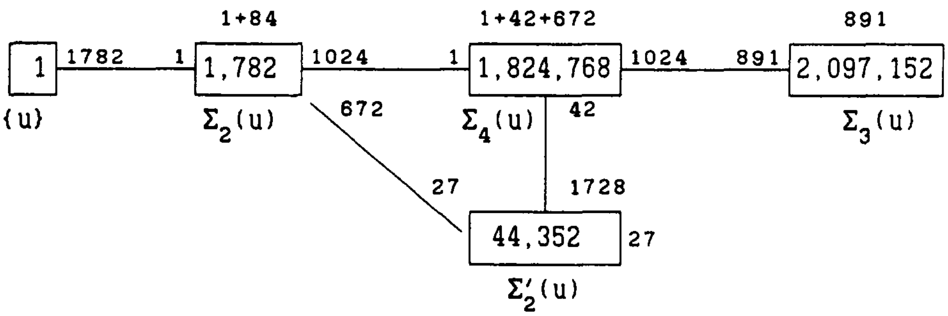


Figure 1.

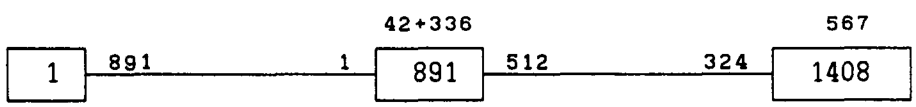


Figure 2.

LEMMA 6.6. *Let $u, w \in \Gamma(v)$. Then u and w are adjacent if and only if $w \in \Sigma_2(u) \cup \Sigma_2'(u)$.*

By Lemma 6.6 the structure of the subgraph of Γ induced by $\Gamma(v)$ is determined uniquely up to isomorphism.

Let $w \in \Sigma_i(u)$ for i =3 or 4 (cf. Figure 1). Then by Lemma 6.6 w is at distance 2 from u and by Lemma 6.3 the order of $u \cdot w$ is either i or $2i$. So we have the following.

LEMMA 6.7. *Subgroup $K(v)$ has exactly two orbits on the set of vertices at distance 2 from v in Γ. If $\Gamma_2^3(v)$ and $\Gamma_2^4(v)$ are these orbits then for $w \in \Gamma_2^i(v)$ the order of the product $v \cdot w$ is either i or $2i$. In addition $K(v, w)$ acts transitively on $\Gamma(v) \cap \Gamma(w)$.*

Let us now study the subgraph of Γ induced by the set $\Pi = \Pi(\alpha_1)$, consisting of the subgeometries that contain the element α_1. As we see above, $O_2(K_1)$ has all orbits of length 2 on Π. Let $\overline{\Pi}$ be the set of these orbits. Then K_1 induces on $\overline{\Pi}$ a primitive rank 3 action of $Co_2 \cong K_1/O_2(K_1)$ with subdegrees 1, 891, and 1408. The intersection diagram of the graph of valency 891 on $\overline{\Pi}$ invariant under K_1 is given on Figure 2.

It is easy to see that the action induced by $O_2(K_1)$ on the union of any two of its orbits on Π is of order 4. This implies the following.

LEMMA 6.8. *The action of K on $\Pi = \Pi(\alpha_1)$ is of rank 4 with the subdegrees 1, 1, 1782, and 2816.*

By our construction, the vertices from the same orbit of $O_2(K_1)$ are adjacent. Let us show that there are more adjacencies in Π. Let S be the stabilizer of an $Sp_8(2)$-subgeometry containing α_1 and Σ be the orbit of length 120 of S on Γ. Then $\Sigma \cap \Pi \neq \emptyset$. An analysis of the 2-parts in the orders of $S \cap K_1$ and E shows that $\Sigma \cap \Pi$ is of size at least 8. Since the subgraph induced by Σ is complete we have the following (compare Lemmas 6.2 and 6.6).

