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Bruhat-Chevalley Order in Reductive Monoids

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Abstract. Let M be a reductive monoid with unit group G. Let Λ denote the idempotent cross-section of the $G \times G$-orbits on M. If W is the Weyl group of G and $e, f \in \Lambda$ with $e \leq f$, we introduce a projection map from WeW to WfW. We use these projection maps to obtain a new description of the Bruhat-Chevalley order on the Renner monoid of M. For the canonical compactification X of a semisimple group G_0 with Borel subgroup B_0 of G_0, we show that the poset of $B_0 \times B_0$-orbits of X (with respect to Zariski closure inclusion) is Eulerian.

Keywords: reductive monoid, Renner monoid, Bruhat-Chevalley order, projections

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Introduction

Reductive monoids M are Zariski closures of reductive groups G. By this we mean that if G is a closed subgroup of $GL_n(k)$, then the closure M of G in $M_n(k)$ is a reductive monoid. The Bruhat-Chevalley order in G has a natural extension to reductive monoids. The Renner monoid R takes the place of the Weyl group W. Associated with the Bruhat decomposition of a $G \times G$-orbit of M is a $W \times W$-orbit of R that is a graded poset and has been explicitly determined by the author [7]. The ordering on R is more removed from the ordering on W and hence harder to understand. It is the detailed study of the ordering on R that is the purpose of this paper.

The $W \times W$-orbits are indexed by the cross-section lattice Λ of M. For two $W \times W$-orbits WeW, WfW with $e \leq f$, we define an upward projection map $p : WeW \to WfW$. These are order-preserving maps with some pleasing properties. If $\sigma \in WeW, \theta \in WfW$, then we show that $\sigma \leq \theta$ in R if and only if $p(\sigma) \leq \theta$. Combining with the description of the order on the $W \times W$-orbits in [7], we obtain a new description of the order on R which enables us to obtain several consequences. In particular we show that any length 2 interval in R is either a chain or a diamond. This leads to a conjecture on the Möbius function on R.

We go on to study canonical reductive monoids associated with the canonical compactification of semisimple groups, cf. [13]. We prove that for a canonical monoid, the poset $R^* = R\backslash\{0\}$ is Eulerian. This extends a classical result of Verma [18] that W is Eulerian and a recent result of the author [7] that the $W \times W$-orbits in R^* are Eulerian.

1. Preliminaries

Let P be a finite partially ordered set with a maximum and minimum element such that all maximal chains have the same length. Then P admits a rank function with the minimal element having rank zero. If $X \subseteq P$, we will say that X is *balanced* if the number of even rank elements of X is equal to the number of odd rank elements of X. P is said to be *Eulerian* if for $\alpha, \beta \in P$ with $\alpha < \beta$, the interval $[\alpha, \beta]$ is *balanced*. Eulerian posets were first defined by Stanley [16] and have been extensively studied. The Dehn-Somerville equations are valid in these posets and as stated in [17; Section 3.14], Eulerian posets enjoy remarkable duality properties.

Let k be an algebraically closed field and let G be a reductive group defined over k. Let T be a maximal torus contained in a Borel subgroup B of G. Let $W = N_G(T)/T$ denote the Weyl group of G and let S denote the generating set of simple reflections of W. Then G has the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB \tag{1}$$

As in [1], the Bruhat-Chevalley order on W is defined as:

$$x \leq y \quad \text{if } BxB \subseteq \overline{ByB} \tag{2}$$

As is well known, this is equivalent x being a subword of a reduced expression $y = \alpha_1 \dots \alpha_m, \alpha_1, \dots, \alpha_m \in S$. The length $l(y)$ is defined to be m. If w_0 is the longest element of W, then $B^- = w_0 B w_0$ is the Borel subgroup of G opposite to B relative to T. If $x_1, \dots, x_n \in W$, then let

$$x_1 * \cdots * x_n = \begin{cases} x_1 \dots x_n & \text{if } l(x_1 \dots x_n) = l(x_1) + \cdots + l(x_n) \\ \text{undefined} & \text{otherwise} \end{cases}$$

For $x, y \in W$, let $x \circ y, x \bigtriangleup y \in W$ be defined as:

$$\overline{B(x \circ y)B} = \overline{BxByB}, \ \overline{B^-(x \bigtriangleup y)B} = \overline{B^- x ByB} \tag{3}$$

Lemma 1.1 *Let $x, y \in W$. Then*

(i) $x \circ y = x_1 * y = x * y_1$ *for some* $x_1 \leq x, y_1 \leq y$
(ii) $x \circ y = \max\{xy' \mid y' \leq y\} = \max\{x'y \mid x' \leq x\} = \max\{x'y' \mid x' \leq x, y' \leq y\}$
(iii) $x \bigtriangleup y = \min\{xy' \mid y' \leq y\} = \min\{x'y \mid x' \geq x\} = \min\{x'y' \mid x' \geq x, y' \leq y\}$
(iv) $x \bigtriangleup y = xy_1$ *with* $y_1 \leq y$ *and* $x = (xy_1) * y_1^{-1}$
(v) $x \bigtriangleup y = xy$ *if and only if* $l(xy) = l(x) - l(y)$
(vi) $(x \bigtriangleup y) \circ y^{-1} = (x \bigtriangleup y)y^{-1}$ *and* $(x \circ y) \bigtriangleup y^{-1} = (x \circ y)y^{-1}$

Proof: (i) follows the Tits axioms and induction on length. (ii) then follows from (i) and (3). (iii) and (iv) follow from (i) and (ii) by noting that $x \bigtriangleup y = w_0((w_0x) \circ y)$.

(iv) Let $x \vartriangle y = s$. Then by (iv) $x = sy_1^{-1}$ for some $y_1 \leq y$. Then by (i), (ii),

$$x = sy_1^{-1} \leq s \circ y_1^{-1} \leq s \circ y^{-1} = s' * y^{-1}$$

for some $s' \leq s$. So $x = s'' * y_2^{-1}$ for some $s'' \leq s'$, $y_2 \leq y$. So

$$s \geq s' \geq s'' = xy_2^{-1} \geq x \vartriangle y = s$$

Hence $s = s'$ and $(x \vartriangle y) \circ y^{-1} = (x \vartriangle y)y^{-1}$. □

Corollary 1.2 *Let $x, x', y, y' \in W$. If $x \geq x'$ and $x * y = x' * y'$, then $y \leq y'$.*

Proof: By Lemma 1.1,

$$x * y = x' * y' = x' \circ y' \leq x \circ y' = x * y_1$$

for some $y_1 \leq y'$. By [7; Lemma 2.1], $y \leq y_1$. So $y \leq y'$. □

Lemma 1.3 *Let $x_0, y_0, s \in W$ such that $s_0 = x_0 \vartriangle y_0 \leq s$. Then*

$$Y = \left\{ y \leq y_0^{-1} \,\middle|\, s \circ y = x_0 \right\}$$

is a balanced subset of W.

