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The Regular Near Polygons of Order $(s, 2)$

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Abstract. In this note we classify the regular near polygons of order $(s, 2)$.

Keywords: regular near polygon, distance-regular graph

1. Introduction

Regular near polygons were introduced by Shult and Yanushka [18] as point-line geometries satisfying certain axioms. It is well known that (the collinearity graph of) a regular near polygon of order (s, t) is a distance-regular graph of valency $s(t + 1)$, diameter d and $a_i = c_i(s - 1)$ for all $1 \leq i \leq d - 1$ such that for any vertex x the subgraph induced by the neighbors of x is the disjoint union of $t + 1$ complete graphs of size s.

Let Γ be (the collinearity graph of) a regular near polygon of order (s, t). If $t = 0$, it is clear that Γ is a complete graph. If $t = 1$, then Γ is a line graph and we have a classification of such graphs. (See [6, 17].)

In this note we consider the case $t = 2$ and classify the regular near polygons of order $(s, 2)$.

First we recall our notation and terminology.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph without loops or multiple edges. For vertices x and y in Γ we denote by $\partial_\Gamma(x, y)$ the distance between x and y in Γ. The *diameter* of Γ, denoted by d, is the maximal distance of two vertices in Γ. We denote by $\Gamma_i(x)$ the set of vertices which are at distance i from x.

A connected graph Γ with diameter d is said to be *distance-regular* if there are numbers c_i $(1 \leq i \leq d)$, a_i $(0 \leq i \leq d)$ and b_i $(0 \leq i \leq d - 1)$ such that for any two vertices x and

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y in Γ at distance i the sets

$$\Gamma_{i-1}(x) \cap \Gamma_1(y), \Gamma_i(x) \cap \Gamma_1(y) \quad \text{and} \quad \Gamma_{i+1}(x) \cap \Gamma_1(y)$$

have cardinalities c_i, a_i and b_i, respectively. Then Γ is regular with valency $k := b_0$.

Let Γ be a distance-regular graph with diameter d. The array

$$\iota(\Gamma) = \begin{Bmatrix} * & c_1 & \dots & c_i & \dots & c_{d-1} & c_d \\ a_0 & a_1 & \dots & a_i & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \dots & b_i & \dots & b_{d-1} & * \end{Bmatrix}$$

is called the *intersection array* of Γ. Define $r := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$.

Let $k_i := |\Gamma_i(x)|$ for all $0 \le i \le d$ which does not depend on the choice of x.

By an eigenvalue of Γ we will mean an eigenvalue of its adjacency matrix A. Its multiplicity is its multiplicity as eigenvalue of A.

Define the polynomials $u_i(x)$ $(0 \le i \le d)$ by $u_0(x) := 1$, $u_1(x) := \frac{x}{k}$ and

$$c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) = x u_i(x), \quad \text{for } i = 1, 2, \dots, d-1.$$

Let θ be an eigenvalue of Γ with multiplicity $m(\theta)$. It is well known that

$$m(\theta) = \frac{|V\Gamma|}{\sum_{i=0}^{d} k_i u_i(\theta)^2}.$$

For more information on distance-regular graphs we would like to refer to the books [1, 3, 6, 10].

A graph Γ is said to be *of order* (s, t) if $\Gamma_1(x)$ is a disjoint union of $t+1$ complete graphs of size s for every vertex x in Γ. In this case, Γ is a regular graph of valency $k = s(t+1)$.

A graph Γ is called (the collinearity graph of) *a regular near polygon of order* (s, t) if it is a distance-regular graph of order (s, t) with diameter d and $a_i = c_i(s-1)$ for all $1 \le i \le d-1$.

For a regular near polygon of order (s, t) with diameter d it is known that $c_i \le t+1$ holds for all $1 \le i \le d$ and equality implies $i = d$.

A regular near polygon is called a *regular near* $2d$*-gon* if $c_d = t+1$, a *regular near* $(2d+1)$*-gon,* otherwise.

A regular near $2d$-gon of order (s, t) with $c_1 = \dots = c_{d-1} = 1$ and $c_d = t+1$ is called a *generalized* $2d$*-gon of order* (s, t). When $d = 2, 3$ and 4 a generalized $2d$-gon of order (s, t) is denoted by $GQ(s, t)$, $GH(s, t)$ and $GO(s, t)$, respectively.

More information on regular near polygons and generalized polygons will be found in [6, Sections 6.4–6.6].

The following is our main result.

Theorem 1 *A regular near polygon of order* $(s, 2)$ *is isomorphic to one of the following graphs.*

	Graph	k	d	$\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$	v
(1-a)	$K_{3,3}$	3	2	$\{3, 2; 1, 3\}$	6
(1-b)	O_3	3	2	$\{3, 2; 1, 1\}$	10
(1-c)	The Heawood graph	3	3	$\{3, 2, 2; 1, 1, 3\}$	14
(1-d)	The Pappus graph	3	4	$\{3, 2, 2, 1; 1, 1, 2, 3\}$	18
(1-e)	Tutte's 8 cage	3	4	$\{3, 2, 2, 2; 1, 1, 1, 3\}$	30
(1-f)	The Desargues graph	3	5	$\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}$	20
(1-g)	Tutte's 12 cage	3	6	$\{3, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 3\}$	126
(1-h)	The Foster graph	3	8	$\{3, 2, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 2, 2, 2, 3\}$	90
(2-i)	GQ(2,2)	6	2	$\{6, 4; 1, 3\}$	15
(2-j)	GH(2,2)	6	3	$\{6, 4, 4; 1, 1, 3\}$	63
(2-k)	GQ(4,2)	12	2	$\{12, 8; 1, 3\}$	45
(2-l)	GO(4,2)	12	4	$\{12, 8, 8, 8; 1, 1, 1, 3\}$	2925
(2-m)	GH(8,2)	24	3	$\{24, 16, 16; 1, 1, 3\}$	2457
(3)	H(3,s+1)	$3s$	3	$\{3s, 2s, s; 1, 2, 3\}$	$(s+1)^3$

Let Γ be a distance-regular graph of order $(s, 2)$. We have $c_d \leq 3$.