LEMMA 6.9. *The subgraph of Γ induced by Π is of valency 1 +1782.*

Let v and w be nonadjacent vertices from Π. Then by Lemma 6.9 and Figure 2, v and w are at distance 2 from each other. On the other hand, v and w as involutions are contained in $O_2(K_1)$. It follows from the structure of $O_2(K_1)$ that $z = v \cdot w$ is of order 4 and $\{z^2\} = E_1^{\#}$. By Lemma 6.7 $w \in \Gamma_2^4(v)$, whereas by Lemma 6.9 and Figure 2, $\Gamma(v) \cap \Gamma(w) \cap \Pi$ is of size 648. From the structure of K_1 we see that $K_1 \cap K(v,w) \cong 2^{1+20}.U_4(3) : 2^2$. On the other hand, $K(v,w) \leq C_{K(v)}(w) \leq C_{K(v)}(z^2) = K_1 \cap K(v)$. Since $K(v,w)$ is transitive on $\Gamma(v) \cap \Gamma(w)$, by Lemma 6.7 we obtain the following.

LEMMA 6.10. *Let $w \in \Gamma_2^4(v)$. Then there is a unique element α of type 1 in $\mathcal{G}(K)$ such that $\{v, w\} \subset \Pi(\alpha)$. Moreover, $K(v,w) \cong 2^{1+20}.U_4(3) : 2^2$ and this subgroup stabilizes a vertex $u \in \Gamma(v)$ that is equivalent to v in $\Pi(\alpha)$. Set $\Gamma(v) \cap \Gamma(w)$ is contained in $\Pi(\alpha)$ and has cardinality 648.*

Remark. Let v, w, u be as in Lemma 6.10. Then it is easy to see that $O_2(K(v,w))$ and $O_2(K(v,u))$ have the same image in $K(v,u)/\langle v\rangle$. In addition, $U_6(2) : 2 \cong K(v,u)/O_2(K(v,u))$ has a unique conjugacy class of subgroups $U_4(3) : 2^2$. This means that the action of $K(v,w)$ on $\Gamma(v)$ is uniquely determined.

7. Some properties of the Baby Monster graph

In this section we present some properties of the Baby Monster graph $\Gamma(F_2)$. This graph can be obtained by application of the procedure from Section 6 to the case $K \cong F_2$. These results are contained in [6], [15] and, mainly, in [22]. In Section 8 we deduce from the results certain information concerning the structure of the subgraph induced by $\Gamma(v)$ and the action of $K(v)$ on this subgraph. By the results proved above, the subgraph and the action do not depend on the particular choice of the group K satisfying Theorem A.

We denote $\Gamma(F_2)$ by Φ and F_2 by F. The vertices of Φ are involutions of F that form a conjugacy class of $\{3,4\}$-transpositions in F (the class $2A$ in [4]). The term $\{3,4\}$-transpositions means that the product of any two noncommuting

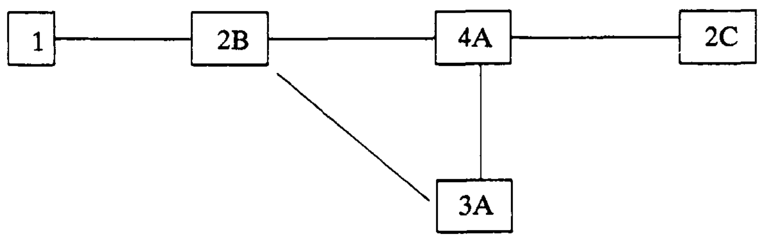


Figure 3.

involutions from the class has order either 3 or 4. Two vertices of Φ are adjacent if their product is a central involution in F ($2B$-involution in [4]). In what follows we do not distinguish vertices of Φ and $\{3,4\}$-transpositions in F.

The permutational rank of F acting on Φ is 5. If (x,y) is a pair of vertices of Φ then the orbital of F containing this pair is uniquely determined by the conjugacy class containing the product $x \cdot y$. Thus there are five possibilities for the product corresponding to classes 1, $2B$, $3A$, $4A$, and $2C$. The corresponding 2-point stabilizers are isomorphic to $2 \cdot {}^2E_6(2) : 2$, $2^{2+20}.U_6(2) : 2$, $Fi_{22} : 2, 2^{1+20}.U_4(3) : 2^2$ and $2^2 \times F_4(2)$, respectively.

A rough structure of Γ with respect to a fixed vertex is given in Figure 3 where we join boxes only if there are edges between vertices in the boxes.

