Journal of Algebraic Combinatorics 1 (1992), 275–282


# Half-Transitive Graphs of Prime-Cube Order

MING-YAO XU
*Institute of Mathematics, Peking University, Beijing 100871, China*

*Received November 14, 1991; Revised June 16, 1992.*

**Abstract.** We call an undirected graph $X$ *half-transitive* if the automorphism group Aut $X$ of $X$ acts transitively on the vertex set and edge set but not on the set of ordered pairs of adjacent vertices of $X$. In this paper we determine all half-transitive graphs of order $p^3$ and degree 4, where $p$ is an odd prime; namely, we prove that all such graphs are Cayley graphs on the non-Abelian group of order $p^3$ and exponent $p^2$, and up to isomorphism there are exactly $(p-1)/2$ such graphs. As a byproduct, this proves the uniqueness of Holt's half-transitive graph with 27 vertices.

**Keywords:** half-transitive graphs, Cayley graphs, simple groups, Schur multiplier

## 1. Introduction

Let $X = (V(X), E(X))$ be a simple undirected graph. We call an ordered pair of adjacent vertices an *arc* of $X$. Let $G$ be a subgroup of Aut $X$. $X$ is said to be *G-vertex-transitive, G-edge-transitive,* or *G-arc-transitive* if $G$ acts transitively on the set of vertices, edges, or arcs of $X$, respectively. $X$ is said to be *vertex-transitive, edge-transitive,* or *arc-transitive* if it is Aut $X$-vertex-transitive, Aut $X$-edge-transitive, or Aut $X$-arc-transitive, respectively. We call a graph $X$ *half-transitive*, or *$\frac{1}{2}$-transitive*, if it is vertex-transitive and edge-transitive but not arc-transitive.

The first examples of half-transitive graphs were found in 1970 by Bouwer [5], who found an infinite family of them. The smallest graph in his family has 54 vertices. In 1981 Holt [10] found an example with 27 vertices and degree 4. Recently, Alspach *et al.* [2] proved that Holt's graph is the smallest $\frac{1}{2}$-transitive graph in the sense that there are no $\frac{1}{2}$-transitive graphs with fewer than 27 vertices or with degree less than 4. In [2] they asked, how many $\frac{1}{2}$-transitive graphs of order 27 and degree 4 are there up to isomorphism? We shall give an answer to that question in this paper. To speak precisely, the purpose of this paper is to determine all $\frac{1}{2}$-transitive graphs of order $p^3$ and degree 4, where $p$ is an odd prime. We shall prove that there are exactly $(p-1)/2$ half-transitive graphs of order $p^3$ and degree 4 up to isomorphism. Thus if $p = 3$, there is only one such graph, so Holt's graph is only the smallest half-transitive graph. Moreover, our graphs are all metacirculants as defined in [3]. This supports a conjecture of Alspach and Marušič [1] which claims that every half-transitive graph of degree 4 is a metacirculant.

The group- and graph-theoretic notation and terminology used here are generally standard, and the reader can refer to [9] and [11] when necessary. Two adjacent vertices $u$ and $v$ in $X$ are denoted by $u \sim v$ or $uv \in E(X)$. For $v \in V(X)$, $X_1(v)$ denotes the neighborhood of $v$ in $X$, that is, the set of vertices adjacent to $v$ in $X$.

Let $G$ be a finite group, and let $S$ be a subset of $G$ not containing the identity element 1. The Cayley *digraph* $X = X(G, S)$ on $G$ with respect to $S$ is defined by

$$\begin{aligned} V(X) &= G, \\ E(X) &= \{(g, sg) \mid g \in G, s \in S\}. \end{aligned}$$

If $S^{-1} = S$, then $X(G, S)$ can be viewed as an undirected graph, identifying an undirected edge $gh$ with two arcs $(g, h)$ and $(h, g)$. This graph is called the Cayley *graph* on $G$ with respect to $S$. It is well known that any Cayley digraph on $G$ is vertex-transitive and its automorphism group contains the right regular representation $R(G)$ of $G$.