If $s = 1$, then $k = 3$ and $a_1 = a_2 = \cdots = a_{d-1} = 0$. The result easily follows from the result of Ito [16]. See also [4]. If $s = 2$, then $k = 6$ and $a_1 = 1$. Such distance-regular graphs were classified by Hiraki et al. in [15]. This shows that our theorem is true for the case $s = 2$. Hence we may assume $s \geq 3$. In Section 2 we will show that if $d = r + 1$, then Γ has to be a generalized $2d$-gon and those are easy to classify. For $d \geq r + 2$ and $s \geq 3$ we show in Section 3 that $c_{r+2} \geq 3$, and hence under the assumption that Γ is a regular near polygon of order $(s, 2)$ it follows that $c_{r+1} = 2$, $c_{r+2} = 3$ and $d = r + 2$. To finish our classification we only need to show the following proposition.

Proposition 2 *Let Γ be a distance-regular graph with the intersection array*

$$\iota(\Gamma) = \left\{ \begin{matrix} * & 1 & \cdots & 1 & 2 & 3 \\ 0 & s-1 & \cdots & s-1 & 2(s-1) & 3(s-1) \\ 3s & 2s & \cdots & 2s & s & * \end{matrix} \right\},$$

where $r = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. Suppose $s \geq 3$. Then $r = 1$.

It is known that a distance-regular graph of order $(s, 2)$ with the above intersection array is isomorphic to the Hamming graph $H(3, s+1)$ if $r = 1$. (See [7] or [6, Section 9.2].)

Our theorem is a direct consequence of Proposition 2.

Proposition 2 will be shown in Sections 4 and 5. In Section 4 we treat the case $s \neq 3, 6$ and show that $r = 1$ by looking at the integrality of the multiplicity of the smallest eigenvalue. In Section 5 we treat the case $s = 3, 6$. In here we will use the eigenvalue method of Bannai-Ito to show $r \leq 21$. Then $r = 1$ follows by looking at the integrality of the multiplicity of the smallest eigenvalue. We prove Theorem 1 in Section 6.

2. Preliminaries

In this section first we introduce the following famous result.

Proposition 3 *Let Γ be a distance-regular graph of diameter d with the intersection array*

$$\iota(\Gamma) = \left\{ \begin{array}{ccccc} * & 1 & \cdots & 1 & c_d \\ 0 & s-1 & \cdots & s-1 & a_d \\ s(t+1) & st & \cdots & st & * \end{array} \right\}.$$

Suppose $t \geq 2$. Then $d \leq 13$ and the following hold.
(1) *If $c_d = 1$, then $d = 2$.*
(2) *If $c_d = t + 1$ then $d \in \{2, 3, 4, 6\}$. Moreover if $s \geq 2$, then $d \neq 6$ and the following hold.*
  (i) *If $d = 2$, then $s \leq t^2$ and $t \leq s^2$.*
  (ii) *If $d = 3$, then $s \leq t^3$, $t \leq s^3$ and st is a square.*
  (iii) *If $d = 4$, then $s \leq t^2$, $t \leq s^2$ and $2st$ is a square.*

Proof: The first assertion is proved by Fuglister [9]. (See also [6, pp. 208–209].) The rest of the assertions are proved by Feit and Higman [8], Higman [12, 13] and Haemers and Roos [11]. (See also [6, Theorem 6.5.1].) □

Lemma 4 *Let Γ be a distance-regular graph of order $(s, 2)$ with diameter d and the intersection array*

$$\iota(\Gamma) = \left\{ \begin{array}{ccccc} * & 1 & \cdots & 1 & c_d \\ 0 & s-1 & \cdots & s-1 & a_d \\ 3s & 2s & \cdots & 2s & * \end{array} \right\}.$$

Suppose $s \geq 2$. Then $c_d = 3$ and $(d, s) = (2, 2), (2, 4), (3, 2), (3, 8)$ or $(4, 4)$.

Proof: By counting the number of complete subgraphs of size $s + 1$ in Γ we have

$$3|V\Gamma| \equiv 0 \pmod{s+1}.$$

Suppose $c_d = 1$. Then it follows, by Proposition 3, that $d = 2$ and thus $|V\Gamma| = 1 + 3s + 6s^2$. We have $s = 2, 3, 5$ or 11 from the first assertion. We can show that no such graphs exist by calculating the multiplicity of the eigenvalues.

Suppose $c_d = 2$. Then we have $d \leq 13$ from Proposition 3. We have

$$3|V\Gamma| = \frac{3}{2s-1}[-1 - s + 3s^2(2s+1)(2s)^{d-2}] \equiv 0 \pmod{s+1}$$

from the first assertion. For given d with $d \leq 13$ there are only finitely many possible values for s. All of them are ruled out by integrality of the multiplicities of eigenvalues.