Following [22] for a vertex x of Φ by $2B_x$, $3A_x$, $4A_x$, and $2C_x$ we denote the set of vertices of Γ whose product with x is in the class $2B$, $3A$, $4A$, $2C$, respectively.

LEMMA 7.1. *Let $u \in 3A_v$. Then $F(u,v)$ acting on $2B_v$ has exactly four orbits Δ_i, $1 \le i \le 4$. Moreover, $\Delta_1 \in 2B_u$; $\Delta_2 \cup \Delta_3 \in 3A_u$; $\Delta_4 \in 4A_u$. An information concerning these orbits and the action of $F(u,v)$ on them is given in Table 1 where $L_i = F(u,v,w_i)$ for $w_i \in \Delta_i$, $1 \le i \le 4$.*

Table 1.

i	$\lvert\Delta_i\rvert$	L_i
1	3,510	$2.U_6(2){:}2$
2	142,155	$2^{10}.M_{22}{:}2$
3	694,980	$2^7.Sp_6(2)$
4	3,127,410	$2.(2^9.L_3(4)){:}2$

LEMMA 7.2. *Let* $u \in 4A_v$. *Then* $F(u,v)$ *has on* $2B_v$ *exactly eight orbits* $\mathcal{O}_i$, $1 \leq i \leq 8$, *whose lengths are 648, 8064, 663552, 1, 1134, 36288, 1161216, and 2097152, respectively. If* $\{w\} = \mathcal{O}_4$ *then* $\Sigma_2(w) = \mathcal{O}_1 \cup \mathcal{O}_5$, $\Sigma_2'(w) = \mathcal{O}_2 \cup \mathcal{O}_6$, $\Sigma_4(w) = \mathcal{O}_3 \cup \mathcal{O}_7$, $\Sigma_3(w) = \mathcal{O}_8$. *Here,* $\mathcal{O}_1 = 2B_v \cap 2B_w$, $\mathcal{O}_2 = 2B_v \cap 2C_w$, $\mathcal{O}_3 = 2B_v \cap 3A_w$ *and* $\mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6 \cup \mathcal{O}_7 \cup \mathcal{O}_8 = 2B_v \cap 4A_u$. *If* $w \in \mathcal{O}_2$ *then* $F(u,v,w) \cong 2^{1+14} \cdot (2 \times U_4(2):2)$.

Remarks. Since $\mathcal{O}_2 \in 2C_u$ there are no edges between $\mathcal{O}_2$ and $\mathcal{O}_1$.

LEMMA 7.3. *Let* $u \in 2C_v$. *Then* $F(u,v)$ *has on* $2B_v$ *exactly two orbits* Ξ_1 *and* Ξ_2 *of length 69615 and 3898440, respectively. Moreover,* $\Xi_2 = 2B_v \cap 4A_u$.

We need some results concerning Fischer's group Fi_{22} from [4], which we adapt to the notation of Lemma 7.1.

LEMMA 7.4. *Group* $Fi_{22} : 2$ *acting on* Δ_1 *induces a primitive rank three group with the subdegrees 1, 693, and 2816 and with 2-points stabilizers isomorphic to* $2 \cdot U_6(2){:}2$, $[2^{10}].U_4(2){:}2$, $3 \times U_4(3){:}2$, *respectively. This action is similar to the action by conjugation on the class of 3-transpositions and the subdegree 693 corresponds to commuting transpositions.*

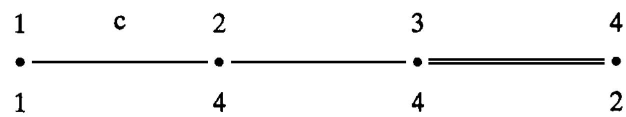
LEMMA 7.5. *Subgroup* L_2 *acting on* Δ_1 *has exactly three orbits whose lengths are* 22, $2^5 \cdot 77$, *and* 2^{10}.

The following lemma is a direct consequence of two previous ones.

LEMMA 7.6. *Subgroup* L_4 *stabilizes at most one point from* Δ_1.