From elementary group theory we know that up to isomorphism there are only five groups of order $p^3$, that is, three Abelian groups $Z_{p^3}$, $Z_{p^2} \times Z_p$, and $Z_p \times Z_p \times Z_p$, where $Z_n$ denotes the cyclic group of order $n$, and two non-Abelian groups $G_1(p)$ and $G_2(p)$ defined as

$$G_1(p) = \langle a, b \mid a^{p^2} = 1, b^p = 1, b^{-1}ab = a^{1+p} \rangle \tag{1}$$

and

$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, [b, a] = c, [c, a] = [c, b] = 1 \rangle. \tag{2}$$

The $\frac{1}{2}$-transitive graphs of order $p^3$ and degree 4 we found are Cayley graphs on $G_1(p)$ with respect to four-element sets $S_i = \{b^i a, b^i a^{-1}, (b^i a)^{-1}, (b^i a^{-1})^{-1}\}$ for $1 \leq i \leq (p-1)/2$. These Cayley graphs will be denoted by $\Gamma_i(p)$, that is, $\Gamma_i(p) = X(G_1(p), S_i)$. Holt's graph is $\Gamma_1(3)$ in our notation. Since $G_1(p)$ is metacyclic, all Cayley graphs on $G_1(p)$ are metacirculants.

The main result of this paper is the following theorem:

THEOREM 1.1. *The $\Gamma_i(p)$ defined above are $\frac{1}{2}$-transitive graphs of order $p^3$ and degree 4, and any $\frac{1}{2}$-transitive graph of order $p^3$ and degree 4 is isomorphic to a $\Gamma_i(p)$.*

As a consequence of this theorem we have the following corollary:

COROLLARY 1.1. *Up to isomorphism there is only one $\frac{1}{2}$-transitive graph of order 27 and degree 4, namely, Holt's graph.*

## 2. Quoted and Preliminary Results

In this section we list some quoted and preliminary results which we need in the proof of Theorem 1.1.

LEMMA 2.1 ([12]). *Any vertex-transitive graph of order* $p^3$ *is a Cayley graph on a group of order* $p^3$.

LEMMA 2.2 ([2]). *Every edge-transitive Cayley graph on an Abelian group is also arc-transitive.*

LEMMA 2.3 *Any edge-transitive Cayley graph on* $G_2(p)$ *of degree 4 is also arc-transitive.*

*Proof.* Let $X = X(G_2(p), S)$ be an edge-transitive Cayley graph on $G_2(p)$ with respect to $S = \{x, x^{-1}, y, y^{-1}\}$. If $\langle x, y\rangle < G_2(p)$, then every component of $X$ is an edge-transitive graph with fewer vertices. Thus it must be a Cayley graph on a group of order dividing $p^2$, hence on an Abelian group. Therefore it is arc-transitive. Thus we may assume that $\langle x, y\rangle = G_2(p)$.

It is easy to verify that $x$ and $y$ and also $x^{-1}$ and $y^{-1}$ satisfy the same relations as do $a$ and $b$. It follows that the mapping $\sigma : x \longmapsto x^{-1}, y \longmapsto y^{-1}$ is an automorphism of $G_2(p)$. Hence $\sigma R(x)$, where $R(x)$ is the right multiplication transformation by $x$, is a graph automorphism mapping the arc $(1, x)$ to the arc $(x, 1)$. Since $X$ is edge-transitive, $X$ also is arc-transitive. □

LEMMA 2.4 ([7]) . *Let* $X(G, S)$ *be the Cayley graph on* $G$ *with respect to a subset* $S$, *and let* $A$=Aut $X$. *Let* $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}\, G | S^\alpha = S\}$. *Then the normalizer* $N_A(R(G))$ *of* $R(G)$ *in* $A$ *is the semidirect product of* $R(G)$ *by* $\mathrm{Aut}(G, S)$.

LEMMA 2.5 ([4]). *Let* $X(G, S)$ *and* $X(G, T)$ *be two connected Cayley graphs on a p-group* $G$ *with respect to subsets* $S$ *and* $T$, *and let* $|S| = |T| < p$. *Then* $X(G, S)$ *and* $X(G, T)$ *are isomorphic if and only if there is an automorphism* $\alpha$ *of* $G$ *such that* $S^\alpha = T$.

We call a group $G$ a *central extension* of $C$ by $T$ if $C \leq Z(G)$ and $G/C \cong T$. The following result is a consequence of the finite simple group classification.