Suppose $c_d = 3$. Then Proposition 3 (2) shows that $(d, s) = (2, 2), (2, 3), (2, 4), (3, 2), (3, 8)$ or $(4, 4)$. We can show that the case $(d, s) = (2, 3)$ is impossible by calculating the multiplicity of the eigenvalues. The desired result is proved. □

Remark There are unique $GQ(2, 2)$, $GQ(4, 2)$ and $GH(8, 2)$. There are exactly two $GH(2, 2)$ and those are dual each other. There exists a $GO(4, 2)$ but the uniqueness problem has not been settled yet.

3. Circuit chasing

In this section we prove the following result.

Proposition 5 *Let Γ be a distance-regular graph with $r = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$ and $(c_{r+1}, a_{r+1}) = (2, 2a_1)$. If $a_1 > 0$, then $c_{r+2} \neq 2$.*

In [14] we have shown that $(c_{r+2}, a_{r+2}) \neq (2, 2a_1)$ by using the circuit chasing technique.

Let Γ be a distance-regular graph of diameter d and let (u, v) be an edge in Γ. Set $D^i_j = D^i_j(u, v) := \Gamma_i(u) \cap \Gamma_j(v)$. The *intersection diagram with respect to* (u, v), is the collection $\{D^i_j\}_{0 \leq i,j \leq d}$ with lines between them. If there is no line between D^i_j and D^s_t, it means that there is no edge (x, y) with $x \in D^i_j$ and $y \in D^s_t$. We write $e(x, D^i_j)$ for the number of neighbors of a vertex x in D^i_j.

Take a circuit and write down the distance distribution, which is called the *profile,* with respect to one of its edges and then to derive the profile with respect to the next edge, using the intersection diagram. We continue this procedure successively to obtain some information for Γ.

More information on the intersection diagram and circuit chasing can be found in [5, 14].

We recall the following lemma, which was proved in [14, Section 3] except for the statement (3).

Lemma 6 *Let Γ be a distance-regular graph as in Proposition 5 with $r \geq 2$, $a_1 > 0$ and $c_{r+2} = 2$. Let (u, v) be an edge of Γ. Then the intersection diagram with respect to (u, v) has the shape as in Figure 1. Moreover the following hold.*

(1) *Let $x \in D^{r+1}_{r+1}$. Then $e(x, D^r_r) = e(x, D^r_{r+1}) = e(x, D^{r+1}_r) = 1$. Let $\{\alpha\} = D^r_{r+1} \cap \Gamma_1(x)$ and $\{\beta\} = D^{r+1}_r \cap \Gamma_1(x)$. Then α and β are adjacent.*
(2) *Let $y \in D^{r+1}_r$, $\{y'\} = D^r_{r+1} \cap \Gamma_1(y)$ and $B = D^{r+1}_{r+1} \cap \Gamma_1(y)$. Then $\{y, y'\} \cup B$ is a clique.*

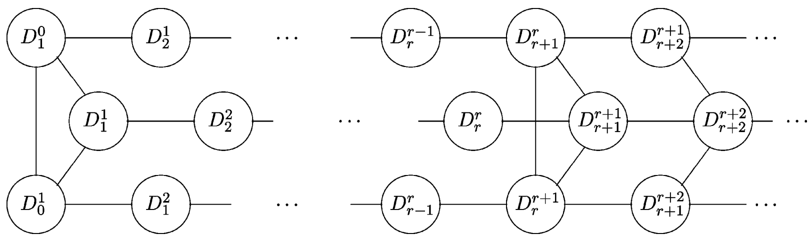


Figure 1.

(3) *Let* $z \in D_{r+2}^{r+2}$ *with* $e(z, D_{r+1}^{r+1}) \neq 0$. *Then* $e(z, D_{r+1}^{r+1}) = 2$. *Let* $\{z', z''\} = D_{r+1}^{r+1} \cap \Gamma_1(z)$. *Then* z' *and* z'' *are not adjacent.*

Proof: (3) There exist $z' \in D_{r+1}^{r+1} \cap \Gamma_1(z)$ and $z_r \in D_r^r \cap \Gamma_1(z')$ from (1). Then there exists $z_i \in D_i^i$ such that $\partial_\Gamma(z, z_i) = r + 2 - i$ for all $i = r, r-1, \ldots, 1$.

It is clear that $\Gamma_r(z_1) \cap \Gamma_1(z) \subseteq D_{r+1}^{r+1}$. Thus we have

$$2 = c_{r+1} = |\Gamma_r(z_1) \cap \Gamma_1(z)| \leq e\big(z, D_{r+1}^{r+1}\big) \leq c_{r+2} = 2.$$

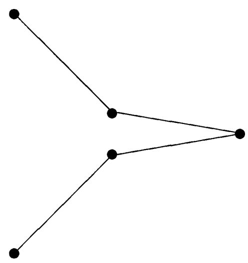
Hence we have $e(z, D_{r+1}^{r+1}) = 2$ and $\{z', z''\} = D_{r+1}^{r+1} \cap \Gamma_1(z) = \Gamma_r(z_1) \cap \Gamma_1(z)$.

Consider the intersection diagram with respect to (z_1, z_2). Then $z' \in D_{r-1}^r$, $z \in D_r^{r+1}$ and $z'' \in D_{r+1}^r$. The lemma is proved. $\square$

Proof of Proposition 5: Since $1 < c_2$ implies $c_2 < c_3$, we may assume $r \geq 2$.