LEMMA 7.7. *Let* Ω *be a undirected graph and* W *be a group acting vertex- and edge-transitively on* Ω. *Suppose that (1) the valency of* Ω *is 693; (2) the stabilizer* $W(x)$ *of a vertex* x *of* Ω *is isomorphic to* $2 \cdot U_6(2) : 2$; *(3)* $W(x)$ *induces on the set* $\Omega(x)$ *of vertices adjacent to* x *a rank 3 action of* $U_6(2) : 2$; *(4) the subgraph induces by* $\Omega(x)$ *has valency 180. Then* $W \cong Fi_{22} : 2$ *and* Ω *is a graph on 3-transpositions of* W.

Proof. The action induced by $W(x)$ on $\Omega(x)$ is similar to the action of $U_6(2) : 2$ on the set of isotropic points of the corresponding unitary space. In the subgraph induced by $\Omega(x)$ two vertices are adjacent if the corresponding points determine an isotropic line. Let $L, P \subseteq \Omega(x)$ be an isotropic line and an isotropic plane, respectively. Let us define a rank 4 geometry $\mathcal{F}$ in which elements of type 1 and 2 are the vertices and the edges of Ω, where elements of type 3 and 4 are the images under W of $L \cup \{x\}$ and $P \cup \{x\}$, respectively. Suppose that incidence relation is defined by inclusion. Then, by the hypothesis of the lemma, $\mathcal{F}$ belongs to the diagram



and that W acts flag-transitively on $\mathcal{F}$. By [16] (see also [19]) the claim of the lemma follows. □

8. The isomorphism $K \cong F_2$

In this section we continue consideration of the graph $\Gamma(K)$ associated with an arbitrary group K satisfying Theorem A. We obtain a considerable information about its structure. This information will enable us to apply known characterizations of the Baby Monster group.

LEMMA 8.1. *Let* $u \in \Gamma_2^3(v)$. *Then* $K(v,u) \cong Fi_{22} : 2$ *and* $|\Gamma(v) \cap \Gamma(u)| = 3510$.

Proof. Let $w \in \Gamma(v) \cap \Gamma(u)$. Then the structure of $\Gamma(v) \cap \Gamma(w) \cap \Gamma(u)$ and the action of $K(v,w,u)$ on this set do not depend on the particular choice of K. So they are as in the Baby Monster case. On the other hand, by Lemma 7.7 in the situation under consideration, the local isomorphism implies the global one. Thus the result follows. □

It is known (Lemma 7 in [17]) that ${}^2E_6(2){:}2$ contains a unique conjugacy class of subgroups isomorphic to $Fi_{22}{:}2$. This means that the action of $K(u,v)$ on $\Gamma(v)$ is uniquely determined. Let Δ_i, $1 \le i \le 4$ be the orbits of $K(u,v)$ on $\Gamma(v)$ as in Lemma 7.1 and $L_i = K(u,v,w_i)$ for $w_i \in \Delta_i$. Notice that $\Delta_1 = \Gamma(v) \cap \Gamma(u)$.

LEMMA 8.2. *In the above notation,* L_1 *has four orbits* Π_i, $1 \le i \le 4$ *on the set of vertices from* $\Gamma(v)$ *that are adjacent to* w_1 *and do not lie in* Δ_1. *The lengths of these orbits are* 891, 891, $693 \cdot 27$, *and* $693 \cdot 36$.

Proof. Consider the set $\Gamma(w_1)$. Then $u \in \Sigma_3(v)$. Since $L_1 \cong 2 \cdot U_6(2) : 2$ we see that L_1 covers $K(w_1,v)/O_2(K(w_1,v))$. Now from Figure 1 one can see that L_1 has two orbits on the points from $\Gamma(w_1)$ collinear to u, both of length 891. Let us consider the action of $K(w_1,v)$ on $\Sigma_2'(v)$. The latter set consists of points that are not collinear to v but lie on a common simplecton with v. $O_2(K(w_1,v))$ preserving each simplecton passing through v, has all orbits of length 64 on $\Sigma_2'(v)$ and on the set of these orbits a primitive action of $U_6(2) : 2$ of degree 693 is induced. The stabilizer of an orbit of $O_2(K(w_1,v))$ induces on this orbit a rank 3 action with the subdegrees 1, 27, 36, and the image of $O_2(K(w_1,v))$ in this action induces a regular elementary abelian normal subgroup. Now, by Lemmas