LEMMA 2.6 *Assume that a non-Abelian simple group* $T$ *has order* $2^m 3^n p^l$, *where* $p > 3$, *a prime. Then* $l = 1$, *and* $p$ *does not divide the order of* $\mathrm{Out}\,T = \mathrm{Aut}\,T/\mathrm{Inn}\,T$, *the outer automorphism group of* $T$. *Moreover, if a group* $G$ *is a central extension of* $C \cong Z_p$ *by such a simple group* $T$, *then* $G \cong C \times T$.

*Proof.* By [8, pp. 12–14] $T$ is one of the following groups: $A_5$, $A_6$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$. Checking the atlas [6] for all groups listed above, we have $l = 1$, and $p$ does not divide the order of Out $T$.

Also by [6] we have that for all these simple groups $p$ does not divide the order of their Schur multiplier. Then we get the last assertion of this lemma. □

LEMMA 2.7 *Let $G$ be a finite group, and let $|G| = 2^m 3^n p^3$, where $p > 3$ is a prime. Assume that a Sylow $p$-subgroup $P$ of $G$ is non-Abelian and of exponent $p^2$ and that $O_{p'}(G) = 1$. Then $P \lhd G$.*

*Proof (L.G. Kovács).* (i) $G$ has no subnormal subgroup which is non-Abelian simple, and hence $O_p(G) > 1$: If $G$ had such a subnormal subgroup $T_1 \cong T$, where $T$ is non-Abelian simple, then by Lemma 2.6 $p \parallel |T|$ and $p \nmid |\text{Out } T|$. The normal closure $T_1^G$ of $T_1$ would be generated by some $G$-conjugates of $T_1$, all of which are subnormal in $G$ and hence in $T_1^G$. It follows that the only composition factors of $T_1^G$ are $T$, and hence $T_1^G$ has the form

$$T_1^G = T_1 \times \ldots \times T_k,$$

where $T_i \cong T$ is conjugate to $T_1$ for any $i$. Since $p^3 \parallel |G|$, we have $k \leq 3$, implying $|G : N_G(T_1)| \leq 3$. Since $N_G(T_1)/T_1 C_G(T_1)$ is isomorphic to a subgroup of Out $T_1$ and $p \nmid |\text{Out } T_1|$, we have $p^3 \mid |T_1 C_G(T_1)|$. Then Sylow $p$-subgroups of $T_1 C_G(T_1)$ are also Sylow $p$-subgroups of $G$. Since $T_1 \cap C_G(T_1) = 1$, $T_1 C_G(T_1) = T_1 \times C_G(T_1)$. Take $P_1 \in \text{Syl}_p(T_1)$ and $P_2 \in \text{Syl}_p(C_G(T_1))$. We have that $P_1 \times P_2$ is conjugate to $P$, contradicting the assumption that $P$ is non-Abelian.

Thus we have proved that $G$ has no minimal normal subgroup which is insoluble, so $O_p(G) > 1$ since $O_{p'}(G) = 1$.

If $O_p(G) = P$, then $P \lhd G$ and we are done. Thus we may assume that $O_p(G) < P$. Then $O_p(G)$ has order $p$ or $p^2$, and hence $O_p(G)$ is Abelian. We have the following:

(ii) $C = C_G(O_p(G)) = O_p(G)$: Assume that $C > O_p(G)$. First we claim that $C/O_p(G)$ has no nontrivial normal $p'$-subgroup. If it had one, say, $M/O_p(G)$, taking the complement $K$ of $O_p(G)$ in $M$ would produce $M = O_p(G) \times K$; then $K \leq O_{p'}(G) = 1$, a contradiction. Thus $C/O_p(G)$ has a normal subgroup which is a direct product of isomorphic non-Abelian simple groups. It follows that $C$ has a subnormal subgroup which is a central extension of $O_p(G)$ by a simple group $T$. By Lemma 2.6 this extension is a direct product of $O_p(G)$ and $T$. (If $|O_p(G)| = p^2$, taking a subgroup $N$ of $O_p(G)$ of order $p$, Lemma 2.6 gives $C/N = O_p(G)/N \times \overline{T}/N$, where $\overline{T}$ is a central extension of $Z_p$ by $\overline{T}$. Then use Lemma 2.6 again.) It follows that $C$, and then $G$, has a subnormal subgroup which is non-Abelian simple, contradicting (i).