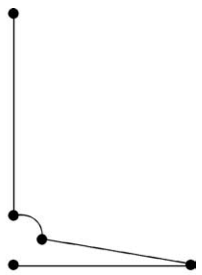
Suppose $c_{r+2} = 2$ and derive a contradiction. Let $C = (x_0, x_1, \ldots, x_{2r+4})$ be a circuit of length $2r + 5$ whose profile with respect to (x_0, x_1) is as follows.

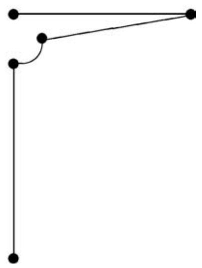
	x_{r+1}	x_{r+2}	x_{r+3}	x_{r+4}	x_{r+5}
x_0	$r+1$	$r+1$	$r+2$	$r+1$	r
x_1	r	$r+1$	$r+2$	$r+1$	$r+1$



(This circuit is the same to the first circuit in the proof of the theorem in [14]. We may only consider the *middle part* of the profiles. See [14, Section 3].) It is not hard to see that there exists such a circuit C and that no three vertices of C do not form a triangle by Lemma 6 (3). Now we can uniquely determine the profiles of C with respect to (x_1, x_2) and with respect to (x_2, x_3) as follows:

	x_{r+2}	x_{r+3}	x_{r+4}	x_{r+5}	x_{r+6}
x_1	$r+1$	$r+2$	$r+1$	$r+1$	r
x_2	r	$r+1$	r	r	$r+1$



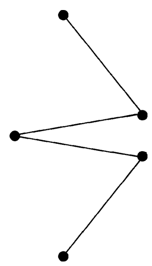


	x_{r+3}	x_{r+4}	x_{r+5}	x_{r+6}	x_{r+7}
x_2	$r+1$	r	r	$r+1$	r
x_3	r	$r+1$	$r+1$	$r+2$	$r+1$

And the profiles of C with respect to (x_3, x_4) is the same to the profile with respect to (x_0, x_1). It follows that the profile of C with respect to (x_i, x_{i+1}) is the same as one of these three types of profile for any $0 \leq i \leq 2r+4$.

The profiles with respect to (x_0, x_1), (x_1, x_2), (x_2, x_3) and (x_3, x_4) give us the distance relation between $\{x_{r+4}, x_{r+5}\}$ and $\{x_0, x_1, x_2, x_3, x_4\}$ as follows.

	x_0	x_1	x_2	x_3	x_4
x_{r+4}	$r+1$	$r+1$	r	$r+1$	r
x_{r+5}	r	$r+1$	r	$r+1$	$r+1$



Then the profile of C with respect to (x_{r+4}, x_{r+5}) is different from the above three types of profile. This is a contradiction. $\square$

4. The case of $s \neq 3, 6$

Let Γ be a regular near polygon of order $(s, 2)$ with $r = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. Assume $d \geq r+2$. Then we have $c_{r+1} = 2,\ c_{r+2} = 3$ and $d = r+2$ from Proposition 5.

Therefore we only need to consider the regular near $2d$-gon as in Proposition 2.

Throughout this section Γ denotes a distance-regular graph as in Proposition 2 with $s \geq 3$.

It is known that regular near $2d$-gon of order (s, t) has the smallest eigenvalue $-t-1$. So we have the following result by a well known multiplicity formula.

Lemma 7 *Let Γ be a regular near $2d$-gon as in Proposition 2. Then -3 is the smallest eigenvalue of Γ with multiplicity*

$$m(-3) = \frac{s^{r+2}(s-2)\{2^{r-1}s^{r+1}(2s+3)-1\}}{(2s-1)\{s^{r+2}-(3s+2)2^{r-1}\}}.$$

Proof: We have $k_0 = 1$, $k_i = 3s(2s)^{i-1}$ for $1 \leq i \leq r$, $k_{r+1} = \frac{3s}{2}(2s)^r$ and $k_{r+2} = \frac{s^2}{2}(2s)^r$. It is straightforward to see that $u_i(-3) = (-s)^{-i}$ for all $0 \leq i \leq r+2$. Hence

$$|V\Gamma| = \frac{s+1}{2s-1}\{2^{r-1}s^{r+1}(2s+3)-1\}$$

and

$$\sum_{i=0}^{d}\left(\frac{k_i}{s^{2i}}\right) = \frac{s+1}{s^{r+2}(s-2)}\{s^{r+2}-(3s+2)2^{r-1}\}.$$

The desired result is proved. □

Lemma 8

(1) *Suppose $s = 4n$ for some integer n. Let $q := 2^{r+4}n^{r+2} - 6n - 1$. Then*

$$(2n-1)\{2^{3r+1}n^{r+1}(8n+3)-1\} \equiv 0 \pmod{q}.$$

(2) *If s is odd, then*

$$\left(\frac{s}{4}\right)^r < \frac{3s+2}{2}.$$

(3) *If $s = 2z$ for some odd integer z, then*

$$\left(\frac{z}{4}\right)^r < \frac{(z-1)(3z+1)(4z+3)}{2z^2}.$$

Proof: (1) Lemma 7 implies that

$$m(-3) = \frac{2^{r+5}n^{r+2}(2n-1)\{2^{3r+1}n^{r+1}(8n+3)-1\}}{(8n-1)(2^{r+4}n^{r+2}-6n-1)}.$$

Since $2^{r+5}n^{r+2}$ and q are relatively prime, the assertion follows from the integrality of $m(-3)$.