7.1 and 7.4, L_1 has an orbits of length 693 on the vertices that are adjacent to w_1 and lie in Δ_1. Thus the result follows. □

LEMMA 8.3. *Let $x \in \Gamma(v) - \Delta_1$. Then x is adjacent to a vertex from Δ_1.*

Proof. By Lemma 7.1 $\Gamma(v) - \Delta_1 = \Delta_2 \cap \Delta_3 \cap \Delta_4$. Let Π_i be the orbits of L_1 from Lemma 8.2 and let $x_i \in \Pi_i$, $1 \leq i \leq 4$. Then, clearly, $x_i \in \Delta_{\alpha(i)}$ for some function α. We will show that for each j, $2 \leq j \leq 4$ there is i, $1 \leq i \leq 4$ such that $\alpha(i) = j$. This will imply the claim of the lemma.

Let Ψ_i be the orbit of $L = K(v,u)$ on the set of edges that contains the edge w_1, x_i, $1 \leq i \leq 4$. Let $r(i, \alpha(i))$ be cardinality of the set of edges from Ψ_i containing a fixed vertex from $\Delta_{\alpha(i)}$, $1 \leq i \leq 4$. Then the following equality holds:

$$|\Delta_1| \cdot |\Pi_i| = |\Delta_{\alpha(i)}| \cdot r(i, \alpha(i))$$

By Lemmas 7.1 and 8.2, the integrality condition gives us the following possibilities:

$$\begin{array}{llllll} \alpha(1) = 2 & r(1,2) = 22 & \alpha(1) = 4 & r(1,4) = 1 & & \\ \alpha(2) = 2 & r(2,2) = 22 & \alpha(2) = 4 & r(2,4) = 1 & & \\ \alpha(3) = 2 & r(3,2) = 462 & \alpha(3) = 4 & r(3,4) = 21 & & \\ \alpha(4) = 2 & r(4,2) = 616 & \alpha(4) = 3 & r(4,3) = 126 & \alpha(4) = 4 & r(4,4) = 28. \end{array}$$

By Lemmas 7.5 and 7.6, by the fact that L_4 does not have transitive representations of degree 28 and interchanging, if necessary, the indexes 1 and 2, we come to the unique possibility:

$$\begin{array}{llll} \alpha(1) = 2 & r(1,2) = 22 & \alpha(2) = 4 & r(2,4) = 1 \\ \alpha(3) = 4 & r(3,4) = 1 & \alpha(4) = 3 & r(4,3) = 126 \end{array}$$

So the proof is done. □

As a consequence of the above lemma, we have the following.

LEMMA 8.4. *Let $u \in \Gamma_2^3(v)$ and $w \in \Gamma(u)$. Then w is at distance at most 2 from v.*

Let $u \in \Gamma_2^4(v)$. Then $K(v,u) \cong 2^{1+20} \cdot U_4(3){:}2^2$ and it has eight orbits on $\Gamma(v)$. Let $\mathcal{O}_i$, $1 \leq i \leq 8$ be these orbits as in Lemma 7.2. Let $y \in \Gamma(v)$. We will show that unless $y \in \mathcal{O}_2$ the distance between u and y is at most 2. Let w be the vertex from $\Gamma(v)$ such that $\{w\} = \mathcal{O}_4$. The graph of valency 1782 on $\Gamma(v)$ as on Figure 1 will be denoted by Ω.

LEMMA 8.5. *Let $y \in \mathcal{O}_i$ for i=4, 5, or 6. Then in the subgraph of Γ induced by $\Gamma(v)$ the vertex y is adjacent to a vertex from $\mathcal{O}_1$.*

Proof. If i=4 then the claim is obvious since $\mathcal{O}_1 \in \Sigma_2(w)$.

If i=5 then the claim follows from the fact that the subgraph of Ω induced by $\Sigma_2(w) = \mathcal{O}_1 \cup \mathcal{O}_5$ is connected.