(iii) The completion of the proof is as follows: By (ii) $G/O_p(G)$ is isomorphic to a subgroup of Out $O_p(G)$. If $O_p(G)$ is cyclic, then the Sylow $p$-subgroup of Out $O_p(G)$ is normal, implying that the Sylow $p$-subgroup of $G/O_p(G)$, and then of $G$, is normal, contradicting our assumption. Thus $O_p(G) \cong Z_p \times Z_p$, and $G/O_p(G)$ is isomorphic to a subgroup $L$ of $GL(2, p)$. Assume that $\overline{L} = (L \cap SL(2, p))/(L \cap Z(SL(2, p)))$. Then $\overline{L}$ is a subgroup of $PSL(2, p)$, whose Sylow

$p$-subgroup is nontrivial and nonnormal. By Dickson's theorem [11, II.8.27] the only subgroup with these properties is $PSL(2, p)$ itself. Therefore $L \geq SL(2, p)$. Write $R = O_{p,p'}(G)$, which is defined by $O_{p,p'}(G)/O_p(G) = O_{p'}(G/O_p(G))$. We have $R > O_p(G)$. Let $Q$ be a complement of $O_p(G)$ in $R$. By the Frattini argument $G = RN_G(Q) = O_p(G)N_G(Q)$. Write $U = N_G(Q) \cap O_p(G)$, and note that $U$ and $Q$ commute elementwise, so $U \leq O_p(Z(R))$. Obviously, $O_p(Z(R))$ is normal in $G$. From $C_G(O_p(G)) = O_p(G) < R$ we know that $O_p(Z(R)) < O_p(G)$. Since $L \geq SL(2, p)$, $O_p(G)$ is minimal normal in $G$; so the only option is that $U = 1$, and therefore $O_p(G)$ is complemented in $G$. However, we know that $O_p(G)$ is not complemented in $P$, a contradiction. □

The final lemma gives information about the automorphism group of the group $G_1(p)$, defined in Section 1.

LEMMA 2.8 Aut $G_1(p)$ *has order* $p^3(p-1)$ *and is generated by the four automorphisms* $\alpha$, $\beta$, $\gamma$, *and* $\delta$ *defined by*

$$\begin{aligned} \alpha &: a \longmapsto a, & b &\longmapsto ba^p, \\ \beta &: a \longmapsto a^{1+p}, & b &\longmapsto b, \\ \gamma &: a \longmapsto ba, & b &\longmapsto b, \\ \delta &: a \longmapsto a^{\varepsilon}, & b &\longmapsto b, \end{aligned}$$

*where* $\varepsilon$ *is an element of order* $p-1$ *in* $\mathbf{Z}^*_{p^2}$, *the group of units in the ring* $\mathbf{Z}_{p^2}$ *of integers modulo* $p^2$. *Moreover, we may write* $P = \langle \alpha, \beta, \gamma \rangle$ *and* $H = \langle \delta \rangle$, *where* $P$ *is the normal Sylow p-subgroup of* Aut $G_1(p)$ *and* $H$ *is the complement of* $P$ *in* Aut $G_1(p)$. *Furthermore, all p-complements in* Aut $G_1(p)$ *are conjugate. In particular,* Aut $G_1(p)$ *has one class of involutions.*

*Proof.* Direct calculation shows that $\alpha$, $\beta$, $\gamma$, and $\delta$ are automorphisms of $G_1(p)$ and that $\langle \alpha, \beta \rangle = \text{Inn}\ G_1(p)$, the inner automorphism group of $G_1(p)$.

Now assume that $\tau$ is an arbitrary automorphism of $G_1(p)$:$a^\tau = b^i a^j$, $b^\tau = b^k a^s$. Since $a^\tau$ has order $p^2$ and $b^\tau$ has order $p$, we have $p \nmid j$ and $p | s$. Since $(b^\tau)^{-1} a^\tau b^\tau = (a^\tau)^{1+p}$ and since $a^p \in Z(G_1(p))$ and $G_1(p)$ is $p$-Abelian, i.e., $(xy)^p = x^p y^p$ for all $x, y \in G_1(p)$, we have

$$\begin{aligned} b^{-k}(b^i a^j) b^k &= (b^i a^j)^{1+p}, \\ b^i a^{j(1+p)^k} &= b^i a^{j(1+p)}. \end{aligned}$$

So $j(1+p)^k \equiv j(1+p) \pmod{p^2}$, implying $k \equiv 1 \pmod{p}$. Thus we may assume $a^\tau = b^i a^j$, $b^\tau = ba^{tp}$, where $i$ and $t$ have at most $p$ choices and $j$ has $p^2 - p$ choices. So $|\text{Aut}\ G_1(p)| \leq p^3(p-1)$. However, $|\langle \alpha, \beta, \gamma, \delta \rangle| = p^3(p-1)$, implying Aut $G_1(p) = \langle \alpha, \beta, \gamma, \delta \rangle$.