(2) Let $q' := s^{r+2} - (3s+2)2^{r-1}$. Then s and q' are relatively prime. By the integrality of $m(-3)$ and Lemma 7 we have

$$(s-2)\{2^{r-1}s^{r+1}(2s+3)-1\} \equiv 0 \pmod{q'}.$$

Since $s^{r+2} \equiv (3s+2)2^{r-1} \pmod{q'}$, we have

$$\begin{aligned} 0 &\equiv (s-2)\{2^{r-1}s^{r+1}(2s+3)-1\}s \\ &\equiv (s-2)\{(2s+3)(3s+2)4^{r-1}-s\} \pmod{q'} \end{aligned}$$

and hence

$$\{s^{r+2} - (3s+2)2^{r-1}\} = q' \le (s-2)\{(2s+3)(3s+2)4^{r-1}-s\}.$$

This implies

$$s^{r+2} < (s-2)(2s+3)(3s+2)4^{r-1} + (3s+2)2^{r-1} < 2s^2(3s+2)4^{r-1}.$$

The desired result is proved.

(3) It follows, by Lemma 7, that

$$m(-3) = \frac{8z^{r+2}(z-1)\{4^r z^{r+1}(4z+3)-1\}}{(4z-1)(4z^{r+2}-3z-1)}.$$

Let $q'' := 4z^{r+2} - 3z - 1$. Then z and q'' are relatively prime and thus

$$\begin{aligned} 0 &\equiv 8(z-1)\{4^r z^{r+1}(4z+3)-1\}z \\ &\equiv 8(z-1)\{4^{r-1}(3z+1)(4z+3)-z\} \pmod{q''}. \end{aligned}$$

Hence we have

$$(4z^{r+2} - 3z - 1) = q'' \le 8(z-1)\{4^{r-1}(3z+1)(4z+3)-z\}.$$

The desired result is proved. □

Lemma 9 *If* $s \neq 3, 6$, *then* $r = 1$.

Proof: Assume $r \geq 2$.

Suppose s is odd. Then $s \geq 5$ from our assumption. It follows, by Lemma 8(2), that $s < 25$. For given odd integer s with $5 \leq s \leq 23$ there are only finitely many possible values for r. All of them are ruled out by integrality of $m(-3)$.

Suppose there exists an odd integer z such that $s = 2z$. Then $z \geq 5$ from our assumption. It follows, by Lemma 8(3), that $z < 97$. For given odd integer z with $5 \leq z \leq 95$ there are only finitely many possible values for r. All of them are ruled out by integrality of $m(-3)$.

Suppose there exists an integer n such that $s = 4n$. First we assume $n = 1$. Then it follows, by Lemma 8(1), that

$$0 \equiv \{11 \cdot 2^{3r+1} - 1\}2^{11} \equiv \{11 \cdot 7^3 - 2^{11}\} \pmod{2^{r+4} - 7}.$$

We have $2^{r+4} - 7 \leq 11 \cdot 7^3 - 2^{11}$ and thus $r < 8$. They are ruled out by integrality of $m(-3)$. Next we assume $n = 2$. Then Lemma 8(1) implies that

$$0 \equiv 3\{19 \cdot 2^{4r+2} - 1\}2^{10} \equiv 3\{19 \cdot 13^2 - 2^{10}\} \pmod{2^{2r+6} - 13}.$$

We have $2^{2r+6} - 13 \leq 3\{19 \cdot 13^2 - 2^{10}\}$ and thus $r = 2$ which is impossible as $3\{19 \cdot 13^2 - 2^{10}\} \not\equiv 0 \pmod{2^{10} - 13}$. Finally we assume $n \geq 3$. Let $q := 2^{r+4}n^{r+2} - 6n - 1$. Then

$$\begin{aligned} 0 &\equiv (2n-1)\{2^{3r+1}n^{r+1}(8n+3) - 1\}n \\ &\equiv (2n-1)\{2^{2r-3}(6n+1)(8n+3) - n\} \pmod{q}. \end{aligned}$$

It follows that

$$(2^{r+4}n^{r+2} - 6n - 1) = q \leq (2n-1)\{2^{2r-3}(6n+1)(8n+3) - n\}.$$

This is a contradiction as $n \geq 3$ and $r \geq 2$. The desired result is proved. □

5. The case of $s = 3, 6$

In this section we prove the remaining case $s = 3, 6$ of Proposition 2.

First we recall some basic results of distance-regular graphs.

Let Γ be a distance-regular graph of diameter $d \geq 3$ and valency $k \geq 3$. Let $\theta_0 = k, \theta_1, \ldots, \theta_d$ be the distinct eigenvalues of Γ.

The monic polynomials $F_i(x)$ $(0 \leq i \leq d)$ are defined by the recurrence relation

$$F_i(x) := (x - k + b_{i-1} + c_i)F_{i-1}(x) - b_{i-1}c_{i-1}F_{i-2}(x) \quad \text{for } i = 2, \ldots, d$$

with $F_0(x) = 1$ and $F_1(x) = x + 1$. It is well known that

$$F_d(x) = (x - \theta_1)(x - \theta_2)\ldots(x - \theta_d)$$

and

$$m(\theta_i) = \frac{|V\Gamma| b_0 b_1 \ldots b_{d-1} c_2 \ldots c_{d-1}}{(k-\theta_i) F_d'(\theta_i) F_{d-1}(\theta_i)}.$$

for all $1 \le i \le d$.