Proof: By Lemma 1.1 (iv), $s_0 = x_0 y_1$, with $y_1 \leq y_0$ and $x_0 = s_0 * y_1^{-1}$. Let $y \in Y$. Then $y \leq y_0^{-1}$ and $s \circ y = x_0$. By Lemma 1.1 (i), $x_0 = s' * y$ with $s' \leq s$. Then

$$s_0 = x_0 \vartriangle y_0 \leq x_0 y^{-1} = s'$$

Since $s_0 * y_1^{-1} = x_0 = s' * y$, we see by Corollary 1.2 that $y \leq y_1^{-1}$. Hence

$$Y = \left\{ y \in W \mid y \leq y_1^{-1}, x_0 = s \circ y \right\}$$

We prove by induction on $l(y_1)$ that Y is balanced. If $l(y_1) = 0$, then $x_0 = s_0 < s$ and $Y = \emptyset$. So let $l(y_1) > 0$. Let $y_1 = \alpha * y_2$, $\alpha \in S$. Let

$$Y_1 = \{y \in Y \mid y\alpha > y\}$$
$$Y_2 = \{y \in Y \mid y\alpha < y, y\alpha \in Y\}$$

Let $y \in Y_1$. Then $y \leq y_1^{-1}$. So

$$y\alpha = y \circ \alpha \leq y_1^{-1} \circ \alpha = y_1^{-1}$$

and

$$s \circ (y\alpha) = s \circ y \circ \alpha = x_0 \circ \alpha = s_0 \circ y_1^{-1} \circ \alpha = s_0 \circ y_1^{-1} = x_0$$

Hence $y\alpha \in Y_2$. Thus $Y_1 \sqcup Y_2$ is balanced. Let

$$Y_3 = \{y \in Y \mid y\alpha < y, y\alpha \notin Y\}$$

So $Y = Y_1 \sqcup Y_2 \sqcup Y_3$. Let $x_1 = s_0 y_2^{-1} = s_0 * y_2^{-1}$. Then by induction hypothesis,

$$Z = \left\{y \in Y \mid y \leq y_2^{-1}, s \circ y = x_1\right\}$$

is balanced. Let $y \in Z$. Then $x_1 = s \circ y = s_1 * y$ for some $s_1 \leq s$. Since $x_1 < x_1\alpha$, we see that $y < y\alpha$. Then

$$y\alpha = y \circ \alpha \leq y_2^{-1} \circ \alpha = y_1^{-1}$$

and

$$s \circ (y\alpha) = s \circ y \circ \alpha = x_1 \circ \alpha = x_0$$

Thus $y\alpha \in Y_3$. Conversely let $y \in Y_3$. Then

$$(y\alpha) * \alpha = y \leq y_1^{-1} = y_2^{-1} * \alpha$$

Hence $y\alpha \leq y_2^{-1}$. Also $s \circ (y\alpha) \neq x_0$ and

$$(s \circ (y\alpha)) \circ \alpha = s \circ ((y\alpha) \circ \alpha) = s \circ y = x_0$$

Thus

$$(s \circ (y\alpha)) * \alpha = x_0 = x_1 * \alpha$$

So $s \circ (y\alpha) = x_1$ and $y\alpha \in Z$. Since Z is balanced, we see that Y_3 is balanced. Hence $Y = Y_1 \sqcup Y_2 \sqcup Y_3$ is balanced. □

If $I \subseteq S$, then as usual let W_I denote the parabolic subgroup of W generated by I and let

$$D_I = \{x \in W \mid xw = x * w \quad \text{for all } w \in W_I\}$$

denote the set of minimal length left coset representatives of W_I.

Let M be a *reductive monoid* having G as its unit group. Thus M is the Zariski closure of G in some $M_n(k)$, where $M_n(k)$ is the monoid of all $n \times n$ matrices over k. We refer to [6, 14] for details. The idempotent set $E(\bar{T})$ of $\bar{T}$ is a finite lattice isomorphic to the face lattice of a rational polytope. As in [5], let

$$\Lambda = \{e \in E(\bar{T}) \mid Be = eBe\}$$

Then Λ is a cross-section of the $G \times G$-orbits of M such that for all $e, f \in \Lambda$,

$$e \leq f \Leftrightarrow e \in MfM$$

Here as usual, $e \leq f$ means $ef = e = fe$. Λ is called the *cross-section lattice* of M. All maximal chains in Λ have the same length. We note that for $M_n(k)$,

$$\Lambda = \left\{ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \middle| 0 \leq r \leq n \right\}$$

is the usual set of idempotent representatives of matrices of different ranks.

By [10], the Bruhat decomposition (1) is extended to M as

$$M = \bigsqcup_{\sigma \in R} B\sigma B \tag{4}$$

where $R = \overline{N_G(T)}/T$ is the *Renner monoid* of M. W is the unit group of R and

$$R = \bigsqcup_{e \in \Lambda} WeW \tag{5}$$

The Bruhat-Chevalley order (2) on W extends to R as:

$$\sigma \leq \theta \quad \text{if } B\sigma B \subseteq \overline{B\theta B} \tag{6}$$

Then each WeW is an interval in R and by [10] all maximal chains in R have the same length. R is an inverse semigroup. This means that the map, $\sigma \to \sigma^{-1}$ is an involution of R. Here if $\sigma = xey \in WeW$, then $\sigma^{-1} = y^{-1}ex^{-1}$. Unlike in W, this involution is not order preserving. However by [11], the map

$$\sigma \to w_0\sigma^{-1}w_0 \tag{7}$$

is an order preserving involution of R. Let $e \in \Lambda$. Then as in [8], consider (in R),

$$\lambda(e) = \{s \in S \mid se = es\} \tag{8}$$

and

$$\lambda^*(e) = \bigcap_{f \geq e} \lambda(f), \quad \lambda_*(e) = \bigcap_{f \leq e} \lambda(f) \tag{9}$$