Let i=6. A vertex from $\Sigma_2'(w)$ is adjacent in Ω to 27 vertices from $\Sigma_2(w)$ (cf. Figure 1). By the remark after Lemma 7.2 there are no edges between $\mathcal{O}_2$ and $\mathcal{O}_1$. Now an easy counting shows that $y \in \mathcal{O}_6$ is adjacent to 15 vertices from $\mathcal{O}_5$ and to 12 vertices from $\mathcal{O}_1$. So the result follows. □

LEMMA 8.6. *Let $y \in \mathcal{O}_7 \cup \mathcal{O}_8$. Then there is a vertex x from $\mathcal{O}_1$ such that $x \cdot y$ is of order 3.*

Proof. It is easy to see from Figure 1 that for vertices a and b of Ω the product $a \cdot b$ is of order 3 if and only if a and b are at distance 3 in Ω. Let $y \in \mathcal{O}_8$. Then one can see from Figure 1 that y is at distance 3 from exactly half of the vertices 3 from $\Sigma_2(w)$. Since $K(v,w,y)$ covers $K(v,w)/O_2(K(v,w))$, the vertices at distance 3 from y are in distinct orbits of $O_2(K(v,w))$ on $\Sigma_2(w)$. On the other hand, such an orbit is either contained in $\mathcal{O}_1$ or disjoint from $\mathcal{O}_1$. This implies that exactly half of the vertices from $\mathcal{O}_1$ are at distance 3 from y. Thus the claim is proved for $y \in \mathcal{O}_8$.

For a vertex $x \in \mathcal{O}_1$ there are $|\mathcal{O}_8| = 2{,}097{,}152$ vertices in Ω that are at distance 3 from x. By the above paragraph one-half of them is contained in $\mathcal{O}_8$. Suppose that there no vertices in $\mathcal{O}_7$ that are at distance 3 from x. Since the union of $\mathcal{O}_i$ for $1 \le i \le 6$ has size less than $|\mathcal{O}_8|/2$ this leads to a contradiction. Thus the proof is complete. □

Now Lemmas 8.5 and 8.6 imply the following.

LEMMA 8.7. *Let $u \in \Gamma_2^4(v)$ and $y \in \Gamma(v)$. Then either $y \in \mathcal{O}_2$ or the distance between u and y is less than 3.*

Let us show now that Γ contains vertices at distance 3 from each other.

Let $u \in \Gamma_2^4(v)$, $\mathcal{O}_i$ be the orbits of $K(u,v)$ on $\Gamma(v)$, $1 \le i \le 8$, $\{w\} = \mathcal{O}_4$ and Π be the subgraph containing the pair $\{u,v\}$ (compare Lemma 6.10). Let $x \in \mathcal{O}_2$. Let us determine the number of vertices in Π adjacent to x. By Lemma 6.10 $\Gamma(x) \cap \Pi$ is contained in $\{v\} \cup \Gamma(v)$. By consideration of the subgraph induced by $\Gamma(v)$ it is easy to show that x is adjacent exactly to those vertices of $\Pi - \{v\}$ that are contained in the (unique) simplecton passing through w and x. This means that $|\Gamma(x) \cap \Pi| = 56$. Let us show that Π contains a vertex z such that $[x,z] = 1$ and x is not adjacent to z. Without loss of generality we assume that Π is stabilized by K_1. The subgroup $O_2(K_1)$ stabilizes $\Sigma_2'(w)$ as a whole and all its orbits on this set are of length 64. This means that at least 4600/64 involutions from Π commute with x. Since the given number is greater than 56, the result follows. It is easy to see that z should be at distance 3 from x. Now, in view of Lemma 8.7, we have the following.

LEMMA 8.8. *$K(v)$ has a unique orbit $\Gamma_3(v)$ on the set of vertices at distance 3 from v in Γ. A vertex from $\Gamma_2^4(v)$ is adjacent to 8064 vertices from $\Gamma_3(v)$. If $u \in \Gamma_3(v)$ then u is an involution commuting with v and $K(v,u)$ acts transitively on $\Gamma(v) \cap \Gamma_2^4(u)$.*

Let $u \in \Gamma_3(v)$ and $w \in \Gamma_2^4(v) \cap \Gamma(u)$. Then u is adjacent to 1782+44352 vertices from $\Gamma(w)$ and at most 8063 of them are in $\Gamma_3(v)$. Thus we have the following.