The remaining assertions of this lemma are obvious. □

### 3. Proof of Theorem 1.1

Let $X$ be a $\frac{1}{2}$-transitive graph of order $p^3$ and degree 4. By Lemmas 2.1 and 2.3, $X$ is a Cayley graph on $G_1(p)$ with respect to a four-element set $T = \{x, x^{-1}, y, y^{-1}\}$. If $X$ is not connected, then its components would be Cayley graphs on a group of order $p$ or $p^2$, hence on an Abelian group. By Lemma 2.2 $X$ would be arc-transitive, a contradiction. Thus, $X$ is connected. It follows that $\langle x, y\rangle = G_1(p)$.

Now we assume that $X = X(G_1(p), T)$ is a Cayley graph on $G_1(p)$ with respect to $T = \{x, x^{-1}, y, y^{-1}\}$ and that $\langle x, y\rangle = G_1(p)$. First we shall determine the full automorphism group $A = \text{Aut } X$ and find conditions under which $X$ is $\frac{1}{2}$-transitive. All of these will be given in the following three Facts.

FACT 3.1. $\text{Aut}(G_1(p), T)$ *has order at most 2, and if* $|\text{Aut}(G_1(p), T)| = 2$, *the nontrivial element in* $\text{Aut}(G_1(p), T)$ *interchanges* $x$ *and* $y$.

*Proof.* Recall that $\text{Aut}(G_1(p), T)$ is the subgroup of $\text{Aut } G_1(p)$ whose elements fix $T$ setwise. Since $\langle T\rangle = G_1(p)$, an automorphism of $G_1(p)$ fixing $T$ pointwise must be the identity. It follows that $\text{Aut}(G_1(p), T)$ acts on $T$ faithfully. So $\text{Aut}(G_1(p), T)$ is isomorphic to a subgroup of the symmetric group $S_4$.

Further, $\text{Aut}(G_1(p), T)$ has no element of order 3. If it had such an element, say, $\tau$, then $\tau$ would have a fixed point and an orbit of length 3 in $T$. Assume that $x$ is the fixed point, i.e., $x^\tau = x$. We would have that $x^{-1}$, which is in the orbit of length 3, is also a fixed point of $\tau$, a contradiction. Now we have proved that $\text{Aut}(G_1(p), T)$ is a 2-group. By Lemma 2.8 $\text{Aut}(G_1(p), T)$ is a subgroup of some $p$-complement conjugate to $H$ in $\text{Aut } G_1(p)$. Since $H$ is cyclic of order $p-1$, to complete the proof of this fact it suffices to prove that $\text{Aut}(G_1(p), T)$ has no element of order 4.

Assume that $\sigma$ is an automorphism of $G_1(p)$ of order 4. $\sigma$ acts on $T$ cyclically. If $x^\sigma = x^{-1}$, then $(x^{-1})^\sigma = x$, and this is not the case. So we may assume that the action of $\sigma$ on $T$ is $\sigma : x \longmapsto y \longmapsto x^{-1} \longmapsto y^{-1} \longmapsto x$. It follows that $x^{\sigma^2} = x^{-1}$, $y^{\sigma^2} = y^{-1}$. Since all involutions in $\text{Aut } G_1(p)$ are conjugate, we may assume that $\sigma^2 \in H$. However, $H$ fixes a subgroup $\langle b\rangle$, which is not in the Frattini subgroup $\Phi(G_1(p))$, but $\sigma^2$ has no fixed subgroup of order $p$ in the Frattini factor group $G_1(p)/\Phi(G_1(p))$; this is a contradiction.

The above argument also shows that if $\text{Aut}(G_1(p), T) \neq 1$, then its nontrivial element interchanges $x$ and $y$. □

For convenience, in what follows we identify the right regular representation $R(G)$ and $G$ itself. The reader can infer the meaning from the context.

FACT 3.2. $A = G_1(p)\text{Aut}(G_1(p), T)$ *and* $G_1(p)$ *is the normal Sylow p-subgroup of* $A$.