Then

$$\begin{aligned} W(e) &= W_{\lambda(e)} = \{w \in W \mid we = ew\}, \\ W^*(e) &= W_{\lambda^*(e)}, \\ W_*(e) &= W_{\lambda_*(e)} = \{w \in W \mid we = e = ew\} \end{aligned}$$

are parabolic subgroups of W with

$$W(e) = W^*(e) \times W_*(e) \tag{10}$$

Moreover $W^*(e)$ is the Weyl group of the unit group of eMe. See [6; Chapter 10] for details. If $I = \lambda(e)$, $K = \lambda_*(e)$, let

$$D(e) = D_I, \quad D_*(e) = D_K \tag{11}$$

Let

$$\mathcal{W}^*_{I,K} = D_I \times W_{I \setminus K} \times D_I^{-1} \tag{12}$$

For $\sigma = (x, w, y), \sigma' = (x', w', y') \in \mathcal{W}^*_{I,K}$, define

$$\sigma \leq \sigma' \quad \text{if } w = w_1 * w_2 * w_3 \quad \text{with} \quad xw_1 \leq x', w_2 \leq w', w_3 y \leq y' \tag{13}$$

By [7; Theorem 2.5],

$$\mathcal{W}^*_{I,K} \text{ is isomorphic to the dual of } WeW \tag{14}$$

The order on R is more subtle. Let $\sigma \in R$. Then

$$\sigma = xey \quad \text{for unique} \quad e \in \Lambda, x \in D_*(e), y \in D(e)^{-1} \tag{15}$$

This is called the *standard form* of σ. Let $\sigma = xey, \theta = sft \in R$ in standard form. Then by [4],

$$\sigma \leq \theta \Leftrightarrow e \leq f, \quad x \leq sw, \quad w^{-1}t \leq y \quad \text{for some } w \in W(f)W_*(e) \tag{16}$$

Let $\sigma = xey$ be in standard form. Let $x \leq x_1, y_1 \leq y$. Let $y_1 = uy_2, u \in W(e), y_2 \in D(e)^{-1}$. Let $x_1u = x_2z, x_2 \in D_*(e), z \in W_*(e)$. Then $x_1ey_1 = x_2ey_2$ in standard form. Now

$$x \leq x_1 = x_2zu^{-1} = x_2u^{-1} \cdot uzu^{-1}$$

Since $uzu^{-1} \in W_*(e)$ and $x \in D_*(e), x \leq x_2u^{-1}$. Also $uy_2 = y_1 \leq y$. Hence $\sigma \leq x_2ey_2$. Thus,

$$\sigma = xey \text{ in standard form } \Rightarrow \sigma \leq x_1ey_1 \text{ for all } x_1 \geq x, y_1 \leq y \tag{17}$$

If $e, f \in \Lambda$ with $e \leq f$, then $e \in \overline{fT}$ and so we see directly from (6) that

$$xey \leq xfy \quad \text{for all } x, y \in W \tag{18}$$

The length function on R is defined as follows. Let $\sigma = xey$ in standard form. Then

$$l(\sigma) = l(x) + l(e) - l(y) \tag{19}$$

where $l(e)$ is the length of the longest element in $D(e)$. We refer to [4, 7, 11, 14] for further details. In particular,

$$\text{length function } = \text{ rank function on } WeW, e \in \Lambda \tag{20}$$

where the rank function is determined from the grading of WeW.

2. Projections

We wish to better understand the order $\leq$ on R given by (6), (16). For $e \in \Lambda$, let z_e denote the longest element in $W_*(e)$. Let $e, f \in \Lambda$ with $e \leq f$. Let $\sigma = xey \in WeW$ in standard form. Let $z_ey = uy_1, u \in W(f), y_1 \in D(f)^{-1}$. We define the projection of σ in WfW as:

$$p_{e,f}(\sigma) = (x \triangle u)fy_1 \tag{21}$$

We claim that (21) is in standard form. Let $x = x_1v, x_1 \in D(f), v \in W(f)$. Since $x \in D_*(e)$ and $W_*(f) \subseteq W_*(e)$, we see that $v \in W^*(f)$. By (10), $v \triangle u \in W^*(f)$. Thus $x \triangle u = x_1(v \triangle u) \in D_*(f)$. Hence (21) is in standard form. Now $z_fz_ey = u'y_1$ with $u' = z_fu \in W(f)$ and by the above, $v \triangle u' = v \triangle u$. Hence we also have

$$p_{e,f}(\sigma) = (x \triangle u')fy_1 \text{ in standard form} \tag{22}$$

The following result in conjunction with (14) yields a new description of the order on R.

Theorem 2.1 *Let* $e, f \in \Lambda, e \leq f$. *Then*

(i) $p_{e,f} : WeW \to WfW$ *is order preserving and* $\sigma \leq p_{e,f}(\sigma)$ *for all* $\sigma \in WeW$.

(ii) *If* $\sigma \in WeW$, $\theta \in WfW$, *then* $\sigma \leq \theta$ *if and only if* $p_{e,f}(\sigma) \leq \theta$.

(iii) *If* $h \in \Lambda$ *with* $e \leq h \leq f$, *then* $p_{e,f} = p_{h,f} \circ p_{e,h}$.

(iv) $p_{e,f}$ *is onto if and only if* $\lambda_*(e) \subseteq \lambda_*(f)$.

(v) $p_{e,f}$ *is* $1-1$ *if and only if* $\lambda(f) \subseteq \lambda(e)$.