LEMMA 8.9. *Let $u \in \Gamma_3(v)$. Then the subgraph induced by $\Gamma(v) \cap \Gamma_2^4(u)$ has valency at least 1782 +44352-8063.*

Now we are in a position to determine $K(v,u)$. Since u commutes with v, $K(u,v) \leq C_{K(v)}(u)$. On the other hand, by Lemma 7.2, the stabilizer of a vertex in the action of $K(u,v)$ on $\Gamma(v) \cap \Gamma_2^4(u)$ is isomorphic to $2^{1+14}.(2 \times U_4(2){:}2)$. Now by the structure of involution centralizers in $2 \cdot {}^2E_6(2){:}2$ (cf. [22]) and description of the maximal subgroups in $F_4(2)$ (cf. [4],[18]) we have the following (compare Lemma 7.3).

LEMMA 8.10. *Let $u \in \Gamma_3(v)$. Then $K(u,v) \cong 2^2 \times F_4(2)$ and it has exactly two orbits on the set $\Gamma(v)$.*

Now, by Lemmas 8.4, 8.7, and 8.10, we see that diameter of Γ is exactly 3 and that the adjacency structure between the orbits of $K(v)$ on Γ is as given in Figure 3. This implies in particular that distinct vertices of Γ correspond to distinct involutions and hence $K(v) = C_K(v)$.

Now it is not so difficult to show that K is nonabelian simple. First, notice that the action of K on Γ is primitive. Really, the orbits of $K(v)$ on $\Gamma - \{v\}$ are $\Gamma(v) = \Gamma_1(v)$, $\Gamma_2^3(v)$, $\Gamma_2^4(v)$, and $\Gamma_3(v)$. It follows from the construction of Γ (cf. Lemmas 6.5, 6.6, 6.7, 8.10, and Figure 1) that each of these orbits contains a pair of vertices that are adjacent in Γ. On the other hand Γ is connected. Let N be a proper normal subgroup of K. Then N is transitive on Γ as a nontrivial normal subgroup of a primitive group. Since the number of vertices of Γ is even, the order of N is even as well. Let $K = E \cong 2 \cdot {}^2E_6(2){:}2$ and $K_1 \cong 2^{1+22} \cdot Co_2$ be subgroups of K as above, i.e., $E \cap K_1 \cong 2^{2+20}.U_6(2) : 2$. Consider the intersections of N with E and K_1. Since K_1 contains a Sylow 2-subgroup of K and the order of N is even, we see that $N \cap K \neq 1$. But the structure of K_1 implies that any its nontrivial normal subgroup has a nontrivial intersection with $K_1 \cap E$. Hence $N \cap E \neq 1$. Also, $N \cap E \neq E$ since otherwise N would coincide with K due to its transitivity on Γ. The structure of E implies that it has exactly two proper normal subgroups. Namely, if $M = E \cap N$ then either $M = Z(E)$ (the order 2 center) or $M = E'$ (the index 2 commutant). But, in any case, the normal closure of $M \cap K_1$ in K_1 being intersected with E, contains $M \cap K_1$ as a proper subgroup. This is a contradiction and so we have the following.

PROPOSITION 8.11. *Let K be a group satisfying the hypothesis of Theorem A. Then:*

(i) *K is nonabelian simple,*
(ii) $|K| = |F_2|$,
(iii) $K_1 \cong 2^{1+22}.Co_2$ *and* $E \cong 2 \cdot {}^2E_6(2):2$ *are full involution centralizers in K.*

Now we have two possibilities to show that $K \cong F_2$. The first one is to apply a characterization of the maximal parabolics amalgams corresponding to flag-transitive actions on rank 5 P-geometries [24]. The second possibility is just to apply the characterizations of F_2 by the centralizers of involutions [1], [22], [25].