Let θ be an eigenvalue of Γ with $\theta \neq k$. Then the minimal polynomial of θ over the rational field divides $F_d(x)$ and thus its algebraic conjugate ρ is also an eigenvalue of Γ. In particular, $m(\theta) = m(\rho)$. (See [1, Section III.1] and [6, Chapter 4].)

Throughout this section Γ denotes a distance-regular graph as in Proposition 2 with $s = 3, 6$. We assume $r \ge 2$ to derive a contradiction.

Let $x = s - 1 + 2\sqrt{2s}\cos\phi$ and $\sigma = e^{\phi\sqrt{-1}}$. Let

$$h_i = h_i(\sigma) := \begin{cases} \dfrac{\sqrt{2s}^{\,i-1}}{\sigma^i(\sigma^2-1)}[\sqrt{2s}(\sigma^{2i+2}-1) + s\sigma(\sigma^{2i}-1)] & \text{if } \sigma \neq \pm 1, \\ (\sqrt{2s}\,\sigma)^{i-1}[\sqrt{2s}(i+1)\sigma + si] & \text{if } \sigma = \pm 1. \end{cases}$$

Then the sequence $\{h_i\}$ satisfies the recurrence relation

$$h_i = (x - s + 1)h_{i-1} - 2sh_{i-2} \quad \text{for } i = 2, 3, \ldots$$

with $h_0 = 1$ and $h_1 = x + 1$. Let

$$P(\sigma) := 2s\sigma^2 + \sqrt{2s}(1-s)\sigma + s,$$
$$Q(\sigma) := s\sigma^2 + \sqrt{2s}(1-s)\sigma + 2s$$

and

$$R(\sigma) := \left(\sigma + \frac{\sqrt{2s}}{2}\right)\left(\sigma + \frac{2}{\sqrt{2s}}\right) = \sigma^2 + \frac{s+2}{\sqrt{2s}}\sigma + 1.$$

Lemma 10 (1) *$F_i(x) = h_i$ for all $i = 0, 1, \ldots, r$ and $F_{r+1}(x) = h_{r+1} + h_r$.*

(2) *If $\sigma \neq \pm 1$, then*

$$F_{r+1}(x) = \frac{\sqrt{2s}^{\,r-1}}{\sigma^{r+1}(\sigma^2-1)}[\sigma^{2r+2}(P(\sigma) + (\sqrt{2s})^3\sigma) - (Q(\sigma) + (\sqrt{2s})^3\sigma)],$$

$$F_{r+2}(x) = \frac{\sqrt{2s}^{\,r} R(\sigma)}{\sigma^{r+2}(\sigma^2-1)}[\sigma^{2r+2}P(\sigma) - Q(\sigma)].$$

Proof: (1) The first assertion follows by induction on i. Then we have

$$\begin{aligned} F_{r+1}(x) &= (x-s+2)F_r(x) - 2sF_{r-1}(x) \\ &= (x-s+1)F_r(x) - 2sF_{r-1}(x) + F_r(x) \;=\; h_{r+1} + h_r. \end{aligned}$$

(2) We have

$$\begin{aligned} F_{r+2}(x) &= (x-2s+3)F_{r+1}(x) - 2sF_r(x) \\ &= (x-s+1)F_{r+1}(x) - 2sF_r(x) + (2-s)F_{r+1}(x) \\ &= (x-s+1)(h_{r+1}+h_r) - 2sh_r + (2-s)(h_{r+1}+h_r) \\ &= h_{r+2} + (3-s)h_{r+1} + (2-s)h_r + 2sh_{r-1}. \end{aligned}$$

The assertions follow by putting

$$h_i := \frac{\sqrt{2s}^{\,i-1}}{\sigma^i(\sigma^2-1)}[\sqrt{2s}(\sigma^{2i+2}-1) + s\sigma(\sigma^{2i}-1)].$$

□

Remarks (1) $\sigma = \pm 1$ if and only if $x = s - 1 \pm 2\sqrt{2s}$.
(2) We have $\sqrt{2s}R(\sigma) = \sigma(x+3)$. Hence $R(\sigma) = 0$ if and only if $x = -3$.

For functions $p(x)$ and $q(\sigma)$ we denote by $p'(x)$ and $q^*(\sigma)$ the derived functions corresponding to x and σ, respectively.
Let

$$\begin{aligned} f_0(x) &:= x^2 + 5(1-s)x + 6s^2 - 11s + 6, \\ f_1(x) &:= (1-s)x + s^2 + 4s + 1, \\ f_2(x) &:= (x-s+1+2\sqrt{2s})(x-s+1-2\sqrt{2s}) \\ &= x^2 + 2(1-s)x + s^2 - 10s + 1, \\ g_1(x) &:= \frac{f_1(x)}{f_0(x)} \quad \text{and} \quad g_2(x) := \frac{(x-3s)(x+3)}{f_2(x)}. \end{aligned}$$

Then it is straightforward to see that $P(\sigma)Q(\sigma) = s\sigma^2 f_0(x)$, $P^*(\sigma)Q(\sigma) - P(\sigma)Q^*(\sigma) = s\sigma f_1(x)$ and $2s(\sigma^2-1)^2 = \sigma^2 f_2(x)$.