Proof: Let $\sigma = xey$ in standard form. Let $z_e y = uy_1, u \in W(f), y_1 \in D(f)^{-1}$. By Lemma 1.1, $x \triangle u = xu_1$ with $u_1 \leq u$. Then

$$u_1 y_1 \leq u y_1 = z_e y$$

Hence $u_1 y_1 = zy'$ for some $z \leq z_1$, $y' \leq y$. Then $z \in W_*(e)$ and $(z^{-1}u_1)\, y_1 = y' \leq y$. Also since $x \in D_*(e)$,

$$x \leq xz = (xu_1)\left(u_1^{-1}z\right)$$

By (16), (21),

$$\sigma = xey \leq xu_1 f y_1 = p_{e,f}(\sigma) \tag{23}$$

Let $\theta = sft$ in standard form such that $\sigma \leq \theta$. Then by (16),

$$x \leq sw,\ w^{-1}t \leq y \quad \text{for some } w \in W(f)W_*(e)$$

So $x = s_1 * w_1$ for some $s_1 \leq s$, $w_1 \leq w$. Since $x \in D_*(e)$, $w_1 \in D_*(e)$. Now $w = w_2 * z$ for some $w_2 \in W(f)$, $z \in W_*(e)$. Then $w_1 \leq w_2$. Since $t \in D(f)^{-1}$ and $y \in D(e)^{-1}$,

$$w_1^{-1}t \leq w_2^{-1}t = zz^{-1}w_2^{-1}t = zw^{-1}t \leq z \circ (w^{-1}t) \leq z \circ y \leq z_e \circ y = z_e y = uy_1$$

Since $t, y_1 \in D(f)^{-1}$ and $w_1, u \in W(f)$, we see by [7; Lemma 2.2] that $w_1 = w_3 * w_4$ with $w_4^{-1} \leq u$, $w_3^{-1}t \leq y_1$. So

$$x \triangle u \leq xw_4^{-1} = s_1 w_1 w_4^{-1} = s_1 w_3 \leq s \circ w_3 = s * w_5$$

for some $w_5 \leq w_3$. Also $w_5^{-1}t \leq w_3^{-1}t \leq y_1$. Hence by (21),

$$p_{e,f}(\sigma) = (x \triangle u) f y_1 \leq sft = \theta \tag{24}$$

So if $\sigma' \in WeW$ with $\sigma \leq \sigma'$, then by (23), $\sigma \leq \sigma' \leq p_{e,f}(\sigma')$. So by (24), $p_{e,f}(\sigma) \leq p_{e,f}(\sigma')$. This proves (i), (ii).

Let $e \le h \le f$ in Λ. Let $\sigma \in WeW$, $\theta = p_{e,f}(\sigma)$. Then by (i), (ii), $p_{e,f}(\sigma) \le p_{h,f} \circ p_{e,h}(\sigma)$. Let $\sigma = xey$, $\theta = sft$ in standard form. Then by (16), $x \le sw$, $w^{-1}t \le y$ for some $w \in W(f)W_*(e)$. So $w = w_1 * z$ for some $w_1 \in W(f)$, $z \in W_*(e)$. Then by (17), (18),

$$\sigma = xey \le swew^{-1}t = sw_1ew_1^{-1}t \le sw_1hw_1^{-1}t \le sw_1fw_1^{-1}t = \theta$$

So if $\pi = sw_1hw_1^{-1}t \in WhW$, then $\sigma \le \pi \le \theta$. By (ii), $p_{e,f}(\sigma) \le \pi$ and $p_{h,f}(\pi) \le \theta$. So by (i),

$$p_{h,f} \circ p_{e,f}(\sigma) \le p_{h,f}(\pi) \le \theta = p_{e,f}(\sigma)$$

This proves (iii).

(iv) Suppose first that $\lambda_*(e) \subseteq \lambda_*(f)$. By (9), $\lambda^*(e) \subseteq \lambda^*(f)$. So $\lambda(e) \subseteq \lambda(f)$. Hence $W(e) \subseteq W(f)$ and $W_*(e) = W_*(f)$. So $D_*(e) = D_*(f)$ and $D(f) \subseteq D(e)$. Thus if $\theta = xfy \in WfW$ is in standard form, then $\sigma = xey$ is in standard form and $p_{e,f}(\sigma) = \theta$. Thus $p_{e,f}$ is onto.

Assume conversely that $p_{e,f}$ is onto. Let w be the longest element in $W^*(f)$ and let $\theta = wf \in W\,fW$. There exists $\sigma = xey$ in standard form such that $p_{e,f}(\sigma) = \theta$. By (21), $w = x \triangle u$ for some $u \in W(f)$. So $w \le x$ and $x \in W(f)$. By (9), (11), $x \in D_*(e) \subseteq D_*(f)$. So by (10), $x \in W^*(f)$. Thus $x = w$. Since w is the longest element of $W^*(f)x$, $\alpha < x$ for all $\alpha \in \lambda^*(f)$. Since $x \in D_*(e)$, $x\alpha > x$ for all $\alpha \in \lambda_*(e)$. Hence $\lambda^*(e) \cap \lambda_*(f) = \emptyset$. There exists x_1ey_1 in standard form such that $p_{e,f}(x_1ey_1) = f$. By (21), $z_e * y_1 \in W(f)$. Hence by (10), $z_e \in W(f) = W^*(f) \times W_*(f)$. Since $W_*(e) \cap W^*(f) = \{1\}$, $z_e \in W_*(f)$. Hence $\lambda_*(e) \subseteq \lambda_*(f)$.

(v) Let $\lambda(f) \subseteq \lambda(e)$. Then $W(f) \subseteq W(e)$ and $D(e) \subseteq D(f)$. Let xey, $x'ey'$, sft be in standard form such that

$$p_{e,f}(xey) = p_{e,f}(x'ey') = sft \tag{25}$$

Let $z_e = vy_1$, where $v \in W(f) \cap W_*(e)$ and $y_1 \in D(f)^{-1} \cap W_*(e)$. Let $u \in W(f) \subseteq W(e)$. Then

$$\begin{aligned} u(y_1y) &= (uy_1)y \\ &= (u * y_1)y \quad \text{since} \quad y_1 \in D(f)^{-1} \\ &= (u * y_1) * y, \quad \text{since} \quad uy_1 \in W(e),\ y \in D(e)^{-1} \\ &= u * (y_1y) \end{aligned}$$

So $y_1y \in D(f)^{-1}$. Similarly $y_1y' \in D(f)^{-1}$. By (25),

$$y_1y = t = y_1y', \quad x \triangle v = s = x' \triangle v$$

So $y = y'$. Since $v \in W_*(e)$, $x = x \triangle v$ and $x' = x' \triangle v$. So $x = x'$. Thus $p_{e,f}$ is $1-1$.