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Reference

1. J. Bierbrauer, "A characterization of the Baby monster group F_2, including a note on ${}^2E_6(2)$," *Journal of Algebra*, vol. 56, pp. 384–395, 1979.
2. F. Buekenhout, "Diagram geometries for sporadic groups," *Contemporary Mathematics*, vol. 45, pp. 1–32, 1985.
3. J.H. Conway, "Three lectures on exceptional groups," in M.B. Powell and G. Higman (eds.), *Finite Simple Groups*, Academic Press, New York, pp. 215–247, 1971.
4. J. Conway et al.: *Atlas of Finite Groups*, Oxford University Press, New York, 1985.
5. R.L. Griess, "The Friendly Giant," *Invent. Mathematics*, vol. 69, 1982, pp. 1–102.
6. D.G. Higman, "A monomial character of Fischer's Baby Monster," in F. Gross (ed.) *Proc. Conf. Finite Groups*, Academic Press, New York, pp 277–283, 1976.
7. A.A. Ivanov, "On 2-transitive graphs of girth 5," *European Journal of Combinatorics*, vol. 8, pp. 393–420, 1987.
8. A.A. Ivanov, "Graphs of girth 5 and diagram geometries related to the Petersen graph," *Soviet Mathematika Doklady*, vol. 36, pp. 83–87, 1988.
9. A.A Ivanov, "A presentation for J_4," *Proc. London Mathematical Society* (3), vol. 64, pp. 369–396, 1992.
10. A.A Ivanov, "A geometric characterization of the Monster," *Proc. Symp. "Groups and Combinatorics,"* Durham, NC, July 1990 (to appear).

11. A.A Ivanov, "Geometric presentations of groups with an application to the Monster," *Proc. ICM-90,* Kyoto, Japan, 1990, vol. 2, pp. 385–395, Springer-Verlag, 1991.
12. A.A Ivanov, "A characterization of the P-geometry $\mathcal{G}(F_2)$," in *2nd Japan Conference on Graph Theory and Combinatorics,* Hakone, Japan, pp. 1–2, August 1990.
13. A.A. Ivanov and S.V. Shpectorov, "Geometries for sporadic groups related to the Petersen graph. II," *European Journal of Combinatorics,* vol. 10, pp. 347–361, 1989.
14. A.A Ivanov and S.V. Shpectorov, "Universal representations of P-geometries from F_2-series," *Journal of Algebra* (to appear).
15. J.S. Leon and C.C. Sims, "The existence and uniqueness of a simple group generated by $\{3,4\}$-transpositions," *Bulletin of the American Mathematics Society,* vol. 83, pp. 1039–1040, 1977.
16. T. Meixner, "Some polar towers," *European Journal of Combinatorics*, vol. 12, pp. 397–416, 1991.
17. S.P. Norton, "Constructing the Monster," *Proc. Sump. Groups and Combinatorics,* Durham, NC, July 1990 (to appear).
18. S.P. Norton and R.A. Wilson, "The maximal subgroups of $F_4(2)$ and its automorphism group," *Communications in Algebra,* vol. 17, pp. 2809–2824, 1989.
19. A. Pasini, "Flag-transitive Buekenhout geometries," in Combinatorics 90: Recent trends and applications, Annals of Discrete Math., Elsevier Science Publ. (to appear)
20. M.A. Ronan and S. Smith, "2-local geometries for some sporadic groups," in *Proc. Symp. Pure Mathematics*, no. 37, Santa Cruz, CA, 1979.
21. M.A. Ronan and G. Stroth, "Minimal parabolic geometries for the sporadic groups," *European Journal of Combinatorics,* vol. 5, pp. 59–91, 1984.
22. Y. Segev, "On the uniqueness of Fischer's Baby Monster," Preprint, 1990.
23. S.V. Spectorov, "On geometries with the diagram P^n," Preprint, 1988.
24. S.V. Spectorov, Private communication, 1990.
25. G. Stroth, "A characterization of Fischer's sporadic group of order $2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$," *Journal of Algebra,* vol. 40, pp. 499–531, 1976.
26. J. Tits, "Buildings of spherical type and finite BN-pairs," *Lecture Notes Mathematics,* vol. 386, Springer-Verlag, New York, 1974.
27. J. Tits, "A local approach to buildings," in *The Geometric Vein,* Springer-Verlag, New York, pp. 519–547, 1981.
28. R.A. Wilson, "The maximal subgroups of Conway's group .2," *Journal of Algebra,* vol. 84, pp. 107–114, 1983.