Lemma 11 *Let θ be an eigenvalue of Γ with $\theta \neq 3s, -3, s-1 \pm 2\sqrt{2s}$. Let $\theta = s - 1 + 2\sqrt{2s}\cos\psi$ and $\tau = e^{\psi\sqrt{-1}}$. Then the following hold.*
(1) *$\tau^{2r+2}P(\tau) = Q(\tau)$ and $P(\tau) \neq 0$.*
(2) *Let*

$$G(x) := g_2(x)\{2r + 2 + g_1(x)\}.$$

If the multiplicity of θ is m, then θ is a root of the equation $G(x) = \frac{3|V\Gamma|}{m}$.

Proof: (1) The first assertion follows from Lemma 10(2). The second assertion follows from $P(\tau)(\tau^{2r+2} - 1) = Q(\tau) - P(\tau) = -s(\tau^2 - 1) \neq 0$.

(2) By the assertion (1) and Lemma 10(2) we have

$$F_{r+1}(\theta) = \frac{\sqrt{2s}^{\,r-1}}{\tau^{r+1}(\tau^2-1)}[(\sqrt{2s})^3\tau(\tau^{2r+2}-1)] = \frac{\sqrt{2s}^{\,r+2}}{\tau^{r+1}}\left[\frac{-s\tau}{P(\tau)}\right].$$

Let $N(\sigma) := \frac{\sqrt{2s}^{\,r} R(\sigma)}{\sigma^{r+2}(\sigma^2-1)}$ and $L(\sigma) := [\sigma^{2r+2}P(\sigma) - Q(\sigma)]$. Then $F_{r+2}(x) = N(\sigma)L(\sigma)$ and thus

$$F'_{r+2}(x) = N^*(\sigma)\sigma' L(\sigma) + N(\sigma)L^*(\sigma)\sigma'.$$

Since $x = s - 1 + \sqrt{2s}(\sigma + \frac{1}{\sigma})$, we have $\sigma' = \frac{\sigma^2}{\sqrt{2s}(\sigma^2-1)}$. It follows that

$$\begin{aligned}
F'_{r+2}(\theta) &= N(\tau)L^*(\tau)\frac{\tau^2}{\sqrt{2s}(\tau^2-1)} \\
&= \frac{\sqrt{2s}^{\,r-1}\tau^2 R(\tau)}{\tau^{r+2}(\tau^2-1)^2}[(2r+2)\tau^{2r+1}P(\tau) + \tau^{2r+2}P^*(\tau) - Q^*(\tau)] \\
&= \frac{\sqrt{2s}^{\,r-2}\tau^2(\theta+3)}{\tau^{r+2}(\tau^2-1)^2 P(\tau)}[(2r+2)P(\tau)Q(\tau) + \tau\{P^*(\tau)Q(\tau) - P(\tau)Q^*(\tau)\}] \\
&= \frac{\sqrt{2s}^{\,r}(\theta+3)}{\tau^{r+2} f_2(\theta)P(\tau)}[(2r+2)s\tau^2 f_0(\theta) + s\tau^2 f_1(\theta)].
\end{aligned}$$

Since

$$m = \frac{|V\Gamma| b_0 b_1 \ldots b_{d-1} c_2 \ldots c_{d-1}}{(k-\theta)F'_d(\theta)F_{d-1}(\theta)} = \frac{|V\Gamma| 3s(2s)^{r+1}}{(3s-\theta)F'_{r+2}(\theta)F_{r+1}(\theta)},$$

we have

$$\frac{3|V\Gamma|}{m} = \frac{(\theta-3s)(\theta+3)}{f_0(\theta)f_2(\theta)}[(2r+2)f_0(\theta) + f_1(\theta)] = G(\theta).$$

The lemma is proved. □

Lemma 12 (1) *Let θ be an eigenvalue of Γ with $\theta \neq 3s, -3$. Then*

$$s - 1 - 2\sqrt{2s} < \theta < s - 1 + 2\sqrt{2s}.$$

(2) *The second largest eigenvalue* θ_1 *of* Γ *satisfies*

$$\theta_1 > s - 1 + 2\sqrt{2s}\cos\left(\frac{\pi}{r}\right).$$

(3) *If* $\theta_1 > s - 1 + \sqrt{8s-1}$, *then there exists an algebraic conjugate* ρ *of* θ_1 *such that*

$$s - 1 - \sqrt{8s-2} < \rho < s - 1 + \sqrt{8s-2}.$$

Proof: (1) Let $\alpha = s - 1 + 2\sqrt{2s}$. Then $F_i(\alpha) = h_i = \sqrt{2s}^{\,i-1}[\sqrt{2s}(i+1) + si] > 0$ for all $0 \le i \le r$, $F_{r+1}(\alpha) = h_{r+1} + h_r > 0$ and

$$\begin{aligned} F_{r+2}(\alpha) &= (2 + 2\sqrt{2s} - s)F_{r+1}(\alpha) - 2sF_r(\alpha) \\ &= (2 + 2\sqrt{2s} - s)h_{r+1} + (2 + 2\sqrt{2s} - 3s)h_r > 0. \end{aligned}$$

Since $\{F_i(x)\}$ is a Sturm series, the largest root of $F_{r+2}(x)$ is less than α. In particular, α is not an eigenvalue of Γ and hence its algebraic conjugate $s - 1 - 2\sqrt{2s}$ is not an eigenvalue of Γ either.

Suppose there exists an eigenvalue θ with $-3 < \theta < s - 1 - 2\sqrt{2s}$. Then $g_2(\theta) < 0 < g_1(\theta)$. We have $G(\theta) < 0$ which contradicts Lemma 11(2). The assertion is proved.