Assume conversely that $p_{e,f}$ is $1-1$. Let v_e, v_f denote the longest elements of $W(e)$ and $W(f)$ respectively. Thus $v_e w_0$ and $v_f w_0$ are respectively the longest elements of $D(e)^{-1}$ and $D(f)^{-1}$. Let

$$z_e v_e w_0 = vy,\ v \in W(f),\ y \in D(f)^{-1} \tag{26}$$

Then

$$p_{e,f}(ev_e w_0) = fy = p_{e,f}(v^{-1}ev_e w_0)$$

Since $p_{e,f}$ is $1-1$, $ev_e w_0 = v^{-1}ev_e w_0$. So $v \in W_*(e)$. So $z = v^{-1}z_e \in W_*(e)$ and by (26),

$$zv_e w_0 = z * (v_e w_0) = y \le v_f w_0$$

So $v_e w_0 \le v_f w_0$. Hence $v_f \le v_e$. Thus $W(f) \subseteq W(e)$ and $\lambda(f) \subseteq \lambda(e)$. This completes the proof. □

Example 2.2 Let $G = GL_3(k)$, $M = M_3(k)$. Then W is the group of permutation matrices and R is the monoid of partial permutation matrices (rook monoid). Let

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\lambda(e) = \lambda_*(e) = \{(23)\}$, $\lambda(f) = \{(12)\}$, $\lambda_*(f) = \emptyset$. Hence $p_{e,f}$ is not $1-1$ or onto. $p_{e,f}$ is given in Table 1. Since $\lambda_*(I) = \theta$, $p_{f,I}$ is onto. $p_{f,I}$ is given in Table 2. Combining with [7; figures 3 and 4], one obtains the Hasse diagram of R.

Example 2.3 Let $\phi : M_n(k) \to M_N(k)$ be defined as:

$$\phi(A) = A \otimes \wedge^2 A \otimes \cdots \otimes \wedge^n A$$

where

$$N = \prod_{r=1}^{n} \binom{n}{r}$$

Table 1. Projection from rank 1 to rank 2.

σ	$p_{e,f}(\sigma)$
$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$
$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$
$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\1&0&0\\0&0&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}$

Let M denote the Zariski closure of $\phi(M_n(k))$ in $M_N(k)$. Then W is the symmetric group of degree n and $S = \{(12), (23), \ldots, (n-1 \quad n)\}$. Also

$$\Lambda = \{e_I \mid I \subseteq S\} \cup \{0\}$$

with

$$e_K \leq e_I \Leftrightarrow K \subseteq I$$

Table 2. Projection from rank 2 to rank 3.

σ	$p_{f,I}(\sigma)$	σ	$p_{f,I}(\sigma)$
$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&0&1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$
$\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\1&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}0&0&0\\1&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

and

$$\lambda(e_I) = \lambda^*(e_I) = I, \quad \lambda_*(e_I) = \emptyset, \quad I \subseteq S$$

So by Theorem 2.1, p_{e_K,e_I} is onto for $K \subseteq I$.

Example 2.4 Let $\phi : SL_n(k) \to GL_N(k)$ be defined as:

$$\phi(A) = A \oplus \wedge^2 A \oplus \cdots \oplus \wedge^n A$$

where $N = 2^n - 1$. Let M denote Zariski closure in $M_N(k)$ of $k\phi(SL_n(k))$. Again W is symmetric group of degree n and $S = \{(12), (23), (n-1 \quad n)\}$. Then

$$\Lambda = \{1\} \cup \{e_I \mid I \subseteq S\}$$

with

$$e_I \le e_K \Leftrightarrow K \subseteq I$$

and

$$\lambda(e_I) = \lambda_*(e_I) = I, \quad \lambda^*(e_I) = \emptyset, \quad I \subseteq S$$

So by Theorem 2.1, p_{e_I, e_K} is $1 - 1$ for $K \subseteq I$.

Corollary 2.5 *Let $e < f$ in Λ, $\sigma \in WeW$, $\theta \in WfW$. Let $\sigma = xey$ in standard form, $z_f z_e y = uy_1$ with $u \in W(f)$, $y_1 \in D(f)^{-1}$. Then θ covers σ if and only if f covers e in Λ, $p_{e,f}(\sigma) = \theta$ and $l(xu) = l(x) - l(u)$.*

Proof: If θ covers σ, then by Theorem 2.1, f covers e in Λ and $\theta = p_{e,f}(\sigma)$. So assume that f covers e in Λ and $\theta = p_{e,f}(\sigma)$. The maximum elements of *WeW* and *WfW* are respectively $w_0 z_e e$ and $w_0 z_f f$. Since f covers e, we see by (9) and [6; Chapter 10] that

$$\lambda_*(f) = \lambda(f) \cap \lambda_*(e)$$

So by (22),

$$p_{e,f}(w_0 z_e e) = w_0 z_e f z_f z_e \tag{27}$$

covers $w_0 z_e e$. By (19), (20), $[\sigma, w_0 z_e]$ has length

$$l(w_0 z_e) - l(x) + l(y) \tag{28}$$

and $[w_0 f z_f z_e, w_0 z_f f]$ has length

$$l(w_0 z_f) - l(w_0 z_e) + l(z_f z_e) \tag{29}$$

By (27)–(29), $[\sigma, w_0 z_f f]$ has length

$$l(w_0 z_f) - l(x) + l(y) + l(z_f z_e) + 1 \tag{30}$$

By (22), $\theta = (x \triangle u) f y_1$. Also

$$l(u) + l(y_1) = l(z_f z_e y) = l(z_f z_e) + l(y) \tag{31}$$

By (19), (20), $[\theta, w_0 z_f f]$ has length

$$l(w_0 z_f) - l(x \triangle u) + l(y_1) \tag{32}$$

By (30)–(32), $[\sigma, \theta]$ has length

$$l(x \triangle u) + l(u) - l(x) + 1$$

Hence θ covers σ if and only if

$$l(x \triangle u) = l(x) - l(u)$$

By Lemma 1.1, this is true if and only if $l(x \triangle u) = l(xu)$. This completes the proof. $\square$

Corollary 2.6 *Any interval in R of length* 2 *has at most* 4 *elements.*

Proof: Consider an interval $[\sigma, \theta]$ in R of length 2. Let $\sigma \in WeW$, $\theta = WfW$. Then $e \leq f$.