*Proof.* We use $A_1$ to denote the stabilizer of the vertex 1 in $A = \text{Aut } X$. Assume that $\tau \in A_1$ is an element of prime order, say, $q$. If $q > 4$, then $\tau$ must fix the neighborhood $X_1(1)$ of 1 pointwise. By the connectedness of $X$, $\tau$ fixes all vertices of $X$; then $\tau = 1$, a contradiction. Thus $q \nmid |A_1|$. It follows that $|A_1| = 2^m 3^n$ and $|A| = 2^m 3^n p^3$.

If $p > 3$, we have $G_1(p) \in \text{Syl}_p(A)$. Since $A_1$ is core-free, $O_{p'}(A) = 1$. Then Lemma 2.7 gives $G_1(p) \lhd A$ and Lemma 2.4 gives $A = G_1(p)\text{Aut}(G_1(p), T)$.

If $p = 3$ and $G_1(3) < Q \in \text{Syl}_3(A)$, then $N_A(G_1(3)) \geq N_Q(G_1(3)) > G_1(3)$. By Lemma 2.4, $N_A(G_1(3)) = G_1(3)\text{Aut}(G_1(3), T)$, so 3 divides the order of $\text{Aut}(G_1(3), T)$, contradicting Fact 3.1. So we have $G_1(3) \in \text{Syl}_3(A)$ and $|A| = 2^m 3^3$ for some $m$. By Lemma 2.4, to complete the proof it suffices to show that $G_1(3) \lhd A$. In this case $A$ is soluble by Burnside's famous $p^a q^b$ theorem [11, V. 7.3]. Since $|A_1| = 2^m$ and $A_1$ is core-free, $O_2(A) = 1$. It follows that $O_3(A) > 1$. If $O_3(A)$ has a characteristic subgroup of order 3, this subgroup must be normal in $A$, so it must be $G_1(3)'$, the derived subgroup of $G_1(3)$. Then we have $G_1(3)'Q = QG_1(3)'$ for all Sylow 2-subgroups $Q$ of $A$. By [11, VI.6.10] $A$ has a normal Sylow 3-subgroup. So we may assume that $O_3(A) \cong Z_3 \times Z_3$. Since $A$ is soluble and $O_2(A) = 1$, we have $C_A(O_3(A)) = O_3(A)$. It follows that $A/O_3(A)$ is isomorphic to a subgroup of $GL(2, 3)$. Now the same argument as in paragraph (iii) of the proof of Lemma 2.7 works and gives the final contradiction. □

The next fact is a necessary and sufficient condition for $X$ to be $\frac{1}{2}$-transitive.

FACT 3.3. *$X$ is $\frac{1}{2}$-transitive if and only if* $\text{Aut}(G_1(p), T) > 1$, *if and only if* $\text{Aut}(G_1(p), T)$ *has order 2.*

*Proof.* If $X$ is $\frac{1}{2}$-transitive, $A > G_1(p)$. By Fact 3.2 $\text{Aut}(G_1(p), T) > 1$, and by Fact 3.1 $|\text{Aut}(G_1(p), T)| = 2$.

Conversely, if $\text{Aut}(G_1(p), T) = \langle\sigma\rangle$ has order 2, then by Fact 3.1 $\sigma$ interchanges $x$ and $y$. Thus the Cayley digraph $X(G_1(p), \{x, y\})$ is arc-transitive. It follows that $X$, the underlying undirected graph of $X(G_1(p), \{x, y\})$, is edge-transitive. Since the stabilizer $A_1$ of $A$ has order 2 and $A$ has degree 4, $X$ is not arc-transitive. Hence $X$ is $\frac{1}{2}$-transitive. □

Now we shall complete the proof of Theorem 1.1 by the following three steps:

(a) *The $\Gamma_i(p)$ defined in Section 1 are $\frac{1}{2}$-transitive for all $i$:* Since $\delta^{(p-1)/2} \in \text{Aut}(G_1(p), S_i)$, where $\delta$ is defined in Lemma 2.8, by Fact 3.3 we have that $\Gamma_i(p) = X(G_1(p), S_i)$ is $\frac{1}{2}$-transitive.