(2) Let $\psi := \left(\frac{\pi}{r}\right)$, $\tau = e^{\psi\sqrt{-1}}$ and $\beta := s - 1 + 2\sqrt{2s}\cos\psi$. Then Lemma 10 implies that

$$\begin{aligned} F_r(\beta) &= \frac{\sqrt{2s}^{\,r-1}}{\tau^r(\tau^2 - 1)}[\sqrt{2s}(\tau^{2r+2} - 1) + s\tau(\tau^{2r} - 1)] \\ &= \frac{\sqrt{2s}^{\,r-1}}{(\tau - \tau^{-1})}\left[\sqrt{2s}\left(\tau^{r+1} - \tau^{-(r+1)}\right) + s(\tau^r - \tau^{-r})\right] \\ &= \frac{\sqrt{2s}^{\,r-1}}{\sin\psi}[\sqrt{2s}\sin(r+1)\psi + s\sin r\psi] < 0. \end{aligned}$$

This implies that the largest root of $F_r(x)$ is greater than β. Since $\{F_i(x)\}$ is a Sturm series, the largest root of $F_{r+2}(x)$ is greater than the largest root of $F_r(x)$. The desired result is proved.

(3) Let

$$\gamma := \prod_{\theta} |(\theta - s + 1)^2 - (8s - 1)|,$$

where θ through over all algebraic conjugates of θ_1. Then γ has to be a non-zero integer.

Since $|(\theta_1 - s + 1)^2 - (8s - 1)| < 1$, there exists an algebraic conjugates ρ of θ_1 such that $|(\rho - s + 1)^2 - (8s - 1)| > 1$. The assertion follows from (1). □

Lemma 13 (1) *If* $s = 3$ *then* $r \leq 15$.
(2) *If* $s = 6$, *then* $r \leq 21$.

Proof: Note that

$$g_1'(x) = \frac{1}{f_0(x)^2}\{(s-1)x^2 - 2(s^2+4s+1)x - (s-1)(s^2-31s+1)\}$$

and

$$g_2'(x) = \frac{1}{f_2(x)^2}(x-3+3s)\{(s-1)x - s^2+4s-1\}.$$

(1) Suppose $s = 3$ and $r \geq 16$. Then the second largest eigenvalue θ_1 satisfies

$$\theta_1 > 2 + 2\sqrt{6}\cos\left(\frac{\pi}{16}\right) > 2 + \sqrt{23}$$

and there exists an algebraic conjugate ρ of θ_1 such that $2 - \sqrt{22} < \rho < 2 + \sqrt{22}$ from Lemma 12. We remark that $g_2(x)$ is a decreasing function in $2 - 2\sqrt{6} < x < -1$ and an increasing function in $-1 < x < 2 + 2\sqrt{6}$. Hence we have

$$g_2(\theta_1) > g_2(2 + \sqrt{23}) > 21$$

and

$$g_2(\rho) < \max\{g_2(2 - \sqrt{22}), g_2(2 + \sqrt{22})\} < 12.$$

Note that $0 < g_1(x) < 7$ for any $2 - 2\sqrt{6} < x < 2 + 2\sqrt{6}$. It follows, by Lemma 11(2), that

$$\begin{aligned} 21(2r+2) &< g_2(\theta_1)\{2r+2+g_1(\theta_1)\} \\ &= g_2(\rho)\{2r+2+g_1(\rho)\} < 12(2r+2+7). \end{aligned}$$

This is a contradiction.
(2) Suppose $s = 6$ and $r \geq 22$. Then the second largest eigenvalue θ_1 satisfies

$$\theta_1 > 5 + 4\sqrt{3}\cos\left(\frac{\pi}{22}\right) > 5 + \sqrt{47}$$

and that there exists an algebraic conjugate ρ of θ_1 such that $5 - \sqrt{46} < \rho < 5 + \sqrt{46}$ from Lemma 12. Since $g_1'(x) > 0$ and $g_2'(x) = \frac{1}{f_2(x)^2}(5x-13)(x+15)$, we have $g_1(\rho) < g_1(\theta_1)$ and

$$g_2(\rho) < \max\{g_2(5 - \sqrt{46}), g_2(5 + \sqrt{46})\} < g_2(5 + \sqrt{47}) < g_2(\theta_1).$$

Hence we have

$$g_2(\rho)\{2r+2+g_1(\rho)\} < g_2(\theta_1)\{2r+2+g_1(\theta_1)\}.$$

This is a contradiction. The lemma is proved. □

Proof of Proposition 2: The case $s \neq 3, 6$ is proved by Lemma 9.

Suppose $s = 3$ or 6. Then there are only finitely many possible values for r from Lemma 13.

All possible values for r with $r \geq 2$ are ruled out by integrity of $m(-3)$ and Lemma 7. Hence the desired result is proved. □

6. Proof of the theorem

We prove our main theorem.

Proof of Theorem 1: Let Γ be a regular near polygon of order $(s, 2)$ with diameter d.

If $s = 1$ or 2, then our theorem is true by the classifications of distance-regular graphs of valency 3, and of distance-regular graphs with $k = 6$ and $a_1 = 1$. (See [4, 15, 16].) Hence we may assume $s \geq 3$. Let $r = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$.

Suppose $d = r + 1$. Then the assertion follows from Lemma 4.

Suppose $d \geq r + 2$. Then we have $c_{r+1} = 2$, $c_{r+2} = 3$ and $d = r + 2$ from Proposition 5. It follows, by Proposition 2, that $r = 1$ and hence Γ has to be the Hamming graph $H(3, s+1)$.

The theorem is proved. □

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