Case 1. $e = f$. By (14), WeW is isomorphic to the dual of $\mathcal{W}^*_{I,K}$ where $I = \lambda(e)$ and $K = \lambda_*(e)$. Now $\mathcal{W}^*_{I,K}$ is a subposet of $\mathcal{W}^*_{I,\emptyset}$ with the same rank function. By [7; Theorem 3.3], $\mathcal{W}^*_{I,\emptyset}$ is an Eulerian poset. Hence any interval of length 2 in $\mathcal{W}^*_{I,\emptyset}$ has 4 elements. It follows that $|[\sigma, \theta]| \leq 4$ in WeW.

Case 2. $e < f$ and f does not cover e in Λ. Then by Corollary 2.5, $[e, f]$ has length 2 in Λ. Now $E(\bar{T})$ is the face lattice of a polytope. Hence in Λ, $|[e, f]| \leq 4$. So in Λ,

$$[e, f] = \{e, h, h', f\}$$

with $e < h$, $e < h' < f$ and with the possibility that $h = h'$. So by Theorem 2.1,

$$[\sigma, \theta] = \{\sigma, p_{e,h}(\sigma), p_{e,h'}(\sigma), \theta\}$$

in R.

Case 3. f covers e in Λ and $\theta = p_{e,f}(\sigma)$. Let $\sigma = xey$ in standard form. If $\pi \in (\sigma, \theta)$, then $\pi \in WeW$ and π covers σ. So by (14), either $R\pi = R\sigma$ or $\pi R = \sigma R$. Let $\pi_1, \pi_2 \in (\sigma, \theta)$ such that $R\pi_1 = R\sigma = R\pi_2$. Then $\pi_1 = x_1 ey$, $\pi_2 = x_2 ey$ in standard form. Let $z_f z_e y = uy_1$, $u \in W(f)$, $y_1 \in D(f)^{-1}$. Since θ covers π_1 and π_2, we see by Corollary 2.5 that

$$x_1 u f y_1 = p_{e,f}(\pi_1) = \theta = p_{e,f}(\pi_2) = x_2 u f y_1 \tag{33}$$

in standard form. It follows that $x_1 u = x_2 u$. Hence $x_1 = x_2$ and $\pi_1 = \pi_2$. Dually by (7), $\pi_1 R = \pi_2 R$ implies that $\pi_1 = \pi_2$. It follows that $|[\sigma, \theta]| \leq 4$.

Case 4. f covers e in Λ and $p_{e,f}(\sigma) = \theta_1 < \theta$. Then θ_1 covers σ and θ covers θ_1. Let $\pi_1, \pi_2 \in (\sigma, \theta)$, $\pi_1 \neq \pi_2$, $\pi_1 \neq \theta_1$, $\pi_2 \neq \theta_1$. Then $\pi_1, \pi_2 \in WeW$ and θ covers π_1, π_2. So $p_{e,f}(\pi_1) = \theta = p_{e,f}(\pi_2)$. Since π_1, π_2 cover σ, we see by (14) that for $i = 1, 2$, $\pi_i R = \sigma R$, or $R\sigma = R\pi_i$. Since $\pi_1 \neq \pi_2$, we can assume by (33) that $R\pi_1 = R\sigma$, $R\pi_2 \neq R\sigma$. So $\pi_1 = x'ey$, $\pi_2 = xey'$ in standard form, x' covers x and y covers y'. Since θ covers π_1, π_2,

$$z_f z_e y = u y_1,\ z_f z_e y' = v y_1,\ u, v \in W(f),\ y_1 \in D(f)^{-1}$$

Then $\theta_1 = x_1 f y_1$, $\theta = x_1' f y_1$ in standard form with

$$x_1 = xu, \quad x_1' = xv = x'u$$

and by Corollary 2.5,

$$x = x_1 * u^{-1} = x_1' * v^{-1}, \quad x' = x_1' * u^{-1}$$

Now x_1' covers x_1 and hence u^{-1} covers v^{-1} by Corollary 1.2. Since x' covers x, this contradicts the exchange condition for W. So $|[\sigma, \theta]| \leq 4$, completing the proof. □

Corollary 2.6 leads us to the following conjecture concerning the Möbius function μ on R. We refer to [17; Chapter 3] for the theory of Möbius functions on posets:

Conjecture 2.7 Let $\sigma, \theta \in R$, $\sigma \leq \theta$. Then

$$\mu(\sigma, \theta) = \begin{cases} (-1)^{l[\sigma,\theta]} & \text{if every interval of length 2 in } [\sigma, \theta] \text{ has 4 elements} \\ 0 & \text{otherwise} \end{cases}$$

Here $l[\sigma, \theta]$ denotes the length of the interval $[\sigma, \theta]$.

Theorem 3.4 below establishes Conjecture 2.7 for canonical monoids.

3. Canonical monoids

In this section we will assume that M is a canonical monoid. This means that $\Lambda^* = \Lambda \backslash \{0\}$ has a least element e_0 with $\lambda(e_0) = \emptyset$. Then as in Example 2.3, Λ^* is in $1-1$ correspondence with the subsets of S. So we can write:

$$\Lambda^* = \{e_I \mid I \subseteq S\} \tag{34}$$

with

$$\lambda(e_I) = \lambda^*(e_I) = I,\ \lambda_*(e_I) = \emptyset,\ I \subseteq S$$

and

$$e_K \leq e_I \Leftrightarrow K \subseteq I$$

See [9, 13] for details. Example 2.3 is an example of a canonical monoid. More generally if G_0 is a semisimple group and if $\phi : G_0 \to GL_n(k)$ is an irreducible representation with highest weight in the interior of the Weyl chamber, then the Zariski closure in $M_n(k)$ of $k\phi(G_0)$ is a canonical monoid. Canonical monoids are closely related to canonical compactifications of semisimple groups in the sense of [2]. The connection between reductive monoids and embeddings of homogenous spaces is studied in [12]. See also [19]. Basically the canonical compactification is obtained as the projective variety $X = (M\backslash\{0\})/$ center. Then the $B \times B$-orbits of X are indexed by $R^* = R\backslash\{0\}$. See [13]. The Bruhat-Chevalley order on R^* corresponds to the Zariski closure inclusion of $B \times B$-orbits of X, the geometric properties of which have been studied in [15].

Let M be a canonical monoid. For $I \subseteq S$, let $R_I = We_I W = We_I D_I^{-1}$. Then by (5), (34),

$$R^* = R\backslash\{0\} = \bigsqcup_{I \subseteq S} R_I \tag{35}$$

For $K \subseteq I$, we write $p_{K,I}$ for p_{e_K,e_I}. So $p_{K,I}\colon R_K \to R_I$. By [7; Theorem 3.3], each R_I is an Eulerian poset. We will show in this section that R^* is an Eulerian poset.