(b) *Let $X$ be a $\frac{1}{2}$-transitive graph of order $p^3$ and degree 4. Then $X \cong \Gamma_i(p)$ for some $i$:* First, $X = X(G_1(p), T)$, a Cayley graph on $G_1(p)$ with respect to $T = \{x, x^{-1}, y, y^{-1}\}$ and $\langle x, y\rangle = G_1(p)$. Since $X$ is $\frac{1}{2}$-transitive, $\text{Aut } X = G_1(p)\langle\sigma\rangle$, where $\sigma$ is an automorphism of order 2 of $G_1(p)$. By Lemma 2.8 there is a $\tau \in \text{Aut } G_1(p)$ such that $\tau^{-1}\sigma\tau = \delta^{(p-1)/2}$, where $\delta$ is defined in Lemma 2.8. Set

$Y = X(G_1(p), T^\tau)$. It is easy to verify that Aut $Y = \tau^{-1}(\text{Aut } X)\tau = G_1(p)\langle\delta^{(p-1)/2}\rangle$ and $Y \cong X$. Therefore we may assume that $\sigma = \delta^{(p-1)/2}$; thus $a^\sigma = a^{-1}$, $b^\sigma = b$.

Now if $x = b^i a^j$, since $\sigma$ interchanges $x$ and $y$, we have $y = b^i a^{-j}$ and $T = \{b^i a^j, b^i a^{-j}, (b^i a^j)^{-1}, (b^i a^{-j})^{-1}\}$. Without loss of generality we may assume that $i \le (p-1)/2$; otherwise, we use $(b^i a^j)^{-1} = b^{p-i}a^{-j+ijp}$ to replace $b^i a^j$. Since $\langle T\rangle = G_1(p)$, $p \nmid j$, so there is an integer $k$ such that $(a^j)^{\delta^k} = a$; hence $T^{\delta^k} = S_i = \{b^i a, b^i a^{-1}, (b^i a)^{-1}, (b^i a^{-1})^{-1}\}$. Thus $X(G_1(p), T) \cong X(G_1(p), S_i) = \Gamma_i(p)$.

(c) *$\Gamma_i(p) \not\cong \Gamma_j(p)$ when* $1 \le i < j \le (p-1)/2$: If $\Gamma_i(p) \cong \Gamma_j(p)$ for some $i$, $j$ with $1 \le i < j \le (p-1)/2$, by Lemma 2.5 there exists a $\sigma \in$ Aut $G_1(p)$ such that $S_i^\sigma = S_j$. Then $\sigma$ maps $b^i a b^i a^{-1} = b^{2i}a^{ip}$ to a product of two elements in $S_j$. Note that the image of $b$ under any automorphism of $G_1(p)$ is $ba^{tp}$ for some integer $t$ by the proof of Lemma 2.8, so $(b^{2i}a^{ip})^\sigma = b^{2i}a^{kp}$ for some $k$. However, it is easy to check that the product of any two elements in $S_j$ cannot have this form; this is a contradiction.

### Acknowledgments

The author is indebted to L.G. Kovács for several helpful discussions about the proof of Lemma 2.7. In fact, the proof presented here is his. This work was supported by the National Natural Science Foundation of China and by the Natural Science Foundation of Shanxi Province. The author thanks the Australian National University, where he wrote this paper, for its hospitality.

### References

1. B. Alspach, personal communication.
2. B. Alspach, D. Marušič, and L. Nowitz, "Constructing graphs which are $\frac{1}{2}$-transitive," *J. Austral. Math. Soc.*, to appear.
3. B. Alspach and T.D. Parsons, "A construction for vertex-transitive graphs," *Canad. J. Math.* **34** (1982), 307–318.
4. L. Babai, "Isomorphism problem for a class of point-symmetric structures," *Acta Math. Acad. Sci. Hungar.* **29** (1977), 329–336.
5. I.Z. Bouwer, "Vertex and edge-transitive but not 1-transitive graphs," *Canad. Math. Bull.* **13** (1970), 231–237.
6. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.
7. C.D. Godsil, "On the full automorphism group of a graph," *Combinatorica* **1** (1981), 243–256.
8. D. Gorenstein, *Finite Simple Groups*, Plenum Press, New York, 1982.
9. F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
10. D.F. Holt, "A graph which is edge transitive but not arc-transitive," *J. Graph Theory* **5** (1981), 201–204.
11. B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.
12. D. Marušič, "Vertex transitive graphs and digraphs of order $p^k$," *Ann. Discrete Math.* **27** (1985), 115–128.