Lemma 3.1 *Let $\sigma \in R_\emptyset$, $s \in W$ such that $p_{\emptyset,S}(\sigma) < s$. Then $[\sigma, s] \cap R_\emptyset$ is balanced.*

Proof: Let $e = e_\emptyset$, $\sigma = x_0 e y_0$, $s_0 = p_{\emptyset,S}(\sigma) = x_0 \triangle y_0$. Then $s_0 < s$. For $y \leq y_0$, let

$$A_y = [\sigma, s] \cap Wey = \left\{xey \mid x_0 \leq x \leq s \circ y^{-1}\right\}$$

Thus A_y is a non-trivial interval in $R_\emptyset$ unless $x_0 = s \circ y^{-1}$. So A_y is balanced unless $x_0 = s \circ y^{-1}$. By Lemma 1.3,

$$Y = \left\{y \leq y_0 \mid x_0 = s \circ y^{-1}\right\}$$

is balanced. It follows that $[\sigma, s] \cap R_0$ is balanced. □

Corollary 3.2 *Let $\sigma \in R_\emptyset$, $\theta \in R_I$ such that $p_{\emptyset,I}(\sigma) < \theta$. Then $[\sigma, \theta] \cap R_\emptyset$ is balanced.*

Proof: Let $e = e_\emptyset$, $f = e_I$, $\sigma = x_0 e y_0$, $\theta = sft$ in standard form. Suppose $x_0 \notin sW_I$. For $t \leq y \leq y_0$, let

$$A_y = [\sigma, \theta] \cap Wey$$

By [3],

$$w = w_y = \max\{u \in W_I \mid u^{-1}t \leq y\}$$

exists. Let $xey = A_y$. Then $x_0ey_0 \leq xey \leq sft$. So $x_0 \leq x$ and there exists $u \in W_I$ such that $x \leq su$, $u^{-1}t \leq y$. Then $u \leq w$ and $su \leq s \circ u \leq s \circ w$. Also $s \circ w = sw_1$ for some $w_1 \leq w$ and $w_1^{-1}t \leq w^{-1}t \leq y$. Hence

$$\sigma \leq x_0ey \leq xey \leq (s \circ w)ey \leq \theta$$

So

$$A_y = [x_0ey, (s \circ w)ey]$$

Since $x_0 \notin sW_I$, $x_0 \neq s \circ w$. Since $R_\emptyset$ is Eulerian, A_y is balanced. Thus

$$[\sigma, \theta] \cap R_\emptyset = \bigcup_{t \leq y \leq y_0} A_y$$

is balanced. Similarly if $y_0 \notin W_It$, $[\sigma, \theta] \cap R_\emptyset$ is balanced. So let $x_0 \in sW_I$, $y_0 \in W_It$. So $s = s_1v$, $x_0 = s_1x_1$, $y_0 = y_1t$ for some $s_1 \in D_1$, $v, x_1, y_1 \in W_I$. If $\sigma' = x_1ey_1$, $\theta' = ve_I$, then working in e_IRe_I, we see by Lemma 3.1 that $[\sigma', \theta'] \cap R_\emptyset$ is balanced. Hence

$$[\sigma, \theta] \cap R_\emptyset = s_1\big([\sigma', \theta'] \cap R_\emptyset\big)t$$

is balanced. □

Lemma 3.3 *Let $\sigma \in R_\emptyset, \theta \in R_I, \sigma \leq \theta$. Let*

$$Z = \{\pi \in [\sigma, \theta] \cap R_\emptyset \mid p_{\emptyset,I}(\pi) = \theta\}$$

Then

$$(-1)^{l(e_\emptyset)} \sum_{\pi \in Z} (-1)^{l(\pi)} = (-1)^{l(e_I)}(-1)^{l(\theta)}$$

Proof: We prove by induction on $l(\theta)$. Suppose first that $p_{\emptyset,I}(\sigma) = \theta' < \theta$. For $\delta \in [\theta', \theta]$ let

$$Z_\delta = \{\pi \in [\sigma, \theta] \cap R_\emptyset \mid p_{\emptyset,I}(\sigma) = \delta\}$$

Then $Z = Z_\theta$ and

$$[\sigma, \theta] \cap R_\emptyset = \bigsqcup_{\delta \in [\theta', \theta]} Z_\delta \tag{36}$$

By induction hypothesis,

$$(-1)^{l(e_\emptyset)} \sum_{\pi \in Z_\delta} (-1)^{l(\pi)} = (-1)^{l(e_I)}(-1)^{l(\delta)} \tag{37}$$

for $\delta \in [\theta', \theta)$. By Corollary 3.2,

$$\sum_{\pi \in [\sigma,\theta] \cap R_\emptyset} (-1)^{l(\pi)} = 0 = \sum_{\delta \in [\theta',\theta]} (-1)^{l(\delta)} \tag{38}$$

By (36)–(38) we see that (37) is also valid for $\delta = \theta$.

Assume therefore that $p_{\emptyset,I}(\sigma) = \theta$. Then as in the proof of Corollary 3.2, we may assume that $\theta = s \in W$. Let $e = e_\emptyset$, $\sigma = x_0 e y_0$. Then $x_0 \triangle y_0 = s$. So $x_0 y_1 = s$, $s * y^{-1} = x_0$ with $y_1 \leq y_0$. For $y \leq y_0$, let

$$A_y = [\sigma, \theta] \cap Wey$$

Let $xey \in A_y$. Then $x_0 e y_0 \leq xey \leq s$. So $x \leq s \circ y^{-1}$. Hence

$$\sigma \leq x_0 ey \leq xey \leq (s \circ y^{-1})ey \leq s \tag{39}$$

So

$$A_y = [x_0 ey, (s \circ y^{-1})ey] \tag{40}$$

and

$$Z = [\sigma, s] \cap R_\emptyset = \bigsqcup_{y \leq y_0} A_y$$

If $A_y \neq \emptyset$, then since $p_{\emptyset,S}(\sigma) = s$, we see that $x_0 \triangle y = s$. So if $x_0 = s \circ y^{-1}$, then by Lemma 1.1 (vi), $x_0 = s * y^{-1}$. Hence $y = y_1$. Moreover $A_{y_1} = \{x_0 e y_1\}$ and by (19), $l(x_0 e y_1) = l(e) + l(s)$. If $A_y \neq \emptyset$ and $x_0 \neq s \circ y^{-1}$, then A_y is balanced by (40). The result follows. □

Theorem 3.4 *R^* is an Eulerian poset.*

Proof: Let $\sigma, \theta \in S^*$, $\sigma < \theta$. We need to show that $[\sigma, \theta]$ is balanced. Let $\sigma \in R_K$, $\theta \in R_I$. So $K \subseteq I$. First assume that $p_{K,I}(\sigma) < \theta$. By Theorem 2.1 (vi), there exists $\sigma_0 \in R_\emptyset$ such that $p_{\emptyset,K}(\sigma_0) = \sigma$. By Corollary 3.2, $[\sigma_0, \theta] \cap R_\emptyset$ is balanced. So by Lemma 3.3, $[\sigma_0, \theta] \cap R_L$ is balanced for $L \subseteq S$. For $I \subseteq J \subseteq S$, $[\sigma_0, \theta] \cap R_J = [\sigma, \theta] \cap R_J$ by Thoerem 2.1. It follows that $[\sigma, \theta]$ is balanced.

Now assume that $p_{K,I}(\sigma) = \theta$. Let $\theta = s\theta' t$, $s \in D_I$, $t \in D_I^{-1}$, $\theta' \in W_I e_I$. By (21), $\sigma' = s^{-1}\sigma t^{-1} \in W_I e_K W_I$. Then $[\sigma, \theta] = s[\sigma', \theta']t \cong [\sigma', \theta']$. Thus without loss of generality, we may assume that $\theta = s \in W$. Let $\sigma = x_0 e_K y_0$ in standard form, $K \neq S$. Then by (21), $s = x_0 \vartriangle y_0$. Hence $x_0 = s * y_1^{-1}$ for some $y_1 \leq y_0$. By Corollary 2.5,

$$\Omega = \{x_0 e_J y_1 \mid K \subseteq J \subseteq S\} = [x_0 e_K y_1, s] \tag{41}$$

Let

$$y_1 = u_J v_J, \quad u_j \in W(J), \quad v_J \in D_J^{-1}, \quad K \subseteq J \subseteq S \tag{42}$$

For $y \in D_J^{-1}$, let

$$A_J(y) = [\sigma, s] \cap W e_J y$$

Suppose $y \in D_J^{-1}$, $y \neq v_J$ and $x e_J y \in A_J(y)$. Then

$$\sigma \leq x e_J y \leq s \tag{43}$$

Then $y \leq y_0$ and by [3],

$$w = \max\{u \in W_J \mid uy \leq y_0\}$$

exists. By (16), (43), there exists $u \in W_J$ such that $x_0 \leq x * u^{-1}$, $uy \leq y_0$. So $u \leq w$. By Lemma 1.1,

$$x_0 \vartriangle w \leq x_0 \vartriangle u \leq (x * u^{-1}) \vartriangle u = x$$

Hence

$$\sigma \leq (x_0 \vartriangle w) e_J y \leq x e_J y \tag{44}$$

Also by (16), (43),

$$\sigma \leq x e_J y \leq s \circ y^{-1} e_J y \leq s \tag{45}$$

Since $p_{K,S}(\sigma) = s$, we see that $x \vartriangle y = s$. So by Lemma 1.1 (vi), $s \circ y^{-1} = s * y^{-1}$. Thus by (44), (45),

$$A_J(y) = [(x_0 \vartriangle w) e_J y, (s * y^{-1}) e_J y]$$

Suppose $|A_J(y)| = 1$. Then $x_0 \vartriangle w = s * y^{-1}$. By Lemma 1.1, $(x_0 \vartriangle w) * w_1^{-1} = x_0$ for some $w_1 \leq w$. Then

$$s * y_1^{-1} = x_0 = (x_0 \vartriangle w) * w_1^{-1} = s * y^{-1} * w_1^{-1}$$

By Corollary 1.2, $y_1^{-1} = y^{-1} * w_1^{-1}$. So $y = w_1 y$. Since $w_1 \in W(J)$ and $y \in D_J^{-1}$, we see by (42) that $y = v_J$, a contradiction. Hence $A_J(y)$ is a non-trivial interval in R_J and hence balanced.

Assume next that $y = v_J$. Let $xe_J v_J \in R_J$. Then

$$\sigma = x_0 e_K y_0 \leq xe_J v_J \leq s \tag{46}$$

Since $p_{J,S}(xe_J y) = s$, we see by Theorem 2.1 that $x \triangle v_J = s$. By Lemma 1.1,

$$x = s * v^{-1} \quad \text{for some} \quad v \leq v_J \tag{47}$$

Also by (16), $x_0 \leq x * u$ for some $u \in W(J)$. Hence by (47),

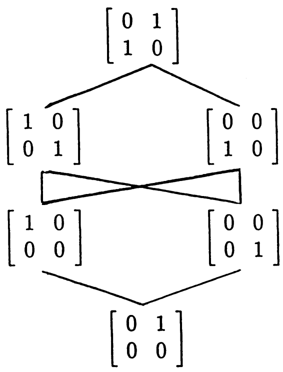
$$s * y_1^{-1} = x_0 \leq x * u = s * v^{-1} * u$$

By Corollary 1.2, $y_1^{-1} \leq v^{-1}u$. So $y_1 \leq u^{-1}v$. So we see by (42), (47) that $v = v_J$. So $x = sv_J^{-1}$. Hence

$$xe_J y = sv_J^{-1}e_J v_J = sv_J^{-1}u_J^{-1}e_J u_J v_J = sy_1^{-1}e_J y_1 = x_0 e_J y_1$$

Thus $A_J(v_J) = \{x_0 e_J y_1\}$. It follows that $[\sigma, s]\backslash\Omega$ is balanced. By (41), $\Omega \cong 2^{S\backslash K}$ is also balanced. Hence $[\sigma, s]$ is balanced, completing the proof. □

Example 3.5 $M_2(k)$ is a canonical monoid. The Eulerian poset R^* is given by:



$M_3(k)$ is not a canonical monoid. In this case, Example 2.2 shows that R^* is not Eulerian. The monoids in Example 2.3 are canonical. With $n = 3$, R^* will be an Eulerian poset with 78 elements.

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